

# THE DIRICHLET PROBLEM FOR $m$ -SUBHARMONIC FUNCTIONS ON COMPACT SETS

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## Abstract

We characterize those compact sets for which the Dirichlet problem has a solution within the class of continuous  $m$ -subharmonic functions defined on a compact set, and then within the class of  $m$ -harmonic functions.

## 1. Introduction

A fundamental tool in the study of uniform algebras is the class of subharmonic functions defined on compact sets, and its dual, the Jensen measures. In [6], Gamelin presented a model that can be used both for subharmonic as well as plurisubharmonic functions defined on compact sets. In this paper we shall use this model to investigate the Dirichlet problem for  $m$ -(sub)harmonic functions. Our inspiration is the work of Poletsky [11] and especially Poletsky-Sigurdsson [13], [14].

Two natural types of boundaries in potential theory are the Choquet boundary (Definition 3.1) with respect to a given class of Jensen measures, and the Šilov boundary (Definition 3.6). In our study of the Dirichlet problem these boundaries have a prominent role, and therefore we shall in §3 put extra attention on them in terms of for example peak points and harmonic  $m$ -measures. We shall then, in §4, characterize those compact sets for which the Dirichlet problem has a solution within the class of continuous  $m$ -subharmonic functions (Theorem 4.2 and Theorem 4.3). We end this paper in §5 with a Dirichlet problem for  $m$ -harmonic functions (Theorem 5.10). In the 1-subharmonic case, these results were obtained by Hansen among others (see e.g. [5], [7], [10] and the references therein), and in the  $n$ -subharmonic case these results were proved by Poletsky-Sigurdsson [13], [14]. In this paper we deal with the remaining cases  $1 < m < n$ . We start in §2 by stating some basic definitions and facts.

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## 2. Jensen measures and envelopes

We start this section with recalling the definition of  $m$ -subharmonic functions defined on domains in  $\mathbb{C}^n$ . For further information about  $m$ -subharmonic functions see e.g. [15], [2], [3], [9].

Let  $\mathbb{C}_{(1,1)}$  to be the set of  $(1, 1)$ -forms with constant coefficients and let

$$\Gamma_m = \{\alpha \in \mathbb{C}_{(1,1)} : \alpha \wedge (dd^c |z|^2)^{n-1} \geq 0, \dots, \alpha^m \wedge (dd^c |z|^2)^{n-m} \geq 0\}.$$

DEFINITION 2.1. Let  $\Omega \subset \mathbb{C}^n$  be a domain,  $1 \leq m \leq n$ , and let  $u$  be a subharmonic function defined on  $\Omega$ . Then we say that  $u$  is  $m$ -subharmonic if the following inequality holds

$$dd^c u \wedge \alpha_1 \wedge \dots \wedge \alpha_{m-1} \wedge (dd^c |z|^2)^{n-m} \geq 0,$$

in the sense of currents for all  $\alpha_1, \dots, \alpha_{m-1} \in \Gamma_m$ . By  $\mathcal{SH}_m(\Omega)$  we denote the set of all  $m$ -subharmonic functions defined on  $\Omega$ .

REMARK. Note that

$$\mathcal{PSH}(\Omega) = \mathcal{SH}_n(\Omega) \subset \dots \subset \mathcal{SH}_1(\Omega) = \mathcal{SH}(\Omega),$$

where  $\mathcal{PSH}(\Omega)$  denotes the set of plurisubharmonic functions,  $\mathcal{SH}(\Omega)$  denotes the set of subharmonic function defined on  $\Omega$ .

REMARK. A smooth  $\mathcal{C}^2$  function  $u$  is  $m$ -subharmonic in  $\Omega$  if the following elementary symmetric functions are non-negative

$$\sigma_k(\lambda(H_u(z))) = \sum_{1 \leq j_1 < \dots < j_k \leq n} \lambda_{j_1}(z) \cdots \lambda_{j_k}(z), \quad \text{for } k = 1, \dots, m, \quad z \in \Omega,$$

where  $\lambda(H_u(z)) = (\lambda_1(z), \dots, \lambda_n(z))$  are eigenvalues of the Hessian matrix

$$H_u(z) = \left[ \frac{\partial^2 u(z)}{\partial z_j \partial \bar{z}_k} \right]_{j,k=1}^n.$$

Assume now that  $X \subset \mathbb{C}^n$  is a compact set. Let  $\mathcal{SH}_m^o(X)$  denote the set of functions that are the restriction to  $X$  of functions that are  $m$ -subharmonic and continuous on some neighborhood of  $X \subseteq \mathbb{C}^n$ . We define a class of Jensen measures.

DEFINITION 2.2. Let  $X$  be a compact set in  $\mathbb{C}^n$ ,  $1 \leq m \leq n$ , and let  $\mu$  be a non-negative regular Borel measure defined on  $X$  with  $\mu(X) = 1$ . We say that  $\mu$  is a *Jensen measure with barycenter*  $z \in X$  with respect to  $\mathcal{SH}_m^o(X)$  if

$$u(z) \leq \int_X u d\mu \quad \text{for all } u \in \mathcal{SH}_m^o(X).$$

The set of such measures will be denoted by  $\mathcal{J}_z^m(X)$  or simply  $\mathcal{J}_z^m$ .

With the help of the Jensen measures defined in Definition 2.2 we can now define  $m$ -subharmonic functions defined on compact sets. For more results about these functions, see [2].

DEFINITION 2.3. Let  $X$  be a compact set in  $\mathbb{C}^n$ . An upper semicontinuous function  $u$  defined on  $X$  is said to be  $m$ -subharmonic on  $X$ ,  $1 \leq m \leq n$ , if

$$u(z) \leq \int_X u \, d\mu, \quad \text{for all } z \in X \text{ and all } \mu \in \mathcal{J}_z^m(X).$$

The set of  $m$ -subharmonic functions defined on  $X$  will be denoted by  $\mathcal{SH}_m(X)$ . A function  $h: X \rightarrow \mathbb{R}$  is called  $m$ -harmonic if  $h$ , and  $-h$ , are  $m$ -subharmonic, i.e.  $h$  is continuous and for every  $z \in X$  and for every  $\mu \in \mathcal{J}_z^m$  we have

$$h(z) = \int h \, d\mu.$$

The set of all  $m$ -harmonic functions defined on  $X$  will be denoted by  $\mathcal{H}_m(X)$ . We shall call  $n$ -harmonic functions *pluriharmonic*, and denote it by  $\mathcal{PH}(X) = \mathcal{H}_n(X)$ .

REMARK. It follows from Definition 2.2 that for any  $z \in X$  we have

$$\mathcal{J}_z^m(X) = \bigcap_{X \subset U} \mathcal{J}_z^m(U) = \bigcap_{X \subset U} \mathcal{J}_z^{m,c}(U),$$

where  $U$  denotes an open set in  $\mathbb{C}^n$ . Recall that  $\mu \in \mathcal{J}_z^m(U)$  ( $\mathcal{J}_z^{m,c}(U)$ ) if  $\mu$  is a probability measure with compact support in  $U$ , and for all  $u \in \mathcal{SH}_m(U)$  ( $u \in \mathcal{SH}_m(U) \cap \mathcal{C}(U)$ ) it holds that

$$u(z) \leq \int u \, d\mu.$$

By Theorem 2.2 in [3] we have that  $\mathcal{J}_z^m(U) = \mathcal{J}_z^{m,c}(U)$ .

From the proof of Theorem 2.8 in [2] it follows that

$$\begin{aligned} \mathcal{J}_z^m(X) &= \left\{ \mu : u(z) \leq \int_X u \, d\mu, \text{ for all } u \in \mathcal{SH}_m(X) \right\} \\ &= \left\{ \mu : u(z) \leq \int_X u \, d\mu, \text{ for all } u \in \mathcal{SH}_m(X) \cap \mathcal{C}(X) \right\}. \end{aligned}$$

Furthermore, for every lower semi-continuous function  $f$  defined on  $X$  and for every  $z \in X$  we have

$$\begin{aligned} \mathbf{S}_f(z) &= \sup\{v(z) : v \in \mathcal{SH}_m^o(X), v \leq f\} \\ &= \sup\{v(z) : v \in \mathcal{SH}_m(X), v \leq f\} \\ &= \sup\{v(z) : v \in \mathcal{SH}_m(X) \cap \mathcal{C}(X), v \leq f\}. \end{aligned}$$

The celebrated Edwards' theorem states that

$$\mathbf{S}_f(z) = \inf \left\{ \int f d\mu : \mu \in \mathcal{J}_z^m(X) \right\}.$$

In Theorem 3.5 we shall use the following lemma.

**LEMMA 2.4.** *Let  $X$  be a compact subset in  $\mathbb{C}^n$ ,  $1 \leq m \leq n$ , and let  $f$  be a real-valued lower semicontinuous function defined on  $X$ . Then there exists a sequence  $u_j \in \mathcal{SH}_m^o(X)$  such that  $u_j \nearrow \mathbf{S}_f$ , as  $j \rightarrow \infty$ .*

**PROOF.** The proof follows from Edwards' theorem, and Choquet's lemma (see e.g. Lemma 2.3.4 in [8]).

### 3. The Choquet and Šilov boundaries of compact sets

The Choquet boundary (Definition 3.1) of a compact set  $X$  with respect to  $\mathcal{J}_z^m$  and its topological closure the Šilov boundary (Definition 3.6) are central concepts in the Dirichlet problems studied in §§4 and 5, so in this section we shall characterize these boundaries in terms of peak points and  $m$ -harmonic measures.

**DEFINITION 3.1.** Let  $1 \leq m \leq n$ , and let  $X$  be a compact set in  $\mathbb{C}^n$ . The *Choquet boundary* of  $X$  with respect to  $\mathcal{J}_z^m$  is defined as

$$O_X^m = \{z \in X : \mathcal{J}_z^m = \{\delta_z\}\}.$$

From Lemma 1.10 in [6] it follows that  $O_X^m$  is a  $G_\delta$ -set. Let  $\Omega$  be a bounded domain in  $\mathbb{C}^n$ , and set  $X = \bar{\Omega}$ . Then the Choquet boundary is contained in the topological boundary, i.e.  $O_X^m \subseteq \partial X$ .

Next we introduce the concept of  $m$ -subharmonic peak points.

**DEFINITION 3.2.** Let  $1 \leq m \leq n$ , and let  $X$  be a compact set in  $\mathbb{C}^n$ . We say that a point  $z \in X$  is a  *$m$ -subharmonic peak point* (or simply a *peak point*) if there exists a function  $u \in \mathcal{SH}_m(X)$  such that  $u(z) = 0$ , and  $u(w) < 0$  for  $w \in X \setminus \{z\}$ . The function  $u$  is then called a *peak function*.

Using Gamelin's more general setting we can, from Theorem 1.13 in [6], draw the conclusion that: a point  $z \in X$  is a  $m$ -subharmonic peak point if, and only if, there exists a function  $u \in \mathcal{SH}_m(X) \cap \mathcal{C}(X)$  such that  $u(z) = 0$  and  $u(w) < 0$  for  $w \in X \setminus \{z\}$ .

LEMMA 3.3. *Let  $1 \leq m \leq n$ , and let  $X$  be a compact set in  $\mathbb{C}^n$ . Then  $z \in O_X^m$  if, and only if, for every  $f \in \mathcal{C}(X)$  we have that  $\mathbf{S}_f(z) = f(z)$ .*

PROOF. Cf. page 10 in [6].

LEMMA 3.4. *Let  $1 \leq m \leq n$ , and let  $X$  be a compact set in  $\mathbb{C}^n$ . A point  $z \in X$  is a  $m$ -subharmonic peak point if, and only if, for any neighborhood  $V$  of  $z$  there exists  $u \in \mathcal{SH}_m(X)$  such that  $u < 0$ ,  $u(z) = -1$  and  $u \leq -2$  on  $X \setminus V$ .*

PROOF. The implication  $\Rightarrow$  is immediate. To prove the converse implication take  $z_0 \in X$ , and let  $\epsilon_j \searrow 0$ , as  $j \rightarrow \infty$ , be a sequence such that

$$\epsilon_j < \frac{1}{2} \frac{(3/4)^j}{1 - (3/4)^j}.$$

Let  $Y_n \subseteq X$  be closed subsets such that

$$X \setminus \{z_0\} = \bigcup_{n=1}^{\infty} Y_n.$$

Now we shall define a sequence of functions from  $\mathcal{SH}_m(X)$ . Let  $u_0 = -1$ . Suppose that we already have chosen  $u_0, \dots, u_j \in \mathcal{SH}_m(X) \cap \mathcal{C}(X)$ , such that for  $1 \leq k \leq j$  we have that  $u_k < 0$ ,  $u_k(z_0) = -1$ , and

$$u_k \leq -2 \quad \text{on } U_{k-1} \cup Y_{k-1},$$

where

$$U_{k-1} = \left\{ w \in X : \max_{1 \leq \ell \leq k-1} u_\ell(w) \geq -1 + \epsilon_{k-1} \right\}.$$

Note that  $\{U_k\}$  is an increasing sequence of closed sets. Now take a function  $u_{j+1} \in \mathcal{SH}_m(X)$  such that  $u_{j+1} \leq -2$  on  $Y_j \cup U_j$ , and  $u_{j+1}(z_0) = -1$ . Let us then define

$$u = \frac{1}{4} \sum_{j=0}^{\infty} \left( \frac{3}{4} \right)^j u_j.$$

This construction then implies that  $u \in \mathcal{SH}_m(X)$  and  $u(z_0) = -1$ . Now suppose that  $w \neq z_0$  and assume first that  $w \notin \bigcup_{k=1}^{\infty} U_k$ . Then  $u_j(w) \leq -1$

for all  $j$ , and therefor  $u_j(w) \leq -2$  for at least one  $j$ , so we get  $u(w) < -1$ . Assume next that  $w \in \bigcup_{k=1}^{\infty} U_k$ . If  $w \in U_\ell \setminus U_{\ell-1}$ , then

$$\begin{aligned} u_j(w) &\leq -1 + \epsilon_{\ell-1}, & \text{for } 1 \leq j \leq \ell - 1, \\ u_\ell(w) &< 0, \\ u_j(w) &\leq -2, & \text{for } j > \ell. \end{aligned}$$

Now we have that

$$\begin{aligned} u(w) &\leq \frac{1}{4} \left( (-1 + \epsilon_\ell) \sum_{j=0}^{\ell-1} \left( \frac{3}{4} \right)^j - 2 \sum_{j=\ell+1}^{\infty} \left( \frac{3}{4} \right)^j \right) \\ &= -1 + \epsilon_\ell \left( 1 - \left( \frac{3}{4} \right)^\ell \right) - \frac{1}{2} \left( \frac{3}{4} \right)^\ell < -1. \end{aligned}$$

This means that  $u + 1$  is a peak function.

In Theorem 3.5 we characterize the Choquet boundary of  $X$  with respect to  $\mathcal{J}_z^m$  in terms of peak points.

**THEOREM 3.5.** *Let  $1 \leq m \leq n$ , and let  $X$  be a compact set in  $\mathbb{C}^n$ . Then  $z \in O_X^m$  if, and only if,  $z$  is a peak point.*

**PROOF.** If  $z$  is a peak point, then there exists a function  $u \in \mathcal{SH}_m(X) \cap \mathcal{C}(X)$  such that  $u(z) = 0$  and  $u(w) < 0$ , for all  $w \neq z$ . Let  $\mu \in \mathcal{J}_z^m$ . Then

$$0 = u(z) \leq \int_X u \, d\mu \leq 0,$$

which implies that  $\text{supp } \mu \subseteq \{w : u(w) = 0\}$ . Hence,  $\mu = \delta_z$ .

On the other hand, let  $z_0 \in O_X^m$  and let  $V$  be any neighborhood of  $z_0$ . Furthermore, let  $f \in \mathcal{C}(X)$  be such that  $f < 0$ ,  $f(z_0) = -1$ , and  $f < -4$  on  $X \setminus V$ . Then,  $\mathbf{S}_f(z_0) = -1$ , and  $\mathbf{S}_f < -4$  on  $X \setminus V$ . From Lemma 2.4 it follows that one can find a function  $v \in \mathcal{SH}_m^o(X)$  such that  $v \leq \mathbf{S}_f$ , and  $-1 > v(z_0) > -2$ . Then the function  $u$  defined by

$$u(z) = -\frac{v(z)}{v(z_0)}$$

satisfies  $u(z_0) = -1$ , and  $u < -2$  on  $X \setminus V$ . The proof is finished by Lemma 3.4.

**REMARK.** It follows from Theorem 3.5 that  $O_X^m$  is non-empty.

Next, we introduce the Šilov boundary of a compact set  $X$  with respect to  $\mathcal{J}_z^m$ .

DEFINITION 3.6. Let  $1 \leq m \leq n$ , and let  $X$  be a compact set in  $\mathbb{C}^n$ . The Šilov boundary,  $B_X^m$ , of  $X$  is defined to be the topological closure of  $O_X^m$ .

It is not always true that a  $m$ -subharmonic function must attain its maximum on the Šilov boundary,  $B_X^m$  (see the example before Theorem 4.3 in [13]). But we have at least the following weak maximum principle that we shall use in our study of the Šilov boundary and the  $m$ -harmonic measure.

THEOREM 3.7. Let  $1 \leq m \leq n$ , and let  $X$  be a compact set in  $\mathbb{C}^n$ . If  $u \in \mathcal{SH}_m(X)$ , then

$$u(z) \leq \sup_{w \in B_X^m} u(w) \quad \text{for all } z \in X.$$

PROOF. See Theorem 1.12 in [6].

DEFINITION 3.8. Let  $1 \leq m \leq n$ , and let  $X$  be a compact set in  $\mathbb{C}^n$ . The  $m$ -harmonic measure of a subset  $E \subseteq X$  is defined as the function

$$\omega(z, E, X) = \inf_{\substack{V \supset E \\ V \text{ open}}} \sup_{\mu \in \mathcal{J}_z^m} \mu(V).$$

We have the following estimate.

THEOREM 3.9. Let  $1 \leq m \leq n$ ,  $K$  and  $k$  be constants and let  $X$  be a compact set in  $\mathbb{C}^n$ . If  $u \in \mathcal{SH}_m(X)$  satisfies  $u \leq K$  on  $X$ , and  $u \leq k < K$  on some set  $Y \subseteq X$ , then

$$u(z) \leq k\omega(z, Y, X) + K(1 - \omega(z, Y, X)), \quad z \in X.$$

PROOF. Fix  $\epsilon > 0$ , and set

$$U = \{z \in X : u(z) < k + \epsilon\}.$$

Then  $U$  is an open set such that  $Y \subseteq U$ . For all  $z \in X$ , there exists a measure  $\mu \in \mathcal{J}_z^m(X)$ , and an open set  $V$ , such that  $Y \subseteq V \subseteq U$  and

$$\omega(z, Y, X) \leq \mu(V) \leq \omega(z, Y, X) + \epsilon.$$

Then we have that

$$\begin{aligned} u(z) &\leq \int_X u \, d\mu = \int_V u \, d\mu + \int_{X \setminus V} u \, d\mu \\ &\leq (k + \epsilon)(\omega(z, Y, X) + \epsilon) + K(1 - \omega(z, Y, X)). \end{aligned}$$

If we let  $\epsilon \rightarrow 0^+$ , then we get the desired conclusion.

**THEOREM 3.10.** *Let  $1 \leq m \leq n$ , and let  $X$  be a compact set in  $\mathbb{C}^n$ . The Šilov boundary,  $B_X^m$ , is the smallest closed set  $E$  such that  $\omega(z, E, X)$  is identically 1.*

**PROOF.** First we shall prove that  $\omega(z, B_X^m, X) = 1$  for all  $z \in X$ . To prove this assume by contradiction that there exists a  $z_0 \in X$  such that  $\omega(z_0, B_X^m, X) < 1$ . Then there exists an open neighborhood  $V$  of  $B_X^m$ , and  $0 < c < 1$ , such that  $\mu(V) < 1 - c$  for all  $\mu \in \mathcal{J}_{z_0}^m$ .

Let  $W$  be an open set such that  $B_X^m \subseteq W \subseteq V$ , and let  $f \in \mathcal{C}(X)$  be such that  $-1 \leq f \leq 0$ ,  $f = -1$  on  $W$ , and  $f = 0$  on  $X \setminus V$ . Then we have that

$$\mathbf{S}_f \leq f = -1 \quad \text{on } W. \quad (3.1)$$

By Edwards' theorem we have that

$$\mathbf{S}_f(z_0) = \inf \left\{ \int f d\mu : \mu \in \mathcal{J}_{z_0}^m(X) \right\}.$$

For given  $\mu \in \mathcal{J}_{z_0}^m$  it holds that

$$\int_X f d\mu = \int_{X \setminus V} f d\mu + \int_V f d\mu > 0 + (-1)(1 - c) = -1 + c,$$

so

$$\mathbf{S}_f(z_0) > -1 + c. \quad (3.2)$$

Therefore, by (3.1) and (3.2) we conclude that there exists a function  $u \in \mathcal{SH}_m(X)$  such that  $u < -1$  on  $W$ , and  $u(z_0) > -1$ . But this is impossible since by Theorem 3.7 each  $m$ -subharmonic function must attain its maximum on  $B_X^m$ . This ends the first part of the proof.

Next, assume that there exists a proper closed subset  $E$  of  $B_X^m$  such that for all  $z \in X$  we have that  $\omega(z, E, X) = 1$ . Then there exist a point  $z_0 \in O_X^m \setminus E$ , and a neighborhood  $V$  of  $E$  such that  $z_0 \notin V$ . Then since  $\mathcal{J}_{z_0}^m = \{\delta_{z_0}\}$ , we get that  $\omega(z_0, E, X) = 0$  and a contradiction is obtained.

**COROLLARY 3.11.** *Let  $1 \leq m \leq n$ , and let  $X$  be a compact set in  $\mathbb{C}^n$ . The Šilov boundary,  $B_X^m$ , is the smallest closed set  $Y$  with the property that for every  $z \in X$  there exists a Jensen measure  $\mu \in \mathcal{J}_z^m$  such that  $\text{supp } \mu \subseteq Y$ .*

**PROOF.** Assume that  $Y$  is a subset of  $X$  such that for every  $z \in X$  there exists a Jensen measure  $\mu \in \mathcal{J}_z^m$  such that  $\text{supp } \mu \subseteq Y$ . For  $z$  from the Choquet boundary we have that  $\mathcal{J}_z^m = \{\delta_z\}$ . Therefore it follows that  $O_X^m \subseteq Y$ , and hence  $B_X^m \subseteq Y$ . For  $z \in X$ , and for any neighborhood  $V$  of  $Y$ , we have that

$$\sup_{\mu \in \mathcal{J}_z^m} \mu(V) = 1,$$



and therefore  $\omega(z, Y, X) = 1$ . If  $Y$  is the smallest closed set with the assumed property it now follows by using Theorem 3.10 that  $B_X^m = Y$ .

DEFINITION 3.12. Let  $1 \leq m \leq n$ ,  $X$  be a compact set in  $\mathbb{C}^n$  and  $z \in X$ . Then we define

$$\mathcal{J}_z^{b,m} = \{\mu \in \mathcal{J}_z^m : \text{supp } \mu \subseteq B_X^m\}.$$

PROPOSITION 3.13. Let  $1 \leq m \leq n$ , and let  $X$  be a compact set in  $\mathbb{C}^n$ . For every  $z \in X$  we have that  $\mathcal{J}_z^{b,m}$  is non-empty.

PROOF. This follows from Corollary 3.11.

DEFINITION 3.14. Let  $1 \leq m \leq n$ , and let  $X$  be a compact set in  $\mathbb{C}^n$ . For  $z \in X$  let us define the following set

$$\mathcal{J}_X^m(z) = \{w \in X : \omega(z, \bar{B}(w, r) \cap X, X) > 0, \text{ for all } r > 0\}.$$

In solving the Dirichlet problem in the case when the Choquet boundary is the whole compact set (Theorem 4.3) we shall need sets  $\mathcal{J}_X^m(z)$  together with Proposition 3.15. The inspiration behind  $\mathcal{J}_X^m(z)$  is from potential theory, and it is explained in the remark after Proposition 3.15.

PROPOSITION 3.15. Let  $1 \leq m \leq n$ , and let  $X$  be a compact set in  $\mathbb{C}^n$ . Then for  $z \in X$  we have that

- (1)  $\mathcal{J}_X^m(z)$  is a closed set;
- (2) if  $\mu \in \mathcal{J}_z^m(X)$  then  $\text{supp } \mu \subseteq \mathcal{J}_X^m(z)$ ;
- (3)  $\mathcal{J}_X^m(z) = \{z\}$  if, and only if,  $z \in O_X^m$ .

PROOF. (1) First note that

$$\mathcal{J}_X^m(z) = \bigcap_{r>0} Y_r, \quad \text{where } Y_r = \{w \in X : \omega(z, \bar{B}(w, r) \cap X, X) > 0\},$$

and therefore it is sufficient to prove that the sets  $Y_r$  are closed. Let  $Y_r \ni x_j \rightarrow x \in X$ . Then for every  $j$  there exists an open set  $V_j \supset \bar{B}(x_j, r) \cap X$ , and  $\mu_j \in \mathcal{J}_z^m(X)$  such that  $\mu_j(V_j) > 0$ . From a compactness argument there exists a  $j_0$  such that  $V_{j_0} \supset \bar{B}(x, r) \cap X$ , and therefore  $\omega(z, \bar{B}(x, r) \cap X, X) > 0$ , so  $x \in Y_r$ .

(2) Fix  $r > 0$ . Let  $\mu \in \mathcal{J}_z^m(X)$ , and let  $w \in \text{supp } \mu$ , then  $\mu(\bar{B}(w, r) \cap X) > 0$ . This means that  $\omega(z, \bar{B}(w, r) \cap X, X) > 0$ , hence  $w \in \mathcal{J}_X^m(z)$ .

(3) It follows from (2) that if  $\mathcal{J}_X^m(z) = \{z\}$ , then  $\mathcal{J}_z^m(X) = \{\delta_z\}$ . Thus,  $z \in O_X^m$ . On the other hand, if  $z \in O_X^m$ , then for  $w \neq z$  we have that  $\omega(z, \bar{B}(w, r), X) = 0$  if  $r < \|z - w\|$ . Therefore,  $w \notin \mathcal{J}_X^m(z)$ , which implies that  $\mathcal{J}_X^m(z) = \{z\}$ .

REMARK. There is a very nice characterization of those points for which  $\mathcal{J}_X^1(z) = \{z\}$  in the case of subharmonic functions. Namely,  $\mathcal{J}_X^1(z) = \{z\}$  if, and only if,  $X^c$  is not thin at  $z$  (see Theorem 3.3 in [12]). A similar result for  $m$ -subharmonic functions,  $m > 1$ , is not possible. To see this look at Example 5.5: then for all  $z \in \partial \mathbb{D}^n$  the set  $(\mathbb{D}^n)^c$  is not  $m$ -thin at  $z$ , but  $\mathcal{J}_{\mathbb{D}^n}^m(z) \neq \{z\}$ ,  $m > 1$ , if e.g.  $z \in \mathbb{D} \times \cdots \times \mathbb{D} \times \partial \mathbb{D}$ .

#### 4. The Dirichlet problem for continuous $m$ -subharmonic functions

In Theorem 4.2, we characterize those compact sets  $X$  for which the Dirichlet problem has a solution within the class of continuous  $m$ -subharmonic functions defined on a compact set. To obtain this we need the notion of  $O^m$ -regular compact sets (Definition 4.1). We end this section with Theorem 4.3 where we consider the case when  $X$  is equal to its Choquet boundary.

DEFINITION 4.1. Let  $1 \leq m \leq n$ . We say that a compact set  $X$  in  $\mathbb{C}^n$  is  $O^m$ -regular if  $O_X^m$  is a closed subset of  $X$ .

The next theorem provides the characterization of  $O^m$ -regular sets.

THEOREM 4.2. Let  $1 \leq m \leq n$ . Let  $X$  be a compact set in  $\mathbb{C}^n$ . Then the following conditions are equivalent:

- (1)  $X$  is a  $O^m$ -regular set;
- (2) for every  $f \in \mathcal{C}(O_X^m)$  there exists a function  $u \in \mathcal{SH}_m(X) \cap \mathcal{C}(X)$  such that  $u = f$  on  $O_X^m$ ;
- (3) for every  $f \in \mathcal{C}(B_X^m)$  there exists a function  $u \in \mathcal{SH}_m(X) \cap \mathcal{C}(X)$  such that  $u = f$  on  $B_X^m$ .

PROOF. The implication (1)  $\Rightarrow$  (2) follows from Theorem 3.3 in [14]. To prove the implication (2)  $\Rightarrow$  (3) note that if  $f \in \mathcal{C}(B_X^m)$ , then  $f \in \mathcal{C}(O_X^m)$ , and therefore there exists  $u \in \mathcal{SH}_m(X) \cap \mathcal{C}(X)$  such that  $u = f$  on  $O_X^m$ . Since both functions  $u$  and  $f$  are continuous we obtain that  $u = f$  on  $B_X^m$ . To prove the last implication (3)  $\Rightarrow$  (1) suppose that there exists  $z_0 \in B_X^m \setminus O_X^m$ . By Lemma 3.3 there exists a function  $f \in \mathcal{C}(X)$  such that  $\mathbf{S}_f(z_0) < f(z_0)$ . By assumption there exists a function  $u \in \mathcal{SH}_m(X) \cap \mathcal{C}(X)$  such that  $u = f$  on  $B_X^m$ . Then

$$\mathbf{S}_f(z_0) \geq u(z_0) = f(z_0),$$

and a contradiction is obtained.

Next, we consider the case when  $X$  is equal to its Choquet boundary.

**THEOREM 4.3.** *Let  $X$  be a compact set in  $\mathbb{C}^n$ , and  $1 \leq m \leq n$ . The following conditions are then equivalent:*

- (1)  $O_X^m = X$ ;
- (2)  $B_X^m = X$ ;
- (3)  $\mathcal{C}(X) = \mathcal{SH}_m(X)$ ;
- (4)  $\mathcal{C}(X) = \mathcal{H}_m(X)$ ;
- (5)  $f(z) = \|z\|^2 \in \mathcal{H}_m(X)$ ;
- (6)  $g(z) = -\|z\|^2 \in \mathcal{SH}_m(X)$ ;
- (7)  $\mathcal{J}_X^m(z) = \{z\}$  for all  $z \in X$ .

**PROOF.** The following implications are obvious: (1)  $\Rightarrow$  (2), (3)  $\Rightarrow$  (4), (4)  $\Rightarrow$  (5), and (3)  $\Rightarrow$  (6). We have that (2)  $\Rightarrow$  (3) follows from Theorem 4.2. For implication (5)  $\Rightarrow$  (1) take  $z_0 \in X$ . Since  $-\|z\|^2 \in \mathcal{H}_m(X)$ , then also  $-\|z - z_0\|^2$  is  $m$ -harmonic and it is also a peak function for  $z_0$ . Thus,  $z_0 \in O_X^m$ . To note implication (6)  $\Rightarrow$  (5): Since  $\|z\|^2$  is  $m$ -subharmonic, and by assumption  $-\|z\|^2$  is also  $m$ -subharmonic we have that  $\|z\|^2$  is  $m$ -harmonic. Finally, the equivalence between (1) and (7) follows from Proposition 3.15.

## 5. The Dirichlet problem for $m$ -harmonic functions

In this section we shall characterize those compact sets for which the Dirichlet problem has a solution for  $m$ -harmonic functions (Theorem 5.10). First let us compare  $m$ -harmonic functions defined on a compact set with  $m$ -harmonic functions defined on an open set.

It was proved in [2] that every  $m$ -harmonic function defined on an *open* set is pluriharmonic. The situation is different for the function theory on compact sets. We give in Example 5.1 an example of a 2-harmonic function defined on a compact set that is not pluriharmonic (3-harmonic). On the other hand, in Proposition 5.2 we show that there are compact sets  $X$  for which  $\mathcal{H}_m(X) = \mathcal{PH}(X)$ .

**EXAMPLE 5.1.** Let  $X = \{(0, 0, z_3) \in \mathbb{C}^3 : |z_3| \leq 1\}$ , and let  $u$  be a function defined on  $X$  by  $u(z_1, z_2, z_3) = -|z_3|^2$ . Then  $-u$  is plurisubharmonic, and also 2-subharmonic. Furthermore,  $u$  is the restriction of a 2-subharmonic function defined in  $\mathbb{C}^3$ ; namely

$$u(z_1, z_2, z_3) = 2(|z_1|^2 + |z_2|^2 - \frac{1}{2}|z_3|^2), \quad (z_1, z_2, z_3) \in X.$$

Finally, note that  $u$  is not plurisubharmonic (3-subharmonic) on  $X$ . To prove this assume by contradiction that  $u \in \mathcal{SH}_3(X)$ . By assumption there exists a decreasing sequence  $u_j \in \mathcal{SH}_3^o(X)$  such that  $u_j \rightarrow u$ , as  $j \rightarrow \infty$ . But then

$u_j$  must be subharmonic on the set  $Y = X \cap \{|z_3| < 1\}$ , and therefore  $u$  must be also subharmonic on  $Y$ , and a contradiction is obtained.

**PROPOSITION 5.2.** *Let  $\Omega \subset \mathbb{C}^n$  be a bounded  $B$ -regular domain in the sense of Sibony [16]. Then we have that  $\mathcal{H}_m(\bar{\Omega}) = \mathcal{PH}(\bar{\Omega})$ .*

**PROOF.** Recall that if  $\Omega$  is a  $B$ -regular domain, then for all  $z \in \partial\Omega$  we have that  $\mathcal{J}_z^n(\bar{\Omega}) = \{\delta_z\}$ . Take any  $h \in \mathcal{H}_m(\bar{\Omega})$ , then  $h \in \mathcal{H}_m(\Omega)$ , so  $h \in \mathcal{PH}(\Omega)$ . By the assumption of  $B_n$ -regularity we have also that  $h \in \mathcal{PH}(\partial\Omega)$ , which implies that  $h \in \mathcal{PH}(\bar{\Omega})$ .

One of the main notions in Theorem 5.10 is so called  $m$ -Poisson sets defined as follows.

**DEFINITION 5.3.** Let  $1 \leq m \leq n$ . A compact set  $X$  in  $\mathbb{C}^n$  is called a  $m$ -Poisson set if for every  $f \in \mathcal{C}(B_X^m)$ , there exists a function  $u \in \mathcal{H}_m(X)$  such that  $u = f$  on  $B_X^m$ .

**EXAMPLE 5.4.** Let  $X$  be the topological closure of the unit ball  $\mathbb{B}$  in  $\mathbb{C}^n$ , and let  $1 \leq m \leq n$ . Then we have that

$$O_X^m = B_X^m = \partial\mathbb{B}.$$

Hence,  $X$  is a  $O^m$ -regular set. But  $X$  is not a  $m$ -Poisson set, since it is not always possible to extend a function  $f \in \mathcal{C}(\partial\mathbb{B})$  to the inside so that it is  $m$ -pluriharmonic (see e.g. [4]).

**EXAMPLE 5.5.** Let  $X$  be the closure of the unit polydisc  $\mathbb{D}^n$  in  $\mathbb{C}^n$ , and let  $1 \leq m \leq n$ . Then

$$O_X^1 = B_X^1 = \partial\bar{\mathbb{D}}^n,$$

$$O_X^n = B_X^n = \partial\mathbb{D} \times \cdots \times \partial\mathbb{D},$$

and for  $1 < m < n$  we get that

$$O_X^m = B_X^m = \bigcup_{1 \leq j_1 < \cdots < j_m \leq n} \partial\mathbb{D} \times \cdots \times \overbrace{\partial\mathbb{D}}^{j_1} \times \cdots \times \overbrace{\partial\mathbb{D}}^{j_m} \times \cdots \times \partial\mathbb{D}.$$

Thus,  $O_X^n$  and  $B_X^n$  are equal to the distinguished boundary of  $\mathbb{D}^n$ . For  $1 < m < n$ , the above statement follows from the fact that any  $m$ -subharmonic function in  $\Omega \subset \mathbb{C}^n$  is  $(m-1)$ -subharmonic on any hyperplane passing by  $\Omega$  (see [15]). In particular,  $X$  is a  $O^m$ -regular set. Furthermore, for  $n \geq 3$  the compact set  $X$  is a  $n$ -Poisson set, since for every  $f \in \mathcal{C}(\partial\mathbb{D} \times \cdots \times \partial\mathbb{D})$  we can always find a pluriharmonic function  $u$  defined on  $\mathbb{D}^n$  such that  $u = f$  on  $\partial\mathbb{D} \times \cdots \times \partial\mathbb{D}$  (see e.g. [1], [2]).

Let us next define an (partial) order in the cone of Jensen measures.

DEFINITION 5.6. Let  $\mu$  and  $\nu$  be Jensen measures. We say that  $\mu$  is *subordinated* to  $\nu$ , and denote it with  $\mu \preceq \nu$ , if for all  $u \in \mathcal{SH}_m(X) \cap \mathcal{C}(X)$  it holds that

$$\int_X u \, d\mu \leq \int_X u \, d\nu.$$

REMARK. Note that  $\preceq$  is indeed an (partial) order, see e.g. [6].

LEMMA 5.7. Let  $1 \leq m \leq n$ , and let  $X$  be a compact set in  $\mathbb{C}^n$ . For every  $\mu \in \mathcal{J}_z^m$  there exists a measure  $\nu \in \mathcal{J}_z^{b,m}$  with  $\mu \preceq \nu$ .

PROOF. Cf. Theorem 1.17 in [6].

DEFINITION 5.8. Let  $1 \leq m \leq n$ , and  $X$  be a compact set in  $\mathbb{C}^n$ . For given  $f \in \mathcal{C}(B_X^m)$  we define the *Perron-Bremermann envelope*,  $PB_f$ , as

$$PB_f = \sup\{v(z) : v \in \mathcal{SH}_m(X) \cap \mathcal{C}(X), v \leq f \text{ on } B_X^m\}.$$

Now we prove some elementary, but useful facts about the Perron-Bremermann envelope.

PROPOSITION 5.9. Let  $1 \leq m \leq n$ , and  $X$  be a compact set in  $\mathbb{C}^n$ . For every  $f \in \mathcal{C}(B_X^m)$  we have that

- (1)  $PB_f$  is a lower semicontinuous function, and that for any  $z \in X$  and for any  $\mu \in \mathcal{J}_z^m$  it holds that

$$PB_f(z) \leq \int PB_f \, d\mu;$$

- (2) if  $X$  is a  $O^m$ -regular set, then

$$PB_f(z) = \inf \left\{ \int f \, d\mu; \mu \in \mathcal{J}_z^{b,m} \right\}.$$

PROOF. Part (1) is an immediate consequence of the definition. Next, to part (2). For  $f \in \mathcal{C}(B_X^m)$ , let

$$I_f(z) = \inf \left\{ \int f \, d\mu; \mu \in \mathcal{J}_z^{b,m} \right\}.$$

By construction we have that if  $v \in \mathcal{SH}_m(X) \cap \mathcal{C}(X)$ , and  $v \leq f$  on  $B_X^m$ , then for any  $\mu \in \mathcal{J}_z^{b,m}$  it holds that

$$v(z) \leq \int v \, d\mu \leq \int f \, d\mu,$$

and therefore  $PB_f(z) \leq I_f(z)$ . By Theorem 4.2, there exists a function  $u$  such that  $-u \in \mathcal{SH}_m(X) \cap \mathcal{C}(X)$  and  $u = f$  on  $B_X^m$ . By Lemma 5.7 it holds that for every  $\mu \in \mathcal{J}_z^m$  there is a Jensen measure  $\nu \in \mathcal{J}_z^{b,m}$  such that

$$\int u \, d\mu \geq \int u \, d\nu.$$

Thus,

$$PB_f(z) \geq S_u(z) = \inf \left\{ \int u \, d\mu; \mu \in \mathcal{J}_z^m \right\} = I_u(z) = I_f(z) \geq PB_f(z).$$

Our final result is a characterization of those compact sets  $X$  for which the Dirichlet problem has a solution within the class of  $m$ -harmonic functions defined on a compact set.

**THEOREM 5.10.** *Let  $X$  be a compact set in  $\mathbb{C}^n$ , and let  $1 \leq m \leq n$ . Then the following conditions are equivalent:*

- (1)  $X$  is a  $m$ -Poisson set;
- (2) for every  $z \in X$ , the set  $\mathcal{J}_z^{b,m}$  contains exactly one measure,  $P_z^m$ ;
- (3) for every  $f \in \mathcal{C}(B_X^m)$  we have that

$$PB_{-f} = -PB_f \quad \text{on } X.$$

**PROOF.** To prove the implication (1)  $\Rightarrow$  (2) assume that  $X$  is a  $m$ -Poisson set,  $z \in X$ , and  $\mu, \nu \in \mathcal{J}_z^{b,m}$ . By the assumptions we have that for any  $f \in \mathcal{C}(B_X^m)$  there exists a function  $h \in \mathcal{H}_m(X)$  with  $h = f$  on  $B_X^m$ . Hence,

$$h(z) = \int f \, d\mu = \int f \, d\nu,$$

which implies that  $\mu = \nu$ . Next, we shall verify the implication (2)  $\Rightarrow$  (3). First we shall prove that  $X$  is an  $O^m$ -regular set. Let  $O_X^m \ni z_j \rightarrow z \in X$ , as  $j \rightarrow \infty$ . Then from the sequence of measures  $\delta_{z_j}$  we can extract a subsequence (denoted also by  $\delta_{z_j}$ ) such that  $\delta_{z_j}$  is weak\*-convergent to some measure  $\mu \in \mathcal{J}_z^{b,m}$ . By assumption the measure  $\mu = P_z^m$  is unique. Then for every  $f \in \mathcal{C}(X)$  we have that

$$\int f \, dP_z^m = \lim_{j \rightarrow \infty} \int f \, d\delta_{z_j} = \lim_{j \rightarrow \infty} f(z_j) = f(z) = \int f \, d\delta_z,$$

which means that  $P_z^m = \delta_z$ , so  $z \in O_X^m$ . Thus,  $X$  is an  $O^m$ -regular set. Now for  $f \in \mathcal{C}(B_X^m)$  let us define the following function

$$h(z) = \int f dP_z^m.$$

We are going to prove that  $h$  is a  $m$ -harmonic function. First we show that  $h$  is continuous. Let  $X \ni z_j \rightarrow z \in X$ . We can assume that  $P_{z_j}^m$  is weak\*-convergent to some measure  $\mu \in \mathcal{J}_z^{b,m}$ , if necessary we extract a subsequence. Therefore, by assumption  $\mu = P_z^m$ , and it follows that

$$\lim_{j \rightarrow \infty} h(z_j) = \lim_{j \rightarrow \infty} \int f dP_{z_j}^m = \int f dP_z^m = h(z).$$

Proposition 5.9(2), gives us that for every  $z \in X$

$$h(z) = \int f dP_z^m = \inf \left\{ \int f d\mu; \mu \in \mathcal{J}_z^{b,m} \right\} = PB_f(z), \quad (5.1)$$

and therefore by Proposition 5.9(1) we conclude that  $h \in \mathcal{H}_m(X) \cap \mathcal{C}(X)$ . On the other hand, for any  $z \in X$  and any  $\mu \in \mathcal{J}_z^m$  we have by Lemma 5.7 that

$$h(z) \leq \int f d\mu \leq \int h dP_z^m = \int f dP_z^m = h(z).$$

Hence,  $h \in \mathcal{H}_m(X)$ . Note also that it follows from (5.1) that

$$PB_{-f}(z) = \int -f dP_z^m = - \int f dP_z^m = -PB_f(z).$$

Next we shall prove the implication (3)  $\Rightarrow$  (1). By Proposition 5.9(1), the envelopes  $PB_f$  and  $PB_{-f}$  are lower semicontinuous and therefore we can conclude that  $PB_f \in \mathcal{C}(X)$ . Furthermore, we have that for any  $z \in X$  and any  $\mu \in \mathcal{J}_z^m$

$$-PB_f(z) = PB_{-f}(z) \leq \int PB_{-f} d\mu = \int -PB_f d\mu \leq -PB_f(z).$$

Hence,  $PB_f \in \mathcal{H}_m(X)$ . To conclude that (1) holds note that  $PB_f \leq f$ , and  $PB_{-f} \leq -f$ , on  $B_X^m$  and therefore we must have that  $PB_f = f$  on  $B_X^m$ .

**COROLLARY 5.11.** *Let  $1 \leq m \leq n$ . Then every compact set  $X$  in  $\mathbb{C}^n$  that is a  $m$ -Poisson set is also an  $O^m$ -regular set.*

**PROOF.** This follows immediately from Theorem 4.2 and Theorem 5.10.

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