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POWERS OF PARTIALLY ORDERED SETS: CANCELLATION AND REFINEMENT PROPERTIES

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1. Introduction.

This paper concerns a general operation on partially ordered sets (posets). Given two posets A and B, A^B denotes the set of all order preserving (monotone, isotone) functions from B into A, i.e., maps $f\colon B \to A$ such that $f(x) \leq f(y)$, whenever $x, y \in B$ and $x \leq y$. The order on A^B is defined pointwise, $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in B$. This operation, called exponentiation, was defined and studied by G. Birkhoff in [2] and [3], but except for one paper, Novotný [15], nothing more was written about it until the subject was revived recently by I. Rival who, in collaboration with others, wrote several papers on various aspects of this operation.

In relation to the operations of disjoint union, or sum, and direct product, exponentiation obeys the same equational laws encountered in the arithmetic of integers,

$$A^{B+C} \cong A^B \cdot A^C, \quad (A^B)^C \cong A^{B \cdot C}, \quad (AB)^C \cong A^C \cdot B^C,$$

where $\cong$ denotes isomorphism. We shall be concerned with four properties that do not hold in general, but will be shown to be valid for large classes of posets.

I. Cancellation law for bases. $A^B \cong A^C$ implies $B \cong C$.

II. Cancellation law for exponents. $A^C \cong B^C$ implies $A \cong B$.

III. Refinement property for powers. If $A^C \cong B^D$, then for some E, X, Y, Z, $A \cong E^X$, $B \cong E^Y$, $C \cong Y \cdot Z$, $D \cong X \cdot Z$.

IV. Mixed refinement property. If $A^B \cong \sqcap\ (C_i, i \in I)$, then for some A_i $(i \in I)$, $A \cong \sqcap\ (A_i, i \in I)$ and $C_i \cong A_i^B$ for all $i \in I$.

The largest part of our efforts will be devoted to the properties II and III. These properties will be shown to hold under a variety of conditions, and

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unfortunately there is no reasonable way of combining all our results into one theorem. See Theorems 5.2, 5.4, 8.1, 8.2, 8.3, and 10.3. One special case is known in the literature: It is shown in Duffus and Rival [8] that II holds if A and B are finite lattices and C is a finite poset.

Property IV turns out to be much easier to handle. The only conditions that have to be imposed on the posets are that B and A^B be connected. See Theorem 9.1. For the special case when A is a bounded lattice and B is finite, this result can be found in Duffus, Jónsson and Rival [7].

We do not make a detailed study of the property I. In Duffus and Rival [8] it is shown that this property holds for finite posets, provided the base is not unordered. Making use of our results concerning III and IV, we obtain I for certain infinite posets. See Theorems 10.1 and 10.2.

Section 11 contains some results about the automorphism group of A^B. However, we do not make a thorough investigation of this subject, and it is clear that much more can be said about the connection between this group and the automorphism groups of A and B. See Theorems 11.2, 11.4 and 11.5. The twelfth and final section contains a short list of open problems.

In [16], Wille considers the properties I–IV for A and B lattices of finite length and C and D finite posets. His results, which were obtained later than ours, cover some situations not included here, but more importantly, his techniques are different and, when applicable, they are simpler.

Several of the results presented here were announced in [13].

2. Background material and examples.

The operations of addition and multiplication are commutative and associative, and multiplication is distributive over addition. Every poset is the sum of its (connected) components, and from this it follows that the cancellation law for addition,

$$A+B \cong A+C \quad \text{implies} \quad B \cong C,$$

holds provided A does not have an infinite set of pairwise isomorphic components. In particular, this holds whenever A is finite. On the other hand, the unique factorization property fails, even for finite posets (Hasimoto and Nakayama [12]). E.g., letting $\mathbf{2}$ denote the 2-element chain, we have

$$\begin{aligned}\mathbf{1}+\mathbf{2}+\mathbf{2}^2+\mathbf{2}^3+\mathbf{2}^4+\mathbf{2}^5 &= (\mathbf{1}+\mathbf{2})(\mathbf{1}+\mathbf{2}^2+\mathbf{2}^4)\\ &= (\mathbf{1}+\mathbf{2}+\mathbf{2}^2)(\mathbf{1}+\mathbf{2}^3),\end{aligned}$$

and all four factors are directly indecomposable. However, a fundamental theorem in Hashimoto [10] states that any two direct decompositions of a connected poset have isomorphic refinements. Thus a connected poset has, up to isomorphism, at most one representation as a direct product of directly indecomposable posets. In particular, every finite connected poset has the unique factorization property. It follows easily that a direct factor that is connected and finitely factorable can be cancelled. I.e., if A is connected, and is isomorphic to a direct product of finitely many directly indecomposable posets, then $A \cdot B \cong A \cdot C$ implies $B \cong C$. The hypothesis that A be finitely factorable obviously cannot be omitted. To show that the connectedness of A is also essential, recall that in [9] Hanf constructed a Boolean algebra X such that $\mathbf{2} \cdot X \ncong X$ but $\mathbf{2} \cdot \mathbf{2} \cdot X \cong X$. Let B be the sum of infinitely many copies of X, so that $B + X \cong B$, and let $C = B + \mathbf{2} \cdot X$ and $A = \mathbf{1} + \mathbf{2}$. Then $A \cdot B \cong A \cdot C$, but $B \ncong C$.

A poset whose components are isomorphic to direct products of directly indecomposable factors can be represented as a polynomial in these factors (Hashimoto [11]). This representation is unique, and the operations of addition and multiplication of posets agree with the operations on polynomials. From this it follows that if A, B and C are finite sums of finitely factorable connected posets, then $A \cdot B \cong A \cdot C$ implies $B \cong C$. A more direct proof of this can be found in Duffus [6]. For A, B and C finite, this is also a special case of a very general cancellation theorem for finite relational structures in Lovász [14].

We shall need a stronger form of the Hashimoto refinement theorem, the so-called strong refinement property. This property is not explicitly formulated in [10], although the argument given there can be used to show that it holds. A formulation using factor relations can be found in Chang, Jónsson, and Tarski [4], but since that concept will not be used here, we choose to express the property more directly in terms of isomorphisms.

THE STRONG REFINEMENT PROPERTY. If φ: $A \cdot B \cong C \cdot D$, where A, B, C, D are connected posets, then there exist isomorphisms

$$\alpha\colon W \cdot X \cong A, \qquad \beta\colon Y \cdot Z \cong B$$
$$\gamma\colon W \cdot Y \cong C, \qquad \delta\colon X \cdot Z \cong D$$

such that, for all $w \in W$, $x \in X$, $y \in Y$, $z \in Z$,

$$\varphi(\alpha(w,x), \beta(y,z)) = (\gamma(w,y), \delta(x,z)) .$$

A similar result holds for infinite products, but this will not be needed here.

It has been noted already that the familiar exponential identities hold in the present setting. Observe also that if C is connected, then

$$(A+B)^C \cong A^C + B^C .$$

It is well known that if A is a finite distributive lattice, then

$$A \cong 2^{J(A)^\delta},$$

where $J(A)$ is the set of all join-irreducible elements of A (not including 0), and the superscript δ denotes the dual. Hence, for finite distributive lattices A and B,

$$A^{J(B)^\delta} \cong 2^{J(A)^\delta \cdot J(B)^\delta} \cong B^{J(A)^\delta} \cong A * B,$$

where $*$ denotes free products in the variety of all distributive lattices. Applied to finite ordinals $\boldsymbol{m}$ and $\boldsymbol{n}$, this yields

$$\mathbf{2}^{\boldsymbol{n}} = \boldsymbol{n} \oplus \mathbf{1}, \quad (\boldsymbol{m} \oplus \mathbf{1})^{\boldsymbol{n}} = (\boldsymbol{n} \oplus \mathbf{1})^{\boldsymbol{m}},$$

where $\oplus$ denotes ordered sums.

We conclude this section with a number of examples that will give some indication of the type of restrictions that must be imposed on the posets involved in order for I–IV to hold.

EXAMPLE 2.1. (Failure of I). If A is unordered, then $A^B \cong A^C$, whenever B and C have the same number of components, for A^B consists of just those functions from B into A that are constant on each component, and similarly for A^C. For a less trivial example, let X be the Boolean algebra constructed in Hanf [9] with $\mathbf{2} \cdot X \ncong X$ but $\mathbf{2} \cdot \mathbf{2} \cdot X \cong X$. Let B be the sum of infinitely many copies of X, and let $C = B + \mathbf{2} \cdot X$. As was noted earlier, $(\mathbf{1}+\mathbf{2}) \cdot B \cong (\mathbf{1}+\mathbf{2}) \cdot C$. Letting $A \cong \mathbf{2}^{\mathbf{1}+\mathbf{2}}$, we therefore have $A^B \cong A^C$, although $B \ncong C$.

EXAMPLE 2.2. (Failure of II and III). Let C be the poset in Fig. 1, let A consist of infinitely many copies of C, and let $B = A + \mathbf{1}$. Then C^C has five components, four singletons (the automorphisms of C), and one with 32 elements. Therefore, A^C has infinitely many one-element components, and hence $B^C \cong A^C + \mathbf{1} \cong A^C$. Note that C is finite and A and B have finite length.

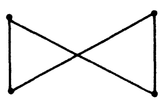

Fig. 1.

EXAMPLE 2.3. (Failure of II and III). Let A and B be the posets in Fig. 2, i.e., $A = \mathbf{1} \oplus (\mathbf{1}+\mathbf{2})$ and $B = \mathbf{1} \oplus (\mathbf{1}+\mathbf{1})$, and let C be the direct product of infinitely many copies of $\mathbf{2}$. Each function from C into A or into B is into one of the arms, because C has a largest element. The right arm of A is $\mathbf{3} = \mathbf{2}^{\mathbf{2}}$, and since $\mathbf{2} \cdot C \cong C$, it follows that each of the posets A^C and B^C consists of two copies of $\mathbf{2}^C$, pasted together by identifying their zero elements. Notice that A and B are triple indecomposable, additively, multiplicatively and exponentially.

Fig. 2.

EXAMPLE 2.4. (Failure of III). Let N be the pentagon (the five-element non-modular lattice), and let

$$A = \mathbf{2}^{\mathbf{2}^3+\mathbf{1}} \cdot N^{\mathbf{2}^4+\mathbf{2}^2+\mathbf{1}}, \qquad B = \mathbf{2}^{\mathbf{2}+\mathbf{1}} \cdot N^{\mathbf{2}^2+\mathbf{2}+\mathbf{1}},$$
$$C = \mathbf{2}+\mathbf{1}, \qquad D = \mathbf{2}^3+\mathbf{1}.$$

Then $A^C \cong B^D$. Since A and B are exponentially indecomposable, III fails.

EXAMPLE 2.5. (Failure of IV). If B is not connected, then IV will in general fail, since $B = B_0 + B_1$ implies $A^B \cong A^{B_0} \cdot A^{B_1}$. For a more interesting example, let X, Y and B be the posets called A, B and C in Example 2.3, and let $A = X + Y + \mathbf{1} + \mathbf{1}$. Then $X^B \cong Y^B$, and therefore $A^B \cong (X^B + \mathbf{1})(\mathbf{1}+\mathbf{1})$, but A is directly indecomposable.

3. Cancellation and refinements: A general criterion.

The cancellation law for exponents will be shown to hold under a number of different conditions that have no reasonable common generalization, and the same situation prevails regarding the refinement property for powers. Nevertheless, the methods used to prove these various results are to a considerable extent the same, and in order to avoid duplicate arguments we therefore formulate some rather technical properties of an isomorphism $\varphi\colon A^C \cong B^D$ that will imply the existence of a refinement for this particular isomorphism, and in case $C \cong D$ will imply that $A \cong B$. First we introduce some notation and terminology that will be in effect throughout the paper.

In most cases we will identify a poset A, as a relational structure $(X, \leq)$, with the set X of its elements, but there will be occasions when two partial ordering relations $\leq$ and $\leq'$ are defined on the same set X, and then we must of course distinguish between the posets $A = (X, \leq)$ and $A' = (X, \leq')$. If $u \leq v$ in A, then $[u, v]$ denotes the interval $\{x \in A : u \leq x \leq v\}$. We write $u \prec v$ if v covers u, i.e., if $u < v$, and there is no element x with $u < x < v$. The following simple observation is very useful: For $f, g \in A^B$, $f \prec g$ iff there exists $b \in B$ such that $f(b) \prec g(b)$ and $f(x) = g(x)$ whenever $b \neq x \in B$.

We say that A is upper bounded if it has a largest element (always denoted by 1), and that A is lower bounded if it has a smallest element (denoted by 0). If A is both upper bounded and lower bounded, then it is said to be bounded. We say that A is updirected (down-directed) if every finite subset has an upper bound (a lower bound). Finally, we say that A is atomic if in every interval $[u, v]$ with $u<v$ there exists an element that covers u.

For brevity we introduce names for certain important classes of posets. We let $\mathscr{P}$, $\mathscr{L}$, $\mathscr{S}_\vee$, and $\mathscr{S}_\wedge$ denote the classes consisting, respectively, of all posets, all lattices, all join semilattices, and all meet semilattices. For any classe $\mathscr{K}$ of posets, $\mathscr{K}_{\max}$ and $\mathscr{K}_{\min}$ will denote the classes consisting of all those membersof $\mathscr{K}$ that satisfy, respectively, the maximal condition, or the a.c.c., and the minimal condition, or the d.c.c., while $\mathscr{K}(0)$, $\mathscr{K}(1)$ and $\mathscr{K}(0,1)$ will denote the classes consisting of all those members of $\mathscr{K}$ that are, respectively, lower bounded, upper bounded, and bounded.

The constant function in A^B with value a is written $\langle a\rangle$. We often use the fact that, for $a_i \in A$ $(i \in I)$, the join $\bigvee(\langle a_i\rangle, i \in I)=f$ exists in A^B just in case the join $\bigvee(a_i, i \in I)=a$ exists in A, and that in this case $f=\langle a\rangle$. In particular, if $\varphi\colon A^C \cong B^D$, and if $\varphi(\langle a_i\rangle)=g_i$ $(i \in I)$, then the join $\bigvee(g_i, i \in I)=g$ exists in B^D just in case the join $\bigvee(a_i, i \in I)=a$ exists in A, and in this case $\varphi(\langle a\rangle)=g$.

We frequently need to define functions in A^B by specifying their values in different ways on different subsets of B. If $f, g \in A^B$ and $X \subseteq B$, we let $\langle f[x \in X], g\rangle$ be the function that agrees with f on X and with g on $B-X$. If $a, a' \in A$ and $b \in B$, we let $\langle a[x \leq b], a'\rangle$ be the function whose value at x is a if $x \leq b$, but a' otherwise. Several obvious variants of this notation will be used. Of course it must be checked in each case whether the new function belongs to A^B.

We now turn to some more technical notation that we prefer to introduce in a more formal manner. Since the concepts about to be introduced will play a central role, we shall try to motivate them by describing without a proof how an isomorphism $\varphi\colon A^{\mathbf{2}} \cong B^{\mathbf{2}}$ can be used to construct an isomorphism $\psi\colon A \cong B$. (The proof will appear in Bergman, McKenzie, and Nagý [1].) The members of $A^{\mathbf{2}}$ are ordered pairs (a_0, a_1) with $a_0, a_1 \in A$ and $a_0 \leq a_1$. In particular, for $a \in A$, the diagonal element (a, a) belongs to $A^{\mathbf{2}}$, and we must use this to define $\psi(a)$. Unfortunately, $\varphi(a, a)$ is not always a diagonal element. E.g., there is an automorphism of $\mathbf{3}^{\mathbf{2}}$ that interchanges $(1,1)$ and $(0,2)$. However, if we let $\varphi(a, a) = (b_0, b_1)$, then it turns out that (b_0, b_0) and (b_1, b_1) correspond to diagonal elements, say $(b_0, b_0)=\varphi(a_0, a_0)$ and $(b_1, b_1)=\varphi(a_1, a_1)$. Furthermore, the element $\varphi(a_0, a_1)$ in $B^{\mathbf{2}}$ also turns out to be a diagonal element, say (b, b). We now have the situation pictured in Fig. 3, and we have in fact associated with each element $a \in A$ an element $\psi(a)=b \in B$. It is a simple matter to show that ψ is an isomorphism.

This suggests an approach to the cancellation problem in general, and indeed this approach will be shown to work under some rather general conditions, although the proof is quite involved. Suppose $\varphi\colon A^C \cong B^C$. Given $a \in A$, let $g=\varphi(\langle a\rangle)$. For $c \in C$, try to prove that $\langle g(c)\rangle$ corresponds to a

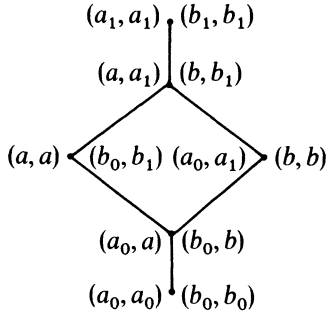

Fig. 3.

constant function, say $\varphi(\langle f(c)\rangle) = \langle g(c)\rangle$. For the function f determined in this manner, which is obviously a member of A^C, try to prove that $\varphi(f)$ is constant, say $\varphi(f) = \langle b\rangle$. The map $a \to b$ will then be an isomorphism of A onto B.

DEFINITION 3.1. Suppose $\varphi\colon A^C \cong B^D$.

(i) $\Delta(\varphi) = \{f \in A^C\colon \varphi(f) \text{ is constant}\}$.
(ii) $R(\varphi) = \{x \in A\colon \langle x\rangle \in \Delta(\varphi)\}$.
(iii) $x \leq_\varphi y$ iff $x, y \in R(\varphi)$, $x \leq y$, and every $f \in A^C$ with $f(C) \subseteqq \{x, y\}$ belongs to $\Delta(\varphi)$.
(iv) $\mathring{\varphi}$ is the map from $R(\varphi)$ to $R(\varphi^{-1})$ such that $\varphi(\langle x\rangle) = \langle \mathring{\varphi}(x)\rangle$ for all $x \in R(\varphi)$.

Four properties of an isomorphism $\varphi\colon A^C \cong B^D$ will be extensively investigated throughout much of this paper, and for easy reference we list these here.

$(\varphi, 1)$ $\leq_\varphi$ is a partial ordering of $R(\varphi)$.
$(\varphi, 2)$ $\mathring{\varphi}\colon (R(\varphi), \leq_\varphi) \cong (R(\varphi^{-1}), \leq_{\infty^{-1}})$.
$(\varphi, 3)$ $\Delta(\varphi)$ is the set of all order preserving maps from C into $(R(\varphi), \leq_\varphi)$.
$(\varphi, 4)$ $\leq_\varphi$ agrees with $\leq$ on $R(\varphi)$.

The next two theorems show the relevance of these properties for our investigations.

THEOREM 3.2. *If* $\varphi\colon A^C \cong B^C$, *and if* $(\varphi, 1)$, $(\varphi, 2)$, $(\varphi, 3)$, *and* $(\varphi^{-1}, 3)$ *hold, then there exists* $\psi\colon A \cong B$ *such that*

$$\langle \psi(a)\rangle = \varphi(\mathring{\varphi}^{-1} \circ \varphi(\langle a\rangle)) \quad \textit{for all } a \in A\,.$$

PROOF. From $(\varphi, 1)$ and $(\varphi, 2)$ it follows that $(\varphi^{-1}, 1)$ and $(\varphi^{-1}, 2)$ also hold.

Given $a \in A$, the function $g = \varphi(\langle a \rangle)$ belongs to $\Delta(\varphi^{-1})$, and therefore by $(\varphi^{-1}, 3)$, it is an order preserving function from C into $(R(\varphi^{-1}), \leq_{\varphi^{-1}})$. Consequently, by $(\varphi, 2)$, $f = \mathring{\varphi}^{-1} \circ g$ is an order preserving function from C into $(R(\varphi), \leq_\varphi)$, and we infer by $(\varphi, 3)$ that $f \in \Delta(\varphi)$, that is, $\varphi(f) = \langle b \rangle$ for some $b \in B$. The map $\psi: a \to b$ is obviously order preserving, and $\langle \psi(a) \rangle = \varphi(\mathring{\varphi}^{-1} \circ \varphi(\langle a \rangle))$. To show that ψ is an isomorphism, we merely observe that the same process in reverse leads from b to a.

THEOREM 3.3. *If* $\varphi: A^C \cong B^D$, *and if* $(\varphi, 1)$, $(\varphi, 2)$, $(\varphi, 3)$, $(\varphi, 4)$ *and* $(\varphi^{-1}, 3)$ *hold, then*

$$A \cong E^D \quad \text{and} \quad B \cong E^C, \quad \text{where } E = (R(\varphi), \leq_\varphi) .$$

PROOF. From $(\varphi, 1)$, $(\varphi, 2)$, and $(\varphi, 4)$ it follows that $(\varphi^{-1}, 1)$, $(\varphi^{-1}, 2)$, and $(\varphi^{-1}, 4)$ also hold. By $(\varphi^{-1}, 3)$ and $(\varphi^{-1}, 4)$, the map $a \to \varphi(\langle a \rangle)$ is an isomorphism of A onto $(R(\varphi^{-1}), \leq_{\varphi^{-1}})^D$, and hence $A \cong E^D$ by $(\varphi, 2)$. Similarly, $B \cong E^C$.

4. Cancellation and refinements: Technical lemmas.

Throughout this section we assume that $\varphi: A^C \cong B^D$. We adopt the convention, which will also be in effect in later sections, of writing $\bar{f} = \varphi(f)$ for $f \in A^C$, and $\bar{x} = \mathring{\varphi}(x)$ for $x \in R(\varphi)$.

LEMMA 4.1. *If D is connected, then* $(\varphi, 1)$ *holds. In fact, given* $x, y, z \in R(\varphi)$ *with* $x \leq y \leq z$, *we have* $x \leq_\varphi z$ *iff* $x \leq_\varphi y$ *and* $y \leq_\varphi z$.

PROOF. First suppose $x \leq_\varphi z$, and consider a function $f \in A^C$ with $f(C) \subseteq \{x, y\}$. The function $g = \langle x[f(k) = x], z \rangle$ is in $\Delta(\varphi)$, that is, $\bar{g}$ is constant. Since $g \wedge \langle y \rangle = f$, hence $\bar{g} \wedge \langle \bar{y} \rangle = \bar{f}$, it follows that $\bar{f}$ is also constant. Thus $x \leq_\varphi y$ and, similarly, $y \leq_\varphi z$.

Now suppose $x \leq_\varphi y$ and $y \leq_\varphi z$, and consider a function $f \in A^C$ with $f(C) \subseteq \{x, z\}$. We complete the proof by showing that $\bar{f}$ is constant. Assuming that this is false, we infer from the connectedness of D that there exist $p, q \in D$ with $p < q$ and $\bar{f}(p) < \bar{f}(q)$. Define

$$\bar{g} = \langle \bar{f}[r < p], \bar{f}(q) \rangle, \qquad h_0 = \langle x[f(k) = x], y \rangle, \qquad h_1 = \langle y[f(k) = x], z \rangle .$$

Since $x \leq_\varphi y$ and $y \leq_\varphi z$, $\bar{h}_i$ is constant, say $\bar{h}_i = \langle u_i \rangle$. Clearly $h_0 \leq f \leq h_1$, hence $\langle u_0 \rangle \leq \bar{f} \leq \langle u_1 \rangle$, $\langle u_0 \rangle \leq \bar{g} \leq \langle u_1 \rangle$, $h_0 \leq g \leq h_1$. From this we infer that

$$g \wedge \langle y \rangle = \langle g[f(k) = x], y \rangle , \tag{1}$$

for if $f(k) = x$, then $g(k) \leq h_1(k) = y$, but if $f(k) = z$, then $g(k) \geq h_0(k) = y$.

We have $\bar{g}(p)=\bar{f}(q)>\bar{f}(p)$, hence $\bar{g}\nleq\bar{f}$, $g\nleq f$, and thus $g(m)\nleq f(m)$ for some $m\in C$. For this element m we must have $f(m)=x<g(m)$, because $g(k)\leq h_1(k)\leq z$ for all $k\in C$. It follows by (1) that $g\wedge\langle y\rangle>h_0$, since $g(m)>x=h_0(m)$. Thus $\bar{g}\wedge\langle\bar{y}\rangle$ exists and is strictly larger than $\langle u_0\rangle$, whence there exists $v\in B$ with $u_0<v\leq\bar{f}(q),\bar{y}$. We infer that

$$\langle u_0\rangle\ <\ \langle u_0[r\ngeq q],v\rangle\ \leq\bar{f},\ \langle\bar{y}\rangle\ ,$$

contrary to the fact that $f\wedge\langle y\rangle=h_0$, hence $\bar{f}\wedge\langle\bar{y}\rangle=\langle u_0\rangle$.

We introduce an ad-hoc property that will in several arguments enable us to combine a number of cases.

DEFINITION 4.2. We say that X^Y has Property (a) if for every non-constant $f\in X^Y$ there exists $g\in X^Y$ such that $f<g$, and g agrees with f on a cofinal filter in Y.

COROLLARY 4.3. *Suppose either* $Y\in\mathscr{P}(1)$, *or* $Y\in\mathscr{S}_\vee$, *or Y is updirected and* $X\in\mathscr{S}_\vee$. *Then X^Y has Property* (a).

PROOF. If $Y\in\mathscr{P}(1)$, take $g=\langle f(1)\rangle$. If $Y\in\mathscr{S}_\vee$, choose $m,n\in Y$ with $f(m)\nleq f(n)$ and let $g(k)=f(k\vee m)$. If Y is updirected and $X\in\mathscr{S}_\vee$, choose $m,n\in Y$ with $f(m)\nleq f(n)$, and let $g(k)=f(k)\vee f(m)$.

The usefulness of Property (a) is in part based on the following simple observation.

LEMMA 4.4. *Suppose $x\in X$ and $f,g\in X^Y$, and suppose $\langle x\rangle<f,g$. If there exists a cofinal filter F in Y such that $f(k)\leq g(k)$ whenever $k\in F$, then $\langle x\rangle$ is not a maximal lower bound for f and g.*

PROOF. Choosing $m\in F$ with $x<f(m)$, we have

$$\langle x\rangle\ <\ \langle x[k\ngeq m],f(m)\rangle\ \leq\ f,g\ .$$

LEMMA 4.5. *If A^C and B^D have Property* (a), *then* $(\varphi,2)$ *holds.*

PROOF. If this fails, we may assume that $x\leq_\varphi y$ but not $\bar{x}\leq_{\varphi^{-1}}\bar{y}$. Then there exists a non-constant $f\in A^C$ with $\bar{f}(D)=\{\bar{x},\bar{y}\}$, and by Property (a) there exists $g\in A^C$ such that $f<g$, and g agrees with f on a cofinal filter F in C. Put $h=\langle x[k\notin F],y\rangle$. Since $x\leq_\varphi y$, $\bar{h}$ is constant, say $\bar{h}=\langle u\rangle$.

Now $\langle\bar{x}\rangle\leq\bar{f}\leq\langle\bar{y}\rangle$, hence $\langle x\rangle\leq f\leq\langle y\rangle$. Also $g\leq\langle y\rangle$, because g agrees with

f on a cofinal filter in C. From this we infer that the meets $g \wedge h$ and $f \wedge h$ exist and are equal to $\langle x[k \notin f], f\rangle$. Hence

$$\bar{g} \wedge \langle u\rangle = \bar{f} \wedge \langle u\rangle = \langle \bar{x}[\bar{f}(r) = \bar{x}], u\rangle .$$

Choose $p \in D$ with $\bar{f}(p) < \bar{g}(p)$, and observe that $\bar{x}$ is a maximal lower bound for $\bar{g}(p)$ and u, for if $\bar{x} < v \leq \bar{g}(p), u$, then $\langle \bar{x}[r \ngeq p], v\rangle$ is a lower bound for $\bar{g}$ and $\langle u\rangle$, contrary to the fact that the value of $\bar{g} \wedge \langle u\rangle$ at p is $\bar{x}$. Letting $\bar{g}_1 = \langle \bar{g}(p)\rangle$, we infer that $\langle x\rangle$ is a maximal lower bound for g_1 and h. This contradicts Lemma 4.4., for $\langle x\rangle < g_1, h$, and $g_1(k) \leq y = h(k)$ for $k \in F$.

LEMMA 4.6. *If $C \in \mathscr{P}(1)$ or $C \in \mathscr{S}_\vee$, then every order preserving function from C into $(R(\varphi), \leq_\varphi)$ belongs to $\Delta(\varphi)$.*

PROOF. We make use of the fact that $\Delta(\varphi)$ is closed under all existing joins and meets in A^C. Suppose f is an order preserving function from C into $(R(\varphi), \leq_\varphi)$. If $C \in \mathscr{P}(1)$, then the functions $\langle f(m)[k \leq m], f(1)\rangle$ $(m \in C)$ belong to $\Delta(\varphi)$, and their meet is f. If $C \in \mathscr{S}_\vee$, then the functions $\langle f(m)[k \ngeq n], f(m \vee n)\rangle$ belong to $\Delta(\varphi)$. For a fixed m, the join of these functions is the function f_m with $f_m(k) = f(m \vee k)$, and the meet of the functions f_m is f.

The conclusion of this lemma is half of the condition $(\varphi, 3)$. The second half is more difficult to establish, and will require a more stringent hypothesis. In particular, we will have to assume that A and B are atomic.

Suppose $x \in R(\varphi)$ and $x \prec y$ in A. Then the interval $[\langle x\rangle, \langle y\rangle]$ is isomorphic to $\mathbf{2}^C$, and letting $f = \langle y\rangle$, we therefore have $[\langle \bar{x}\rangle, \bar{f}] \cong \mathbf{2}^C$. What does this tell us about $\bar{f}$? The next lemma, which will be used in establishing both $(\varphi, 3)$ and $(\varphi, 4)$, addresses this question, but first we make two observations.

As is well known, $\mathbf{2}^X$ is a complete distributive lattice, and X is dually isomorphic to the set of all completely join irreducible elements in $\mathbf{2}^X$. From this it follows that the base $\mathbf{2}$ always cancels: $\mathbf{2}^X \cong \mathbf{2}^Y$ implies $X \cong Y$.

The second observation is somewhat less obvious. The refinement property for powers holds whenever one of the bases is $\mathbf{2}$: If $\mathbf{2}^X \cong U^Y$, then for some Z, $U \cong \mathbf{2}^Z$ and $X \cong Y \cdot Z$. To see this we note that lattices L of the form $\mathbf{2}^Z$ are completely characterized by three properties: (1) L is complete, (2) L satisfies the infinite distributive law

$$x \wedge \bigvee \{y_i : i \in I\} = \bigvee \{x \wedge y_i : i \in I\} ,$$

and (3) every element of L is the join of completely join irreducible elements. It is obvious that $\mathbf{2}^Z$ always has these properties. Conversely, if (1)–(3) hold, then we take Z to be the dual of the set of all completely join irreducible elements of L, and define the isomorphism ψ: $2^Z \cong L$ by letting

$$\psi(f) = \bigvee \{z \in Z : f(z)=1\} .$$

We also note that if a lattice L satisfies (1)–(3), then so does every closed sublattice of L. The proof of our claim is now easy. If $\mathbf{2}^X \cong U^Y$, then U^Y is a lattice that satisfies (1)–(3). Hence the constant functions in U^Y form a closed sublattice, so that U itself is a lattice that satisfies (1)–(3). We therefore conclude that $U \cong \mathbf{2}^Z$ for some poset Z, hence $\mathbf{2}^X \cong \mathbf{2}^{Y \cdot Z}$ and, finally, $X \cong Y \cdot Z$.

LEMMA 4.7. *Suppose C and D are updirected and directly indecomposable, $x \in R(\varphi)$ and $x \prec y \in A$. Then either*

(i) *$y \in R(\varphi)$ and $\bar{x} \prec \bar{y}$ or else*
(ii) *D has a largest element, and $\varphi(\langle y \rangle)(r) = \bar{x}$ whenever $1 \neq r \in D$.*

PROOF. Letting $f = \langle y \rangle$, we first show that the range of $\bar{f}$ contains just one element different from $\bar{x}$. Note that $\langle x \rangle$ is not the meet of two strictly larger members of the interval $[\langle x \rangle, \langle y \rangle]$. In fact, given two such functions, $f_i = \langle x[k \notin F_i], y \rangle$, $i=0,1$, where F_0 and F_1 are filters in C, their meet is $\langle x[k \notin F_0 \cap F_1], y \rangle$, and $F_0 \cap F_1$ is non-empty because C is updirected.

Suppose now that $\bar{f}(D)$ has two members distinct from $\bar{x}$, say $u = \bar{f}(p)$ and $v = \bar{f}(q)$. Since D is updirected, we may assume that $p < q$, and therefore $\bar{x} < u < v$. Let

$$\bar{g} = \langle \bar{x}[r \ngeq q], v \rangle, \qquad \bar{h} = \langle \bar{x}[r \ngeq p], v \rangle .$$

Then $\langle x \rangle < g \leq \langle y \rangle$ and $g < h \nleq \langle y \rangle$. Since C is updirected, it cannot be the case that $h(k) = y$ whenever $g(k) = y$, for then $h \leq \langle y \rangle$. Letting $h_1 = \langle x[g(k)=x], h \rangle$, we therefore have $g < h_1 \leq h$ and $h_1 \wedge \langle y \rangle = g$. Consequently, $\bar{g} < \bar{h}_1 \leq \bar{h}$ and $\bar{h}_1 \wedge \bar{f} = \bar{g}$.

Choose $s \in D$ with $\bar{g}(s) < \bar{h}_1(s)$, and note that $s \geq p$ and $s \ngeq q$. Let $w = \bar{h}_1(s)$. Then $\bar{x} < w \leq v$, and from the fact that $\bar{h}_1 \wedge \bar{f} = \bar{g}$ it follows that $\bar{x}$ is a maximal lower bound for u and w. Finally, letting

$$\bar{g}_0 = \bar{g} \wedge \langle u \rangle = \langle \bar{x}[r \ngeq q], u \rangle ,$$

$$\bar{g}_1 = \bar{g} \wedge \langle w \rangle = \langle \bar{x}[r \ngeq q], w \rangle ,$$

we find that $\langle \bar{x} \rangle$ is a maximal lower bound for $\bar{g}_0$ and $\bar{g}_1$. Since $\langle \bar{x} \rangle < \bar{g}_0, \bar{g}_1 \leq \bar{f}$, this contradicts the fact that $\langle \bar{x} \rangle$ is meet irreducible in $[\langle x \rangle, \langle y \rangle]$.

We have shown that, for some $u \in B$ and some filter F in D, $\bar{f} = \langle \bar{x}[r \notin F], u \rangle$. This implies that $\mathbf{2}^C \cong [\bar{x}, u]^F$, hence $[\bar{x}, u] \cong \mathbf{2}^G$ for some poset G. Thus $\mathbf{2}^C \cong \mathbf{2}^{F \cdot G}$, and we have $C \cong F \cdot G$. In as much as C was assumed to be directly indecomposable, we conclude that either F or G consists of just one element.

If F is trivial, $F=\{1\}$, then the alternative (ii) in the conclusion of the lemma holds. If G is trivial, then $\bar{x} \prec u$. We then apply the results obtained so far to φ^{-1}, with $\bar{x}$ and u in place of x and y. Letting $\bar{g}=\langle u\rangle$, we have two alternatives; either C is upper bounded and $g(k)=x$ whenever $1 \neq k \in C$, or else there exist $z \in A$ with $x \prec z$ and a filter H in C such that $g=\langle x[k \notin H], z\rangle$. However, from the fact that $\bar{f} \leq \langle u\rangle$ we infert that $\langle y\rangle \leq g$. This rules out the first alternative and shows that the second alternative must hold with $H=C$ and $z=y$. I.e., we have $g=\langle y\rangle$, therefore $y \in R(\varphi)$, and $\bar{y}=u$ covers $\bar{x}$.

LEMMA 4.8. *Suppose A is atomic, C and D are updirected and directly indecomposable, and B^D has Property* (a). *If $C \not\cong D$, then* $(\varphi, 4)$ *holds.*

PROOF. Suppose, to the contrary, that there exist $x, y \in R(\varphi)$ such that $x \leq y$ but not $x \leq_{\varphi} y$. Then there exists $f \in A^C$ with $f(C)=\{x, y\}$ such that $\bar{f}$ is not constant. Since B^D has Property (a), there exists $g \in A^C$ such that $f<g$, and $\bar{g}$ agrees with $\bar{f}$ on some cofinal filter F in D. From the fact that $\bar{f} \leq \langle\bar{y}\rangle$ it follows that $\bar{g} \leq\langle\bar{y}\rangle$, hence $g \leq\langle y\rangle$. Choosing $m \in C$ with $f(m)<g(m)$, we infer that $f(m)=x$. Choosing $z \in A$ with $x \prec z \leq g(m)$, we now apply Lemma 4.7 with y replaced by z. Let $h=\langle z\rangle$. Since $C \not\cong D$, the alternative (i) in the conclusion of the lemma cannot apply, and hence D is upper bounded and $\bar{g}(r)=\bar{x}$ whenever $1 \neq r \in D$. Letting $h_1=\langle x[k \ngeq m], z\rangle$, we therefore have $\bar{h}_1(r)=\bar{x}$ whenever $1 \neq r \in D$. But we also have $h_1 \leq g$, hence $\bar{h}_1(1) \leq \bar{g}(1)=\bar{f}(1)$, so that $\bar{h}_1 \leq \bar{f}$, contrary to the fact that $h_1(m)=z>f(m)$ and thus $h_1 \not\leq f$.

LEMMA 4.9. *Suppose $C, D \in \mathscr{P}(1)$, or $C, D \in \mathscr{P}(0)$, or $C, D \in \mathscr{S}_{\vee}$. If $f \in \Delta(\varphi)$, and if $f(C)$ has a largest element a, then $a \in R(\varphi)$.*

PROOF. Let $\bar{f}=\langle b\rangle$ and $g=\langle a\rangle$. First suppose $C, D \in \mathscr{P}(1)$, in which case $a=f(1)$ is the largest element of $f(C)$. Let $\bar{g}_1=\langle\bar{g}(1)\rangle$ and $h=\langle f[r \neq 1], g_1(1)\rangle$. Then $f \leq g \leq g_1$ and $f \leq h \leq g_1$. Since $h(k)=f(k)$ for $k \neq 1$ and $g(1)=a=f(1)$, we have $g \wedge h=f$. Thus $\bar{g} \wedge \bar{h}=\bar{f}=\langle b\rangle$, and since $\bar{g}(1)=\bar{g}_1(1) \geq \bar{h}(1)$, we infer by Lemma 4.4 that $\bar{h}=\langle b\rangle$. Thus $h=f$, $g_1(1)=h(1)=f(1)$, which implies that $g_1 \leq\langle f(1)\rangle=g$, $g=g_1$, $\bar{g}=\bar{g}_1=\langle\bar{g}(1)\rangle$, and therefore $\bar{g}$ is constant, i.e., $a \in R(\varphi)$.

Now suppose $C, D \in \mathscr{P}(0)$. By the dual of what has been proved so far, $\bar{g}(0) \in R(\varphi^{-1})$, that is $\bar{g}(0)=\bar{x}$ for some $x \in R(\varphi)$. Clearly $\langle\bar{x}\rangle \leq \bar{g}$, hence $\langle x\rangle \leq g=\langle a\rangle$, so that $x \leq a$. On the other hand, $f \leq\langle a\rangle=g$, hence $\langle b\rangle=\bar{f} \leq \bar{g}$, $b \leq \bar{g}(0)=\bar{x}$, $f \leq\langle x\rangle$, which implies that $a \leq x$. We therefore have $a=x \in R(\varphi)$.

Finally suppose $C, D \in \mathscr{S}_{\vee}$. If $a \notin R(\varphi)$, then $\bar{g}$ is not constant, and hence there exist $p, q \in D$ with $\bar{g}(p)<\bar{g}(q)$. Letting $\bar{g}_1(r)=\bar{g}(q \vee r)$ for $r \in D$, we have $\bar{g}<\bar{g}_1$, $g<g_1$. Hence for some $m \in C$, $a<g_1(m)$, and clearly m can be so chosen that $f(m)=a$. Letting $h=\langle f[k \ngeq m]\rangle, g_1\rangle$, we then have $g \wedge h=f$, $\bar{g} \wedge \bar{h}=\bar{f}$

$=\langle b\rangle$, which violates Lemma 4.4, since $\bar{h}(r)\leq\bar{g}_1(r)=\bar{g}(r)$ for $r\geq q$, and $\langle b\rangle<\bar{g},\bar{h}$.

LEMMA 4.10. *Suppose A and B are atomic and C and D are directly indecomposable. If either*

(i) *C and D both belong to (the same) one of the classes $\mathscr{L}$, $\mathscr{P}(0,1)$, $\mathscr{S}_\vee(0)$, $\mathscr{S}_\wedge(1)$, $\mathscr{P}_{\max}(1)$, or else*
(ii) *$A,B\in\mathscr{S}_\vee$ and $C,D\in\mathscr{P}(1)$,*

then $(\varphi,3)$ holds.

PROOF. We need to show that if $f\in\Delta(\varphi)$, then $f(C)\subseteq R(\varphi)$, and f is in fact an order preserving function from C into $(R(\varphi),\leq_\varphi)$. Observe that in each of the six cases, C and D are either upper bounded, or else they are join semilattices, and that A^C and B^D therefore have Property (a).

We begin by proving the following statement.

(1) For any $x,y\in A$ with $x\leq y$, if there exists $f\in\Delta(\varphi)$ such that $\{x,y\}\subseteq f(C)\subseteq[x,y]$, then $x\leq_\varphi y$.

Let $\bar{f}=\langle b\rangle$. By Lemma 4.9, $x,y\in R(\varphi)$. Considering a function $g\in A^C$ with $g(C)=\{x,y\}$, we need to show that $\bar{g}$ is constant.

Suppose this is not the case. Since B^D has Property (a), there exists $g_1\in A^C$ such that $g<g_1$, and $\bar{g}_1$ agrees with $\bar{g}$ on a cofinal filter in D. From the fact that $\langle\bar{x}\rangle\leq\bar{g}\leq\langle\bar{y}\rangle$ it follows that $\bar{g}_1\leq\langle\bar{y}\rangle$, hence $\langle x\rangle\leq g<g_1\leq\langle y\rangle$, and hence there exists $m\in C$ with $g(m)=x<g_1(m)$. Choose $z\in A$ with $x\prec z\leq g_1(m)$, and apply Lemma 4.7.

Suppose 4.7(i) applies, i.e., suppose $z\in R(\varphi)$ and $\bar{x}\prec\bar{z}$. Letting $h=\langle x[f(k)\mp y],z\rangle$, we then have $\langle x\rangle<h\leq\langle z\rangle$ and $h\leq f$, hence $\langle\bar{x}\rangle<\bar{h}\leq\langle\bar{z}\rangle$ and $\bar{h}\leq\langle b\rangle$. Since $\bar{x}\prec\bar{z}$, $\bar{h}$ must actually take on the value $\bar{z}$, and we infer that $\bar{z}\leq b$, hence $\langle z\rangle\leq f$, contrary to the fact that $x\in f(C)$.

Now suppose 4.7(ii) applies. In this case D has a largest element, and $\bar{g}(1)=\bar{g}_1(1)$. Letting $h=\langle x[k\ngeq m],z\rangle$, we have $\bar{h}(r)=\bar{x}$ for $1\neq r\in D$, and $\bar{h}(1)\leq\bar{g}_1(1)=\bar{g}(1)$, so that $\bar{h}\leq\bar{g}$, contrary to the fact that $h(m)=z\nleq g(m)$. This contradiction completes the proof of (1).

To complete the proof of the lemma it suffices to show that if $f\in\Delta(\varphi)$, $m,n\in C$, and $m\leq n$, then there exists $g\in\Delta(\varphi)$ such that $f(m)$ and $f(n)$ are, respectively, the smallest and the largest member of $g(C)$. If C has a largest element, we may by Lemma 4.1 assume that $n=1$, and if C has a smallest element, we may assume that $m=0$. The function f in $\Delta(\varphi)$ and the elements $m,n\in C$ with $m\leq n$ will be fixed throughout the remainder of the proof, and we let $\bar{f}=\langle b\rangle$.

The statements (2) and (3) below hold independently of the hypothesis of the lemma, and, in particular, they do not depend on the atomisticity of A and B. This is important because we will also want to apply their duals.

(2) Suppose B^D has Property (a), suppose $g \in A^C$ satisfies the conditions $f \leq g$ and $f(k) = g(k)$ for $k \geq m$, and suppose g is a maximal function satisfying these conditions. Then $g \in \Delta(\varphi)$.

Suppose to the contrary that $\bar{g}$ is not constant, and using Property (a), choose $g_1 \in A^C$ with $g < g_1$ such that g_1 agrees with g on some cofinal filter F in D. We shall arrive at a contradiction by showing that $g_1(k) = f(k)$ for $k \geq m$.

Assuming that this is false, let

$$h = \langle f[k \ngeq m\rangle, g_1\rangle, \qquad \bar{h}_1 = \langle b[r \notin F], \bar{h}\rangle .$$

Then $f < h$, hence $\langle b\rangle < \bar{h}_1$, and consequently $f < h_1$. For $r \in F$ we have

$$\bar{h}_1(r) = \bar{h}(r) \leq \bar{g}_1(r) = \bar{g}(r) ,$$

whence $\bar{h}_1 \leq \bar{g}$. Thus $f < h_1 \leq g, h$, which is impossible because $g \wedge h = f$.

(3) Suppose $D \in \mathscr{P}(1)$, $g, h \in A^C$, $h \leq f$, g, and $f(k) = g(k) = h(k)$ for $k \geq m$. Let $\bar{g}_1 = \langle \bar{g}(1)\rangle$. Then $g_1(k) = f(k)$ for $k \geq m$.

To prove (3), let

$$h_1 = \langle h[k \ngeq m], g_1\rangle, \qquad \bar{h}_2 = \langle \bar{h}[r \neq 1], \bar{h}_1(1)\rangle .$$

Then $g \wedge h_1 = h$ and $h \leq h_2 \leq g, h_1$, hence $h_2 = h$. Therefore

$$\bar{h}_1(1) = \bar{h}_2(1) = \bar{h}(1) \leq \bar{f}(1) = b ,$$

whence it follows that $\bar{h}_1 \leq \langle b\rangle$, $h_1 \leq f$, and hence for $k \geq m$,

$$g_1(k) = h_1(k) \leq f(k) = g(k) \leq g_1(k) .$$

Thus $g_1(k) = f(k)$, as was to be shown.

We now consider the six cases of the lemma.

CASE 1. $C, D \in \mathscr{L}$. Statement (2) applies with $g(k) = f(k \vee m)$ and yields $g \in \Delta(\varphi)$. Next, apply the dual of (2) with f and g replaced by g and g_1, where $g_1(k) = g(k \wedge n)$. This gives $g_1 \in \Delta(\varphi)$. Clearly $f(m)$ and $f(n)$ are, respectively, the smallest and the largest elements of $g_1(C)$. Thus $f(m) \leq_\varphi f(n)$.

CASE 2. $C, D \in \mathscr{P}(0, 1)$. Apply (3) with

$$g = \langle f(m)[k \ngeq m], f\rangle, \qquad h = \langle f(0)[k \ngeq m], f\rangle ,$$

and $\bar{g}_1 = \langle \bar{g}(1)\rangle$. Then $g_1 \in \Delta(\varphi)$, and $f(m)$ and $f(1)$ are, respectively, the smallest and the largest elements of $g_1(C)$, so that $f(m) \leq_\varphi f(1)$.

CASE 3. $C, D \in \mathscr{S}_\wedge(1)$. Apply (3) with g as above and $h = \langle f(k \wedge m)[k \ngeq m], f\rangle$.

CASE 4. $C, D \in \mathscr{S}_\vee(0)$. This is the dual of Case 3.

CASE 5. The condition (ii) holds. Apply (3) with $h = f$ and $g(k) = f(k) \vee f(m)$ for $k \in C$.

CASE 6. $C, D \in \mathscr{P}_{\max}(1)$. The argument in this final case will be quite different. Assuming that $f(m) \leq_\varphi f(1)$ fails. we may further assume that $f(k) \leq_\varphi f(1)$ whenever $m < k \in C$. Let $x = f(m)$ and $y = f(1)$, and for $k \in C$ let $f_k = \langle f(k)[j \leq m], y\rangle$. Then $y \in R(\varphi)$ by Lemma 4.9 and $f_m \notin \Delta(\varphi)$ by (1), but $f_k \in \Delta(\varphi)$ whenever $m < k \in C$.

Let $\bar{g} = \langle \bar{f}_m(1)\rangle$, and note that $f_m < g \leq \langle y\rangle$. The first inclusion is obvious, and the second one follows from the fact that $f_m \leq \langle y\rangle$, hence $\bar{f}_m \leq \langle \bar{y}\rangle$, and therefore $\bar{f}_m(1) \leq \bar{y}$ and, finally, $\bar{g} \leq \langle \bar{y}\rangle$. The element $z = g(m)$ is greater than x. This follows from the fact that $f_m < g$ and $f_m(j) = y = g(j)$ whenever $j \nleq m$, hence for some $j \leq m$,

$$x = f_m(j) < g(j) \leq g(m) = z .$$

It is now easy to see that the function $h = \langle f[k \neq m], z\rangle$ belongs to A^C, for if $k < m$, then $f(k) \leq x < z$, and if $k > m$, then $f_k \in \Delta(\varphi)$, say $\bar{f}_k = \langle b_k\rangle$, hence

$$\bar{f}_m(1) \leq \bar{f}_k(1) = b_k, \quad \bar{g} = \langle \bar{f}_m(1)\rangle \leq \langle b_k\rangle = \bar{f}_k ,$$

so that $g \leq f_k$ and, in particular, $z = g(m) \leq f_k(m) = f(k)$.

We have $\langle b\rangle = \bar{f} < \bar{h}$ because $f(m) = x < z = h(m)$. Therefore $b < \bar{h}(1)$, and letting $\bar{h}_1 = \langle b[r \neq 1], \bar{h}(1)\rangle$, we see that $f < h_1 \leq h$. Also, $h_1 \leq f_m$, because $f \leq g$ and $h(m) = g(m)$, therefore

$$h \leq g, \quad \bar{h}(1) \leq \bar{g}(1) = \bar{f}_m(1) .$$

The inclusions $f < h_1 \leq h$ and $h_1 \leq f_m$ yield the desired contradiction. From the first inclusion we infer that $f(k) < h_1(k)$ for some $k \in C$, and since $f(k) = h(k)$ for $k \neq m$, this means that $f(m) < h_1(m)$. On the other hand, $h_1(m) \leq f_m(m) = f(m)$.

5. Results on cancellation and refinements.

We are now ready to prove some of our principal theorems on cancellation of exponents and refinements for powers.

LEMMA 5.1. *Suppose $A^C \cong B^D$, where A and B are atomic, C and D are directly indecomposable, and either*

(i) *C and D both belong to (the same) one of the classes* $\mathscr{L}$, $\mathscr{P}(0,1)$, $\mathscr{S}_\vee(0)$, $\mathscr{S}_\wedge(1)$, $\mathscr{P}_{\max}(1)$, or else
(ii) $A, B \in \mathscr{S}_\vee$ *and* $C, D \in \mathscr{P}(1)$.

If $C \cong D$, *then* $A \cong B$, *but if* $C \not\cong D$, *then for some* E, $A \cong E^D$ *and* $B \cong E^C$.

PROOF. By Corollary 4.3, A^C and B^D have Property (a). Hence, by Lemmas 4.1, 4.5, and 4.10, the conditions $(\varphi, 1)$, $(\varphi, 2)$, $(\varphi, 3)$, and $(\varphi^{-1}, 3)$ are satisfied by the given isomorphism φ: $A^C \cong B^D$, and if $C \not\cong D$, then $(\varphi, 4)$ also holds by Lemma 4.8. From this the conclusion follows by Theorems 3.2 and 3.3.

THEOREM 5.2. *Suppose* $P \cong A^C \cong B^D$, *where P is atomic and C and D are finitely factorable and upper bounded. Then for some* E, X, Y, Z, $A \cong E^X$, $B \cong E^Y$, $C \cong Y \cdot Z$, $D \cong X \cdot Z$. *In particular, if* $C \cong D$, *then* $A \cong B$.

PROOF. We use induction on $m+n$, where

$$C \cong C_1 \cdot C_2 \dots C_m, \qquad D \cong D_1 \cdot D_2 \dots D_n,$$

with all factors directly indecomposable. Since C and D are connected, m and n are uniquely determined by C and D by Hashimoto's theorems. [We note that the atomicity of $P \cong A^C$ is equivalent to A being atomic and C possessing the a.c.c., unless P is totally unordered. Our conclusion follows readily if P is totally unordered. Thus we assume P is not totally unordered, so that A, B are atomic and $C, D \in \mathscr{P}_{\max}(1)$.] For $m=n=1$ the conclusion holds by Lemma 5.1. We therefore consider a value $m+n>2$, and assume that the theorem holds for all smaller values. We may also assume that $m>1$.

Let $C' = C_1 \cdot C_2 \dots C_{m-1}$. Then $(A^{C_m})^{C'} = B^D$, so that by the inductive hypothesis there exist E_1, X_1, Y_1, Z_1 with

$$A^{C_m} \cong E_1^{X_1}, \qquad B \cong E_1^{Y_1}, \qquad C' \cong Y_1 \cdot Z_1, \qquad D \cong X_1 \cdot Z_1 .$$

Since X_1 has at most n factors, a second application of the inductive hypothesis yields

$$A \cong E_2^{X_2}, \qquad E_1 \cong E_2^{Y_2}, \qquad C_m \cong Y_2 \cdot Z_2, \qquad X_1 \cong X_2 \cdot Z_2 ,$$

and the first part of the conclusion holds with

$$E = E_2, \qquad X = X_2, \qquad Y = Y_1 \cdot Y_2, \qquad Z = Z_1 \cdot Z_2 .$$

Finally, if $C \cong D$, then $C \cong X \cdot Z \cong Y \cdot Z$, and since C is connected and finitely factorable, this implies that $X \cong Y$, hence $A \cong B$.

In certain important cases we are able to drop the assumption of atomisticity in the preceding theorem. This is based on the following simple observation.

LEMMA 5.3. *Suppose* $\varphi\colon A^C \cong B^D$ *and* $\psi\colon A_1^C \cong B_1^D$, *where* $A \subseteqq A_1$, $B \subseteqq B_1$, *and* $\varphi \subseteqq \psi$. *Then* (ψ, i) *implies* (φ, i) *for* $i = 1, 2, 3, 4$.

PROOF. Note that $R(\varphi) = A \cap R(\psi)$, that $\leq_\varphi$ is the restriction of $\leq_\psi$ to $R(\varphi)$, that $\mathring{\varphi}$ is the restriction of $\mathring{\psi}$ to $R(\varphi)$, and that $\Delta(\varphi) = A^C \cap \Delta(\psi)$. From this the conclusion readily follows.

Consider a meet semilattice A, and let A^π be the lattice of semilattice filters in A. Each element of A may be identified with the corresponding principal filter in A, and A thus regarded as a subsemilattice of A^π. If C is a finite poset, then by Theorem 3.1 in Duffus, Jónsson, and Rival [7], there is a unique isomorphism of $(A^C)^\pi$ onto $(A^\pi)^C$ that is the identity on A^C. (This theorem is stated there for lattices only, but the proof is valid more generally for meet semilattices.) Thus if $\varphi\colon A^C \cong B^D$, where $A, B \in \mathscr{S}_\wedge$ and C and D are finite posets, then there is a unique isomorphism $\psi\colon (A^\pi)^C \cong (B^\pi)^D$ with $\varphi \subseteqq \psi$. Of course A^π and B^π need not be atomic, but using an idea from Dilworth and Freese [5], we can iterate this process. Letting $X_0 = A$, $Y_0 = B$, $X_{i+1} = X_i^\pi$, and $Y_{i+1} = Y_i^\pi$, we obtain isomorphisms $\varphi_i\colon X_i^C \cong Y_i^C$ with $\varphi = \varphi_0 \subseteqq \varphi_1 \subseteqq \ldots$. Therefore, if we take A_1 to be the union of the X_i's and B_1 the union of the Y_i's, then there exists an isomorphism $\psi\colon A_1^C \cong B_1^D$ with $\varphi \subseteqq \psi$, and A_1 and B_1 are atomic.

THEOREM 5.4. *Suppose* $A^C \cong B^D$, *where* $C, D \in \mathscr{P}(1)$ *are finite, and* $A, B \in \mathscr{S}_\wedge$. *Then for some* E, X, Y, Z, $A \cong E^X$, $B \cong E^Y$, $C \cong Y \cdot Z$, $D \cong X \cdot Z$. *In particular, if* $C \cong D$, *then* $A \cong B$.

PROOF. First suppose C and D are directly indecomposable. As we have just observed, the given isomorphism $\varphi\colon A^C \cong B^D$ can be extended to an isomorphism $\psi\colon A_1^C \cong B_1^D$, where A_1 and B_1 are atomic lattices. Now $(\varphi, 1)$ holds by Lemma 4.1, and $(\varphi, 2)$ by Lemma 4.5 and Corollary 4.3. By Lemma 4.10, $(\psi, 3)$ and $(\psi^{-1}, 3)$ hold, and hence $(\varphi, 3)$ and $(\varphi^{-1}, 3)$ hold by Lemma 5.3. Finally, if $C \ncong D$, then $(\psi, 4)$ holds by Lemma 4.8, whence $(\varphi, 4)$ follows by Lemma 5.3. For the present case, the conclusion of the theorem follows by Theorems 3.2 and 3.3. For the general case we use straightforward induction, as in the proof of Theorem 5.2.

6. A logarithmic property.

In this section and the next two, a different method will be developed for proving the conditions (φ, i), yielding a refinement for $\varphi\colon A^C \cong B^D$ under conditions that only require C and D to be connected and finitely factorable, but impose additional restrictions on A and B. The basic idea is to associate with each poset A a suitable subset $J'(A)$ in a uniform manner, in such a way that $J'(A^C) \cong J'(A) \cdot C^\delta$. Thus we have here an instance of what is referred to in Duffus and Rival [8] as a logarithmic property.

The familiar notion of a completely join irreducible element can be formulated in a way that is meaningful for arbitrary partially ordered sets.

Definition 6.1. For any poset A, $J(A)$ is the set of all $a \in A$ such that for some $a' \in A$, $a \not\leq a'$, but $x \leq a'$ whenever $x < a$.

It is not generally true that $J(A^C) \cong J(A) \cdot C^\delta$. E.g., let A be the poset in Fig. 1 with a zero added. Then $J(A)$ consists of all four non-zero elements of A, but $J(A^2)$ has only six elements. We therefore use a modified version of this concept.

Definition 6.2. Suppose A is a poset.

(i) Two elements of A are said to be compatible if they have a common upper bound.
(ii) $J'(A)$ is the set of all $a \in A$ such that, for some $a' \in A$, a' is compatible with a and $a \not\leq a'$, but $x \leq a'$ whenever $x \in A$ and $x < a$.
(iii) For any $X \subseteq A$, ∇X is the set of all $a \in A$ such that for every $a' \in A$ that is compatible with a, if $x \leq a'$ whenever $x \in X$ and $x \leq a$, then $a \leq a'$.

Corollary 6.3. *For any poset* A:

(i) $J'(A) \subseteq J(A)$.
(ii) $J'(A) = J(A)$ *if* A *updirected.*
(iii) *If* $A \in \mathscr{S}_\vee$, *then* $J(A)$ *is the set of all completely join irreducible elements of* A.
(iv) $a \in J'(A)$ *iff* $a \in A$ *and* $a \notin \nabla\{x \in A : x < a\}$.
(v) *If* A *satisfies the* d.c.c., *then* $a = \bigvee \{x \in J(A) : x \leq a\}$ *for all* $a \in A$, *and* $A = \nabla J'(A)$.

Proof. All these statements are quite obvious, except perhaps the last part of (v). We prove this by showing that, in general, $A - \nabla J'(A)$ does not have a minimal element. In fact, if such an element a exists, then there must exist an element $a' \in A$ that is compatible with a, such that $a \not\leq a'$, but $x \leq a'$ whenever $x \in J'(A)$ and $x \leq a$. Clearly $a \notin J'(A)$, and this implies that there exists $y \in A$ with $y < a$ and $y \not\leq a'$. By the minimality of a, $y \in \nabla J'(A)$, and since a' is compatible with y, we infer that there exists $x \in J'(A)$ with $x \leq y$ and $x \not\leq a'$. But then $x \leq a$, yielding a contradiction.

Definition 6.4. Suppose $A \in \mathscr{P}(0)$ and T is any poset. For $a \in A$ and $t \in T$ we let

$$j(a,t) = \langle 0[s \not\geq t], a \rangle .$$

LEMMA 6.5. *Suppose $A \in \mathscr{P}(0)$ and T is any poset. Then*
(i) $J'(A^T) = \{j(a,t) : a \in J'(A),\ t \in T\} \cong J'(A) \cdot T^\delta$.
(ii) $J(A^T) = \{j(a,t) : a \in J'(A)$ *or* $(a \in J(A)$ *and* t *is maximal in* $T)\}$.
(iii) *If A is updirected, then*

$$J(A^T) = J'(A^T) = \{j(a,t) : \ a \in J(A) \text{ and } t \in T\} .$$

PROOF. We first remark that, by definition, 0 belongs to neither $J(A)$ nor $J'(A)$. Once this has been noted, the second part of (i) is obvious, for $j(a_0, t_0) \leq j(a_1, t_1)$ iff $a_0 \leq a_1$ and $t_0 \geq t_1$ (provided $a_0 \neq 0$).

Any $f \in A^T$ is the join of the elements $j(f(t), t)$ with $t \in T$. Hence, if $f \in J(A^T)$, then f must be of the form $j(a,t)$. Furthermore, there must exist $g \in A^T$ such that $f \not\leq g$, but $h \leq g$ whenever $h < f$. This implies that $a \not\leq g(t)$, and that $x \leq g(t)$ whenever $x < a$, which shows that $a \in J(A)$. Furthermore, if t is not maximal, say $t < t_1$, then $j(a,t_1) < j(a,t)$, hence $j(a,t_1) \leq g$, $a \leq g(t_1)$. Thus a and $g(t)$ are in this case compatible, and hence $a \in J'(A)$.

Now suppose $a \in J'(A)$ and $t \in T$. There exist $a', b \in A$ such that $a \leq b$, $a' \leq b$, $a \not\leq a'$, and $x \leq a'$ whenever $x < a$. Let

$$g = \langle 0[s \not\geq t], a'[s = t], b \rangle .$$

Then $j(a,t) \not\leq g$, but if $h < j(a,t)$, then $h(t) < a$, hence $h(t) \leq a'$, and consequently $h \leq g$. Thus $j(a,t) \in J(A^T)$ and, in fact, $j(a,t) \in J'(A)$, since $j(b,t)$ is a common upper bound for $j(a,t)$ and g. Next suppose $a \in J(A)$ and t is maximal in T. Choose a' with $a \not\leq a'$ such that $x \leq a'$ whenever $x < a$. In this case, any function $h < j(a,t)$ is of the form $j(x,t)$ with $x < a$, and thus $h \leq j(a',t)$.

This completes the proof of (ii), and to prove (i), it only remains to show that if $j(a,t) \in J'(A^T)$, then $a \in J'(A)$. Choosing $g, g_1 \in A^T$ such that $j(a,t) \leq g_1$, $g \leq g_1$, $j(a,t) \not\leq g$, and $h \leq g$ whenever $h < j(a,t)$, we take $a' = g(t)$ and $b = g_1(t)$, and check that $a \leq b$, $a' \leq b$, $a \not\leq a'$, and $x \leq a'$ whenever $x < a$.

Finally, (iii) follows immediately from (i) and (ii), together with Corollary 6.3(ii).

COROLLARY 6.6. *If $A \in \mathscr{P}(0)$ and T is any poset, then*

$$\nabla J'(A^T) = A^T \text{ iff } \nabla J'(A) = A .$$

PROOF. Since every member of A^T is the join of functions of the form $j(a,t)$, we see that

$$\nabla J'(A^T) = A^T \quad \text{iff } j(a,t) \in \nabla J'(A^T) \quad (a \in A,\ t \in T) .$$

For given $a \in A$ and $t \in T$, it is simple matter to check that

$$j(a,t) \in \nabla J'(A^T) \quad \text{iff } a \in \nabla J'(A)\,.$$

From these two observations the corollary readily follows.

7. Applications of the strong refinement property.

Throughout this section the following assumptions will be in effect:

A is lower bounded, and $\nabla J'(A) = A$.
φ: $A^C \cong B^D$, and C and D are connected.
ψ: $J'(A) \cdot C^\delta \cong J'(B) \cdot D^\delta$ is induced by φ.

The last statement means that ψ is the unique isomorphism such that, for all $a \in J'(A)$, $b \in J'(B)$, $k \in C$ and $r \in D$,

$$\psi(a,k) = (b,r) \quad \text{iff } \varphi j(a,k) = j(b,r)\,.$$

The components of $J'(A) \cdot C^\delta$ are $A_i \cdot C^\delta$ $(i \in I)$, where the A_i's are the components of $J'(A)$, and ψ sends these into the components of $J'(B) \cdot D^\delta$, say

$$\psi_i : \; A_i \cdot C^\delta \cong B_i \cdot D^\delta\,.$$

Applying the strict refinement theorem to this isomorphism, we obtain

$$\alpha_i : \; W_i \cdot X_i \cong A_i, \qquad \gamma_i : \; Y_i \cdot Z_i \cong C^\delta\,,$$

$$\beta_i : \; W_i \cdot Y_i \cong B_i, \qquad \delta_i : \; X_i \cdot Z_i \cong D^\delta\,,$$

such that for all $w \in W_i$, $x \in X_i$, $y \in Y_i$, $z \in Z_i$,

$$\psi_i(\alpha_i(w,x), \gamma_i(y,z)) = (\beta_i(w,y), \delta_i(x,z))$$

or, equivalently,

$$\varphi j(\alpha_i(w,x), \gamma_i(y,z)) = j(\beta_i(w,y), \delta_i(x,z))\,.$$

Observe that, for $f \in A^C$,

$$\alpha_i(w,x) \leq f(\gamma_i(y,z)) \quad \text{iff } \beta_i(w,y) \leq \bar{f}(\delta_i(x,z))\,.$$

We now prove a series of lemmas relating the above assumptions to the concepts introduced in Definition 3.1.

LEMMA 7.1. *An element $a \in A$ belongs to $R(\varphi)$ iff for all $i \in I$ and $w \in W_i$, the condition $\alpha_i(w,x) \leq a$ either holds for all $x \in X_i$, or else for none.*

PROOF. Let $f = \langle a \rangle$, and consider any $m = \gamma_i(y,z)$ in C. Then $\alpha_i(w,x) \leq a$ is equivalent to $\alpha_i(w,x) \leq f(\gamma_i(y,z))$, hence to

$$(1) \qquad \beta_i(w,y) \le \bar{f}(\delta_i(x,z)) .$$

If $\bar{f}$ is constant, then (1) is independent of x. On the other hand, if $\bar{f}$ is not constant, then it follows from the hypothesis $A = \nabla J'(A)$ that i, w, and y can be so chosen that (1) holds for some (x,z), but not for all. Since the original inclusion, $\alpha_i(w,x) \le a$, is independent of z, it must then depend on x.

LEMMA 7.2. *A function $f \in A^C$ belongs to $\Delta(\varphi)$ iff, for all $i \in I$, $w \in W_i$, and $y \in Y_i$, the condition $\alpha_i(w,x) \le f(\gamma_i(y,z))$ either holds for all $x \in X_i$ and $z \in Z_i$, or else for none.*

PROOF. We have

$$\alpha_i(w,x) \le f(\gamma_i(y,z)) \quad \text{iff } \beta_i(w,y) \le \bar{f}(\delta_i(x,z)) .$$

If $\bar{f}$ is constant, then the second condition is independent of x and z. The converse follows from the assumption $B = \nabla J'(B)$.

LEMMA 7.3. *Given $a_0, a_1 \in R(\varphi)$, we have $a_0 \le_\varphi a_1$ iff $a_0 \le a_1$ and, for all $i \in I$ with $|Z_i| > 1$, and for all $a \in A_i$, $a \le a_1$ implies $a \le a_0$.*

PROOF. Suppose $a_0 \le_\varphi a_1$, and consider any $i \in I$ with $|Z_i| > 1$, and an element $a = \alpha_i(w,x)$ in A_i with $a \le a_1$. Choose $z_0, z_1 \in Z_i$ with $z_0 \not\le z_1$, and choose any $y \in Y_i$, and let $m_j = \gamma_i(y, z_j)$ for $j = 0, 1$. Then $m_0 \not\ge m_1$, and letting $f = \langle a_0[k \not\ge m_1], a_1 \rangle$, we have $f(m_0) = a_0$ and $f(m_1) = a_1$. Thus $\alpha_i(w,x) \le f(\gamma_i(y,z_1))$, hence $\beta_i(w,y) \le \bar{f}(\delta_i(x,z_1))$. Since $\bar{f}$ is constant, this gives $\beta_i(w,y) \le \bar{f}(\delta_i(x,z_0))$ or, equivalently, $\alpha_i(w,x) \le f(\gamma_i(y,z_0))$, that is, $\alpha_i(w,x) \le a_0$.

Conversely, suppose $a_0 \le_\varphi a_1$ fails but $a_0 \le a_1$. This means that there exists $f \in A^C$ with $f(C) = \{a_0, a_1\}$ such that $\bar{f}$ is not constant. There exist $p_0, p_1 \in D$ such that $p_0 < p_1$ and $\bar{f}(p_0) < \bar{f}(p_1)$, hence for some $i \in I$ and $b = \beta_i(w,y)$ in B_i, $b \le \bar{f}(p_1)$ and $b \not\le \bar{f}(p_0)$. Write p_j, $j = 0, 1$, in the form $p_j = \delta_i(x_j, z_j)$. Then

$$\beta_i(w,y) \le \bar{f}(\delta_i(x_1,z_1)), \qquad \beta_i(w,y) \not\le \bar{f}(\delta_i(x_0,z_0))$$

or, equivalently,

$$\alpha_i(w,x_1) \le f(\gamma_i(y,z_1)), \qquad \alpha_i(w,x_0) \not\le f(\gamma_i(y,z_0)) .$$

Since $f(C) = \{a_0, a_1\}$ and $a_0 \le a_1$, this implies that

$$\alpha_i(w,x_1) \le a_1, \qquad \alpha_i(w,x_0) \not\le a_0 .$$

Recalling that the condition $\alpha_i(w,x) \le a_j$ is independent of x, we conclude that, for an arbitrary $x \in X_i$, the element $a = \alpha_i(w,x)$ is included in a_1 but not in a_0.

LEMMA 7.4. $\mathring{\varphi}$: $(R(\varphi), \le_\varphi) \cong (R(\varphi^{-1}), \le_{\varphi^{-1}})$.

PROOF. By symmetry it suffices to show that $a_0 \leq_\varphi a_1$ implies $\bar{a}_0 \leq_{\varphi^{-1}} \bar{a}_1$. Considering any $i \in I$ with $|Z_i| > 1$, and $b = \beta_i(w, y)$ in B_i, we need to show that $b \leq \bar{a}_1$ implies $b \leq \bar{a}_0$. Picking any $x \in X_i$, we merely note that the conditions $\beta_i(w, y) \leq \bar{a}_j$ and $\alpha_i(w, x) \leq a_j$ are equivalent, and that by Lemma 7.3 the latter holds for $j = 0$ iff it holds for $j = 1$.

LEMMA 7.5. *For all* $f \in \Delta(\varphi)$, $f(C) \subseteqq R(\varphi)$.

PROOF. By Lemma 7.1 it suffices to show that the condition $\alpha_i(w, x) \leq f(m)$ is independent of x, but if we write $m = \gamma_i(y, z)$, then this condition is equivalent to $\beta_i(w, y) \leq \bar{f}(\delta_i(x, z))$, which does not depend on x since $\bar{f}$ is constant.

LEMMA 7.6. *Every order preserving function from C into $(R(\varphi), \leq_\varphi)$ belongs to $\Delta(\varphi)$.*

PROOF. Let f be such a function. By Lemma 7.2 it suffices to show that the condition $\alpha_i(w, x) \leq f(\gamma_i(y, z))$ is independent of x and z. That it is independent of x follows from Lemma 7.1, since $f(C) \subseteqq R(\varphi)$. In showing that it does not depend on z, we may of course assume that $|Z_i| > 1$, and Lemma 7.3 therefore applies. Since C is connected, $f(C)$ is connected in $(R(\varphi), \leq_\varphi)$. From this the conclusion follows for, by Lemma 7.3, an element $a \in A_i$ that satisfies the condition $a \leq f(m)$ for one $m \in C$ satisfies this condition for all $m \in C$.

LEMMA 7.7. *The set $\Delta(\varphi)$ coincides with the set of all order preserving functions from C into $(R(\varphi), \leq_\varphi)$ iff, for all $b \in B$, for all $i \in I$ with $|Z_i| > 1$, and for all $w \in W_i$, the condition $\beta_i(w, y) \leq b$ either holds for all $y \in Y_i$, or else for none.*

PROOF. Suppose $f \in \Delta(\varphi)$ and $\bar{f} = \langle b \rangle$. By Lemmas 7.1 and 7.3, f is an order preserving function from C into $(R(\varphi), \leq_\varphi)$ iff, for all $i \in I$ with $|Z_i| > 1$, and all $w \in W_i$, the condition $\alpha_i(w, x) \leq f(\gamma_i(y, z))$ is independent of the elements $x \in X_i$, $y \in Y_i$, and $z \in Z_i$. But this condition is equivalent to $\beta_i(w, y) \leq b$. Thus f is order preserving iff this last condition is independent of y.

COROLLARY 7.8. *The conditions $(\varphi, 1)$ and $(\varphi, 2)$ hold. If, for each $i \in I$, either $|Y_i| = 1$ or $|Z_i| = 1$, then $(\varphi, 3)$ holds, and if $|Z_i| = 1$ for all $i \in I$, then $(\varphi, 4)$ holds.*

PROOF. For (φ, i), $i = 1, 2, 3, 4$, see Lemmas 7.3, 7.4, 7.7, and 7.3, respectively.

8. Further results on cancellation and refinements.

We consider here various situations in which Corollary 7.8 applies.

THEOREM 8.1. *Suppose* $A^C \cong B^D$, *where* $A \in \mathscr{P}(0)$, $\nabla J'(A) = A$, C *and* D *are connected, and* $(C, D) = 1$. *Then for some* E, $A \cong E^D$ *and* $B \cong E^C$.

PROOF. The condition $(C, D) = 1$ means that C and D have no common division other than 1. Hence, in the notation of the preceding section, $|Z_i| = 1$ for all $i \in I$, and the conclusion follows from Corollary 7.8 and Theorem 3.3.

THEOREM 8.2. *Suppose* $A^C \cong B^D$, *where* $A \in \mathscr{P}(0)$, $\nabla J'(A) = A$, C *and* D *are connected, and* C *is finitely factorable. Then for some* E, X, Y, Z, $A \cong E^X$, $B \cong E^Y$, $C \cong Y \cdot Z$, $D \cong X \cdot Z$. *In particular, if* $C \cong D$, *then* $A \cong B$.

PROOF. First suppose $C \cong D$. If C is directly indecomposable, then any isomorphism $\varphi \colon A^C \cong B^D$ satisfies $(\varphi, 3)$ and $(\varphi^{-1}, 3)$ as well as $(\varphi, 1)$ and $(\varphi, 2)$ for, in the notation of the preceding section, we have for each $i \in I$ either $|Z_i| = 1$, or else $|X_i| = |Y_i| = 1$. For this case we therefore have $A \cong B$ by Theorem 3.2. For C finitely factorable, the same conclusion is obtained by induction.

Dropping the hypothesis that $C \cong D$, we can find posets X, Y, Z with

$$C \cong Y \cdot Z, \qquad D \cong X \cdot Z, \qquad (X, Y) = 1 .$$

By the first part of the proof, $A^Y \cong B^X$, and hence by Theorem 8.1, $A \cong E^X$ and $B \cong E^Y$ for some poset E.

THEOREM 8.3. *Suppose* $A^C \cong B^D$, *where* $A \in \mathscr{P}(0)$, $\nabla J'(A) = A$, *and* $J'(A)$, C *and* D *are connected. Then for some* E, X, Y, Z, $A \cong E^X$, $B \cong E^Y$, $C \cong Y \cdot Z$ *and* $D \cong X \cdot Z$.

PROOF. In the notation of the preceding section, the set I has just one element, and dropping subscripts we therefore have

$$\alpha : \; W \cdot X \cong J'(A), \qquad \gamma : \; Y \cdot Z \cong C^\delta ,$$

$$\beta : \; W \cdot Y \cong J'(B), \qquad \delta : \; X \cdot Z \cong D^\delta .$$

$$\psi(\alpha(w, x), \gamma(y, z)) = (\beta(w, y), \delta(x, z)) .$$

For $f \in A^C$, the function $f\gamma \colon Y \cdot Z \to A$ is independent of the second argument iff $\bar{f}\delta \colon X \cdot Z \to B$, is independent of its second argument. In fact, suppose $\bar{f}\delta$ does depend on its second argument, say $z_0 > z_1$ and $\bar{f}\delta(x, z_0) < \bar{f}\delta(x, z_1)$. Then there exists $b = \beta(w, y)$ in $J'(B)$ such that

$$\beta(w,y) \leq \bar{f}\delta(x,z_1) \quad \text{and} \quad \beta(w,y) \nleq \bar{f}\delta(x,z_0)\,,$$

hence

$$\alpha(w,x) \leq f\gamma(y,z_1) \quad \text{and} \quad \alpha(w,x) \nleq f\gamma(y,z_0)\,,$$

so that $f\gamma$ also depends on its second argument.

With each function $f \in A^{Y^\delta}$ we associate the unique function $f^* \in A^C$ such that $f^*\gamma(y,z)=f(y)$ for all $y \in Y$ and $z \in Z$. Similarly, for $h \in B^{X^\delta}$, $h^* \in B^D$ is defined by the condition that $h^*\delta(x,z)=h(x)$ for all $x \in X$ and $z \in Z$. The function $\varphi'\colon A^{Y^\delta} \to B^{X^\delta}$ such that $\varphi'(f)^*=\varphi(f^*)$ is easily seen to be an isomorphism, with an induced isomorphism $\psi'\colon J'(A)\cdot Y \cong J'(B)\cdot X$ given by

$$\psi'(\alpha(w,x),y) = (\beta(w,y),x)\,.$$

Applying the strict refinement property to ψ', we obtain the isomorphisms

$$\alpha:\ W\cdot X \cong J'(A), \qquad \gamma':\ Y\cdot \mathbf{1} \cong Y,$$

$$\beta:\ W\cdot Y \cong J'(B), \qquad \delta':\ X\cdot \mathbf{1} \cong X\,,$$

where $\gamma'(y,0)=y$ and $\delta'(x,0)=x$. Thus the poset Z has been replaced by a one-element set, and Corollary 7.8 yields (φ',i) for $i=1,2,3,4$. The conclusion follows by Theorem 3.3.

It is not in general true under the hypothesis of the preceding theorem that $C \cong D$ implies $A \cong B$. E.g., letting $A=\mathbf{2}$ and $B=\mathbf{2}^2$, we can choose $C=D$ so that $\mathbf{2}\cdot C \cong C$.

9. The mixed refinement property.

This section is devoted to the proof of the following result.

THEOREM 9.1. *If B and A^B are connected, then every direct decomposition of A^B is equivalent to one induced by a direct decomposition of A.*

We first make precise the concepts involved.

By a direct decomposition of A we mean an isomorphism φ of A onto a direct product,

$$\varphi:\ A \cong \sqcap\,(A_i,\ i \in I)\,,$$

and we say that φ is equivalent to

$$\varphi':\ A \cong \sqcap\,(A'_i,\ i \in I)$$

if there exist isomorphisms $\alpha_i\colon A_i \cong A'_i$ $(i \in I)$ such that $\varphi'(x)_i=\alpha_i(\varphi(x)_i)$ for all $x \in A$ and $i \in I$, i.e., such that

$$\varphi' = \sqcap(\alpha_i, i \in I) \circ \varphi ,$$

where the direct product of isomorphisms is defined in an obvious manner. This is equivalent to the assertion that, for each $i \in I$, the maps $x \to \varphi(x)_i$ and $x \to \varphi'(x)_i$ have the same kernel, i.e., that $\varphi(x)_i = \varphi(y)_i$ iff $\varphi'(x)_i = \varphi'(y)_i$.

For a given direct decomposition φ of A, the induced direct decomposition

$$\varphi^B: A^B \cong \sqcap (A_i^B, i \in I)$$

is obtained by letting

$$\varphi^B(f)_i(x) = \varphi(f(x))_i \qquad (i \in I, x \in B) .$$

LEMMA 9.2. *A direct decomposition*

$$\psi: A^B \cong \sqcap (C_i, i \in I)$$

is equivalent to some φ^B *iff, for all* $f, g \in A^B$ *and all* $i \in I$,

$$\psi(f)_i = \psi(g)_i \quad \textit{iff } \psi(\langle f(x)\rangle)_i = \psi(\langle g(x)\rangle)_i \quad \textit{for all } x \in B .$$

PROOF. This condition is obviously necessary. Conversely, suppose the condition holds.

The map $\varphi(a) = \psi(\langle a\rangle)$ is an isomorphism of A onto a subdirect product of the posets $A_i \subseteq C_i$, where $A_i = \{\psi(\langle a\rangle)_i : a \in A\}$. To show that this is in fact a direct decomposition of A, it suffices to prove that if $f \in A^B$ and $\psi(f) \in \sqcap (A_i, i \in I)$, then f is constant. For each $i \in I$ there exists $a_i \in A$ such that $\psi(f)_i = \psi(\langle a_i\rangle)_i$, hence for all $x \in B$,

$$\psi(\langle f(x)\rangle)_i = \psi(\langle a_i\rangle)_i ,$$

which shows that $f(x)$ is the same for all values of x.

We thus have $\varphi: A \cong \sqcap (A_i, i \in I)$, hence $\varphi^B: A^B \cong \sqcap (A_i^B, i \in I)$. It remains to check that ψ is equivalent to φ^B, i.e., that for all $f, g \in A^B$ and $i \in I$,

$$\psi(f)_i = \psi(g)_i \quad \text{iff } \varphi^B(f)_i = \varphi^B(g)_i ,$$

but this is but a reformulation of our hypothesis.

COROLLARY 9.3. *A direct decomposition*

$$\psi: A^B \cong \sqcap (C_i, i \in I)$$

is equivalent to one induced by a direct decomposition of A iff the same is true of each of the associated direct decompositions

$$\psi_p: A^B \cong C_p \cdot \sqcap (C_i, p \neq i \in I) .$$

LEMMA 9.4. *Suppose* $\psi\colon A^B \cong C_0 \cdot C_1$ *and* B *is connected. For any* $f, g \in A^B$, *if there exists* $a \in A$ *with* $f, g \leq \langle a \rangle$, *then for* $i = 0, 1$, *the condition* $\psi(f)_i = \psi(g)_i$ *implies that*

$$\psi(\langle f(x) \rangle)_i = \psi(\langle g(x) \rangle)_i \quad \textit{for all } x \in B\,.$$

PROOF. For notational convenience take $i=0$. We first impose very stringent additional conditoons on f and g, and then gradually relax these conditions. The element $x \in B$ will be fixed throughout the proof.

CASE 1. For some $a_0, a_1 \in A$, $a_0 < a_1$ and

$$f = \langle a_1[y > x], a_0 \rangle, \qquad g = \langle a_1[y \geq x], a_0 \rangle\,.$$

Letting $\psi(f) = (p, q)$, $\psi(g) = (p, r)$, and $\psi(\langle a_i \rangle) = (s_i, t_i)$ for $i = 0, 1$, we need to show that $s_0 = p = s_1$. Let $h \in A^B$ be the function with $\psi(h) = (s_0, r)$. Then $\langle a_0 \rangle \leq h \leq \langle a_1 \rangle$, hence $f \vee h = \langle h[f(y) = a_0], a_1 \rangle$. On the other hand, $(p, q) \vee (s_0, r) = (p, r)$, hence $f \vee h = g$. From this it follows that $h(x) = a_1$, $h \geq g$, and therefore $h = g$. Thus $s_0 = p$.

For $y \in B$ let $h_y = \langle a_1[z \geq y], a_0 \rangle$, and let $\psi(h_y) = \langle u_y, v_y \rangle$. We claim that if $y < y'$ and $u_{y'} = p$, then $u_y = p$. Letting $h, h' \in A^B$ be the functions such that $\psi(h) = (u_y, t_0)$ and $\psi(h') = (p, v_y)$ we have $h \wedge h_{y'} = \langle a_0 \rangle$ and $h \vee h' = h_y$. Therefore $h(y') = a_0$, hence $h(y) = a_0$ and, consequently, $h'(y) = a_1$. From this it follows that $h' \geq h_y$, $u_y \leq \psi(h')_0 = p$, and thus $u_y = p$.

Since the equation $u_y = p$ holds for $y = x$, we infer from the connectedness of B that it holds for all $y \in B$. Finally, since $\langle a_1 \rangle = \vee\ (h_y\colon y \in B)$, we conclude that $s_1 = p$.

CASE 2. $f(x) < g(x)$, and $f(y) = g(y)$ whenever $x \neq y \in B$. Let $a_0 = f(x)$, $a_1 = g(x)$, $h = \langle a_0[y \leq x], a \rangle$ and $h' = \langle a_0[y \ngeq x], a_1 \rangle$. Then the joins $f \vee h$ and $g \vee h$ exist, and so do the meets $f' = (f \vee h) \wedge h'$ and $g' = (g \vee h) \wedge h'$. In fact,

$$f' = \langle a_1[y > x], a_0 \rangle, \qquad g' = \langle a_1[y \geq x], a_0 \rangle\,.$$

Of course $\psi(f')_0 = \psi(g')_0$, whence by Case 1, $\psi(\langle f'(x) \rangle)_0 = \psi(\langle g'(x) \rangle)_0$. Since $f'(x) = a_0 = f(x)$ and $g'(x) = a_1 = g(x)$, this is the desired conclusion.

CASE 3. $f \leq g$. Apply Case 2 to the functions

$$f' = \langle f[y \leq x], g \rangle, \qquad g' = \langle f[y < x], g \rangle\,,$$

noting that $f \leq f' \leq g' \leq g$, hence $\psi(f')_0 = \psi(g')_0$.

CASE 4. f and g are arbitrary. Let $h \in A^B$ be the function such that $\psi(h)$

$=(\psi(f)_0, \psi(\langle a\rangle)_1)$. Then $f, g \leq h \leq \langle a\rangle$ and $\psi(h)_0 = \psi(f)_0 = \psi(g)_0$, so that by Case 3,

$$\psi(\langle f(x)\rangle)_0 = \psi(\langle h(x)\rangle)_0 = \psi(\langle g(x)\rangle)_0 .$$

PROOF OF THEOREM 9.1. By Corollary 9.3 it suffices to show that every direct decomposition $\psi\colon A^B \cong C_0 \cdot C_1$ is equivalent to φ^B for some $\varphi\colon A \cong A_0 \cdot A_1$.

For $a \in A$ let $\varphi(a) = \psi(\langle a\rangle)$. Then φ is an isomorphism of A onto a subdirect product of two posets $A_0 \subseteq C_0$ and $A_1 \subseteq C_1$. We begin by showing that φ is in fact an isomorphism of A onto $A_0 \cdot A_1$. I.e., given $a_0, a_1 \in A$, we show that the function $f \in A^B$ with $\psi(f) = (\varphi(a_0)_0, \varphi(a_1)_1)$ is constant.

First suppose there exists $a \in A$ with $a_0, a_1 \leq a$. Taking any $x \in B$, we see by Lemma 9.4 that

$$\psi(\langle f(x)\rangle)_i = \psi(\langle a_i\rangle)_i = \psi(f)_i$$

for $i = 0, 1$, and hence that $f = \langle f(x)\rangle$ is constant.

In the general case we use the fact that A is connected (because A^B is) to obtain elements $u_0, u_1, \ldots, u_n \in A$ with $u_0 = a_0$ and $u_n = a_1$ such that any two successive terms u_i and u_{i+1} with $i < n$ have a common upper bound v_i. We use induction on n. The case $n = 1$ has already been treated, and we therefore consider a value $n > 1$, assuming the conclusion to hold for all smaller values.

Let $\varphi(u_i) = (p_i, q_i)$ and $\varphi(v_i) = (r_i, s_i)$. By the case $n = 1$, there exist $u_i' \in A$ for $i < n$ such that $\varphi(u_i') = (p_i, q_{i+1})$, and by the dual of the same case there exist v_i' for $i < n-1$ such that $\varphi(v_i') = (r_i, s_{i+1})$. Clearly $u_i', u_{i+1}' \leq v_i'$ for $i < n-1$, whence by the inductive hypothesis, the function f with

$$\psi(f)_0 = \varphi(u_0')_0 = r_0 = \varphi(a_0)_0 ,$$

$$\psi(f)_1 = \varphi(u_{n-1}')_1 = s_n = \varphi(a_1)_1$$

is constant, as was to be shown.

The isomorphism φ induces an isomorphism

$$\varphi^B\colon A^B \cong A_0^B \cdot A_1^B ,$$

and by the strong refinement property there exist isomorphisms

$$\alpha_0\colon W \cdot X \cong A_0^B, \quad \alpha_1\colon Y \cdot Z \cong A_1^B ,$$

$$\gamma_0\colon W \cdot Y \cong C_0, \quad \gamma_1\colon X \cdot Z \cong C_1$$

such that, for all $w \in W$, $x \in X$, $y \in Y$, $z \in Z$,

$$(\varphi^B)^{-1}(\alpha_0(w,x), \alpha_1(y,z)) = \psi^{-1}(\gamma_0(w,y), \gamma_1(x,z)) .$$

To complete the proof it suffices to show that X and Z are one-element sets, and by symmetry it suffices to consider one of them, say X.

Choose any element $a \in A$, and let

$$\psi(\langle a\rangle) = (\gamma_0(w,y),\gamma_1(x,z)) = (p,q)\,.$$

Since X is connected, it suffices to show that any element $x' \in X$ that is comparable with x is equal to x. We may assume that $x' \leq x$. Let $f \in A^B$ be the function such that

$$\psi(f) = (\gamma_0(w,y),\gamma_1(x',z)) = (p,q')\,.$$

Clearly $f \leq \langle a\rangle$. We have

$$\varphi^B(\langle a\rangle) = (\alpha_0(w,x),\alpha_1(y,z)) = (\langle p\rangle,\langle q\rangle)\,,$$

$$\varphi^B(\langle f\rangle) = (\alpha_0(w,x'),\alpha_1(y,z)) = (g,\langle q\rangle)\,,$$

where $g \in A_0^B$. Applying Lemma 9.4 to both φ^B and ψ, we find that for any $t \in B$,

$$\varphi^B(\langle f(t)\rangle)_1 = \varphi^B(\langle a\rangle)_1 = \langle q\rangle\,,$$

hence $\psi(\langle f(t)\rangle)_1 = q = \psi(\langle a\rangle)_1$, and that $\psi(\langle f(t)\rangle)_0 = \psi(\langle a\rangle)_0$. Thus $f(t) = a$ for all $t \in B$, and we have $x' = x$, as was to be shown.

10. Cancellation results for bases.

By Duffus and Rival [8], $A^C \cong A^D$ implies $C \cong D$ whenever A, C, and D are finite and A is not unordered, and by Novotný [15] this implication holds whenever A is totally ordered and has more than one element. We state here some results of a related nature that follow rather easily from the theorems in the preceding sections.

THEOREM 10.1. *If $P \in \mathscr{P}(0)$ is of finite length, having more than one element, then $P \cong A^C \cong A^D$ implies $C \cong D$.*

PROOF. Since P is of finite length, it is finitely factorable, whence it follows that

$$A \cong \sqcap\,(A_i,\ i \in I), \qquad C = \textstyle\sum\,(C_j,\ j \in J), \qquad D = \textstyle\sum\,(D_k,\ k \in K)\,,$$

where the index sets I, J, and K are finite, each A_i is directly indecomposable, and the posets C_j and D_k are connected. Since $A_i^{C_j}$ and $A_i^{D_k}$ are connected, they are directly indecomposable by Theorem 9.1. Consequently, by Hashimoto's theorem, the sets J and K have the same number of elements, and

$$A_i^{C_j} \cong A_{\lambda(i,j)}^{D_{\mu(i,j)}}\,,$$

where $(i,j) \to (\lambda(i,j),\mu(i,j))$ is a one-to-one map of $I \cdot J$ onto $I \cdot K$. We can write

$A_i \cong E_i^{U_i}$, where E_i is exponentially indecomposable, and from the fact that A_i is directly indecomposable it follows that U_i is connected. By Theorem 8.2,

$$U_i \cdot C_j \cong U_{\lambda(i,j)} \cdot D_{\mu(i,j)} .$$

Summing over i and j, we get $U \cdot C \cong U \cdot D$, where U is the sum of the posets U_i. Since C, D and U are finite sums of finitely factorable sets, we conclude that $C \cong D$.

According to Novotný's theorem, every totally ordered set with more than one element cancels as a base without any restriction on the exponents. Example 2.2 shows that there are finite bounded posets that do not have this property. The next theorem yields other examples of posets, both finite and infinite, that cancel unconditionally as bases.

THEOREM 10.2. *Suppose $A \in \mathscr{P}(0)$ has more than one element, $\nabla J'(A) = A$, A is either of finite length, or else is exponentially indecomposable, and suppose $J'(A)$ is connected. Then for all C, D, $A^C \cong A^D$ implies $C \cong D$.*

PROOF. Since $J'(A)$ is connected, A is directly indecomposable. Hence, if $C = \sum (C_i,\ i \in I)$ where the posets C_i are connected, then by Theorem 9.1, the posets A^{C_i} are directly indecomposable. It follows that $A^{C_i} \cong A^{D_i}$, where $D = \sum (D_i,\ i \in I)$ and the sets D_i are connected. Applying Theorem 8.3 to this last isomorphism, we obtain

$$A \cong E_i^{X_i} \cong E_i^{Y_i}, \qquad C_i \cong Y_i \cdot Z_i, \qquad D_i \cong X_i \cdot Z_i .$$

If A is exponentially indecomposable, then $X_i \cong Y_i \cong 1$, but if A is of finite length, then $X_i \cong Y_i$ by Theorem 10.1. Thus, in either case, $C_i \cong D_i$, $C \cong D$.

THEOREM 10.3. *Suppose $A \in \mathscr{P}(0)$ is exponentially indecomposable, $\nabla J'(A) = A$, and $J'(A)$ is connected. Then for all B, C, D, $A^C \cong B^D$ implies that, for some Y, $B \cong A^Y$ and $C \cong Y \cdot D$.*

PROOF. Since $J'(A)$ is connected, A is directly indecomposable. Using Theorem 9.1 twice we infer, first, that A^C is isomorphic to a direct product of directly indecomposable factors and, second, that the same is true of B. Writing $B \cong \sqcap (B_i, i \in I)$ and $D = \sum (D_j,\ j \in J)$, where the posets B_i are directly indecomposable and the D_j's connected, we have $A^{C_{i,j}} \cong B_i^{D_j}$, where the $C_{i,j}$'s are the components of C. Applying Theorem 8.3 to this last isomorphism, and using the fact that A is exponentially indecomposable, we obtain $B_i \cong A^{Y_{i,j}}$ and $C_{ij} \cong Y_{i,j} D_j$ for some posets $Y_{i,j}$. By Theorem 10.2, all the posets $Y_{i,j}$, for a fixed i, are isomorphic, say $Y_{i,j} \cong Y_i$, and the conclusion holds with $Y = \sum (Y_i,\ i \in I)$.

11. Automorphisms of A^B.

There is an embedding π of Aut $(A) \times$ Aut (B) into Aut (A^B) defined by $\pi(\alpha, \beta)(f) = \alpha \circ f \circ \beta^{-1}$, but in general π is not onto. Even when our methods apply, i.e., when the automorphisms $\varphi\colon A^B \cong A^B$ satisfy $(\varphi, 1)$, $(\varphi, 2)$, and $(\varphi, 3)$, and we are therefore able to associate with φ an automorphism α of A, it is generally not the case that $(\alpha^B)^{-1} \circ \varphi$ is induced by an automorphism of B. E.g., $(\mathbf{2}^{\mathbf{2}})^{\mathbf{2}}$ has an involutionary automorphism induced by the non-trivial automorphism of $\mathbf{2} \cdot \mathbf{2}$, although both the base $\mathbf{2}^{\mathbf{2}}$ and the exponent $\mathbf{2}$ are rigid. An example with A exponentially indecomposable is $(\mathbf{2}^{\mathbf{2}} \cdot N)^{\mathbf{2}} \cong \mathbf{2}^{\mathbf{2} \cdot \mathbf{2}} \cdot N^{\mathbf{2}}$, where N is the five element non-modular lattice. Here the automorphism derives from the factorization of A. A less trivial example follows.

EXAMPLE 11.1. Let A be as in Fig. 2, and let $B \cong 2$. Then A has a non-trivial automorphism although A and B are rigid and triple indecomposable. The poset A has two arms, $\mathbf{2}$ and $\mathbf{3} \cong \mathbf{2}^{\mathbf{2}}$. If we add more arms, say $\mathbf{4}^{\mathbf{2}}, \mathbf{5}^{\mathbf{2}}, \ldots, \boldsymbol{n}^{\mathbf{2}}$, we get a finite, lower bounded poset A_n that is rigid and triple indecomposable, and such that Aut (A_n^2) is an elementary Abelian 2-group of order 2^{n-2}.

Surprisingly, the above example represents the worst possible situation, in a sense.

THEOREM 11.2. *Suppose $A \in \mathscr{P}(0)$ is rigid, $A = \nabla J'(A)$, and B is connected and directly indecomposable. If $J'(A)$ has n components, then* Aut (A^B) *is isomorphic to an extension of a subgroup of* $(\mathrm{Aut}\,(B))^n$ *by an elementary Abelian 2-group.*

PROOF. By Corollary 7.8, every automorphism φ of A^B satisfies (φ, i) for $i = 1, 2, 3$, whence by Theorem 3.2 there exists an automorphism α of A such that

$$\langle \alpha(a) \rangle = \varphi(\mathring{\varphi}^{-1} \circ \varphi(\langle a \rangle)) \quad \text{for } a \in A .$$

Since A is rigid, α is the identity automorphism of A, and $\mathring{\varphi}$, which agrees with α on $R(\varphi)$, is the identity map. Thus $\varphi^2(\langle a \rangle) = \langle a \rangle$ for all $a \in A$.

We have shown that the normal subgroup H of Aut (A^B) that is generated by the squares of all the automorphisms is contained in the group K consisting of all those automorphisms that leave every constant function fixed. We complete the proof by constructing an embedding of K into $(\mathrm{Aut}\,(B))^n$.

Consider any $\varphi \in K$. For $a \in J'(A)$ and $b \in B$, $j(a, b)$ belongs to $J'(A^B)$, and hence so does $\varphi j(a, b)$. Therefore $\varphi j(a, b) = j(a', b')$ for some $a' \in J'(A)$ and $b' \in B$. Now $j(a, c) \leq \langle a \rangle$, hence $j(a', b') \leq \varphi(\langle a \rangle) = \langle a \rangle$, so that $a' \leq a$. By symmetry, $a \leq a'$, and therefore $a' = a$. We therefore have $\varphi j(a, b) = j(a, b')$ for some $b' \in B$. For a fixed $a \in J'(A)$, the map $b \to b'$ is an automorphism λ_a of B, i.e.,

$$\varphi j(a, b) = j(a, \lambda_a(b)) \quad (a \in J'(A),\ b \in B) .$$

The automorphisms λ_a may vary with a, but we claim that they are fixed on each component of $J'(A)$. To prove this it suffices to show that if $a', a'' \in J'(A)$ and $a' < a''$, then $\lambda_{a'} = \lambda_{a''}$. Let $\lambda_{a'}(b) = b'$ and $\lambda_{a''}(b) = b''$. From the fact that $j(a', b) < j(a'', b)$, hence $j(a', b') < j(a'', b'')$, we see that $b'' \leq b'$. For some $x \in B$,

$$\varphi j(a', x) = j(a', b''),$$

and from the inequalities.

$$j(a', b') \leq j(a', b'') < j(a'', b'')$$

we infer that

$$j(a', b) \leq j(a', x) < j(a'', b).$$

and therefore $x = b$, $b'' = b'$.

Let A_i, $i = 0, 1, \ldots, n-1$, be the components of $J'(A)$, and write $\lambda_i = \lambda_a$ for $a \in A_i$. With each member of K we have associated a member $\lambda = \langle \lambda_i, i < n \rangle$ of $(\text{Aut}\,(B))^n$. This map is a homomorphism because if $\varphi' \to \lambda' = \langle \lambda_i', i < n \rangle$, then $\varphi \circ \varphi' \to \langle \lambda_i \circ \lambda_i', i < n \rangle$, and it is easily seen to be on embedding.

Example 11.3. If we take $A = \mathbf{1} \oplus (\mathbf{1} + \mathbf{2} + \ldots + \boldsymbol{n})$ and $B = \mathbf{2} \cdot \mathbf{2}$, then the subgroup of $(\text{Aut}\,(B)^n$ in Theorem 11.2 will actually be the whole group, but if we take

$$A = \mathbf{1} \oplus (\mathbf{1} + \mathbf{2} + \ldots + \boldsymbol{n}) \oplus \mathbf{1},$$

then the subgroup will be isomorphic to $\text{Aut}\,(B)$, although $J'(A)$ has n components in this case also.

If the exponent B in Theorem 11.2 is rigid, then $\text{Aut}\,(A^B)$ will be an elementary Abelian 2-group. In the next theorem we obtain the same conclusion under a different hypothesis.

Theorem 11.4. *If A is rigid and atomic, and $B \in \mathscr{P}_{\max}(1)$ is rigid and directly indecomposable, then* $\text{Aut}\,(A^B)$ *is an elentary Abelian 2-group.*

Proof. By Lemmas 4.1, 4.5, and 4.10, every automorphism φ of A^B satisfies (φ, i) for $i = 1, 2, 3$ and as in the proof of Theorem 11.2 we infer from this and the rigidity of A that φ^2 leaves every constant function fixed. To complete the proof, we therefore consider an automorphism φ of A^B with $\varphi(\langle a \rangle) = \langle a \rangle$ for all $a \in A$, and show that φ is the identity.

Since B is rigid, so is $\mathbf{2}^B$. Hence if $a \prec a'$ in A, then φ maps every member of the interval $[\langle a \rangle, \langle a' \rangle]$ onto itself. Now suppose $a \leq a'$ and $f = \langle a[x \leq b], a' \rangle$. Writing, as usual, $\bar{f} = \varphi(f)$, we certainly have $a \leq \bar{f}(b)$. If this inclusion is strict, then there exists $u \in A$ with $a \prec u \leq \bar{f}(b)$, but then φ maps the function g

$=\langle a[x \not\geq b], u\rangle$ onto itself, which is impossible because $g \not\leq f$ and $g \leq \bar{f}$. Thus $\bar{f}(b)=a$.

Any function $f \in A^B$ is the meet of the functions

$$f_b = \langle f(b)[x \leq b], f(1)\rangle .$$

Hence $\bar{f}(b) \leq \bar{f}_b(b) = f(b)$ for all $b \in B$. Thus $\bar{f} \leq f$ and, by symmetry, $f \leq \bar{f}$, so that $\bar{f}=f$.

THEOREM 11.5. *If A is a subdirectly irreducible lattice and B is a finite, connected poset, then*

$$\mathrm{Aut}\,(A^B) \cong \mathrm{Aut}\,(A) \times \mathrm{Aut}\,(B) .$$

PROOF. Given $\varphi \in \mathrm{Aut}\,(A^B)$, we show that there exists $\beta \in \mathrm{Aut}\,(B)$ such that, for all $f, g \in A^B$ and $b \in B$,

$$f(b) = g(b) \quad \text{iff } \bar{f}(\beta(b)) = \bar{g}(\beta(b)) .$$

Here, as usual, $\bar{f}=\varphi(f)$. Define

$$\theta_b = \{(f, g) : f(b)=g(b)\} ,$$

$$\bar{\theta}_b = \{(f, g) : \bar{f}(b)=\bar{g}(b)\} .$$

The meet of the congruence relations θ_b is the zero congruence, and similarly for $\bar{\theta}_b$. Hence, by the distributivity of the congruence lattice,

$$\theta_b = \bigwedge \{\theta_b \vee \bar{\theta}_c : c \in B\} \quad \text{for } b \in B ,$$

$$\bar{\theta}_c = \bigwedge \{\theta_b \vee \bar{\theta}_c : b \in B\} \quad \text{for } c \in B .$$

Since A^B/θ_b is isomorphic to A, and thus subdirectly irreducible, θ_b cannot be the meet of finitely many strictly larger congruence relations, and hence $\bar{\theta}_c \subseteqq \theta_b$ for some $c \in B$. Similarly, for each $c \in B$ there exists $b \in B$ with $\theta_b \subseteqq \bar{\theta}_c$. Note that no two of the relations θ_b are comparable, because if $a < a'$ in A, then the functions $\langle a[x<b], a'\rangle$ and $\langle a[x \leq b], a'\rangle$ differ only at b. We infer that there is a one-to-one map β of B onto B such that

$$\theta_b = \bar{\theta}_{\beta(b)} \quad \text{for all } b \in B .$$

To prove that β is an automorphism, fix $a, a' \in A$ with $a < a'$. Let $f=\langle a\rangle$ and $g=\langle a'\rangle$, and for any filter F in B let $\bar{h}_F=\langle \bar{f}[x \notin F], g\rangle$. Then $h_F=\langle a[x \notin \beta^{-1}(F)], a'\rangle$. Thus $\beta^{-1}(F)$ is a filter whenever F is, and β is therefore an automorphism.

We claim that φ maps constant functions into constant functions. To show this, let $f=\langle a\rangle$, and suppose $b_0 < b_1$ in B. Let $c_i=\beta(b_i)$ and $\bar{f}_i=\langle \bar{f}(c_i)\rangle$ for

$i=0,1$. Then $\bar{f}_i(c_i)=\bar{f}(c_i)$, hence $f_i(b_i)=f(b_i)=a$. Since $f_0(b_0)\leq f_0(b_1)\leq f_1(b_1)$, this gives

$$f_0(b_1) = a = f_1(b_1),$$

from which it follows that $\bar{f}_0(c_1)=\bar{f}_1(c_1)$, that is $\bar{f}(c_0)=\bar{f}(c_1)$. Thus $\bar{f}$ is constant, as was to be shown.

There exists a map α of A into itself such that $\varphi(\langle a\rangle)=\langle\alpha(a)\rangle$ for all $a\in A$, and it is easy to see that α is an automorphism. Finally, to show that $\bar{f}=\alpha\circ f\circ\beta^{-1}$, consider any $c\in B$ and let $b=\beta^{-1}(c)$ and $g=\langle f(b)\rangle$. Then $f(b)=g(b)$, hence

$$\bar{f}(c) = \bar{g}(c) = \alpha(f(b)) = \alpha(f(\beta^{-1}(c))).$$

12. Open problems.

Our investigations suggest many questions concerning possible extensions and unifications of the results, and simplifications of some of the arguments. We list just a few such problems.

PROBLEM 12.1. Find counter examples (or prove that none exist) to the refinement of $A^C\cong B^D$ under any of the following conditions:

(i) A, B, C, and D are finite and connected.
(ii) C, D, and A^C are finite and connected.
(iii) A^C is a directly indecomposable lattice.
(iv) C and D are bounded and satisfy both chain conditions.
(v) A and B are atomic, and C and D are finitely factorable and belong both to (the same) one of the classes $\mathscr{L}$, $\mathscr{P}(0,1)$, $\mathscr{S}_\vee(0)$, $\mathscr{S}_\wedge(1)$, and $\mathscr{P}_{\max}(1)$.

PROBLEM 12.2. There now exist three main refinement theorems dealing with direct products and powers of posets. They deal with the relations $\sqcap(A_i, i\in I)\cong\sqcap(B_j, j\in J)$ (the products are connected); $A^B\cong\sqcap(C_i, i\in I)$ (B and A^B are connected); $A^C\cong B^D$ (under a variety of relatively complex conditions). Does there exist a reasonable refinement theorem regarding some kind of special subdirect products that includes all three of these theorems as corollaries?

PROBLEM 12.3. Does the relation

$$\mathrm{Aut}\,(A^B) \cong \mathrm{Aut}\,(A)\times\mathrm{Aut}\,(B)$$

necessarily hold when A is an exponentially and directly indecomposable lattice and B is connected (and possibly also finite)?

PROBLEM 12.4. We have associated with each automorphism φ of A^B (where A and B satisfy certain conditions) a unique automorphism α of A. In general, this correspondence is not a homomorphism. Does it have any interesting properties? (This question seems particularly attractive, when A is a lattice.)

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