MATH. SCAND. 126 (2020), 339–366

ON THE CONVERGENCE OF ITERATES OF CONVOLUTION OPERATORS IN BANACH SPACES

HEYBETKULU MUSTAFAYEV

Abstract

Let G be a locally compact abelian group and let $M(G)$ be the measure algebra of G. A measure $\mu \in M(G)$ is said to be power bounded if $\sup_{n\geq 0}\|\mu^n\|_1 < \infty$. Let $\mathbf{T} = \{T_g : g \in G\}$ be a bounded and continuous representation of G on a Banach space X. For any $\mu \in M(G)$, there is a bounded linear operator on X associated with μ, denoted by $\mathbf{T}_\mu$, which integrates T_g with respect to μ. In this paper, we study norm and almost everywhere behavior of the sequences $\{\mathbf{T}_\mu^n x\}$ ($x \in X$) in the case when μ is power bounded. Some related problems are also discussed.

1. Introduction

For a complex Banach space X, we denote by $B(X)$ the algebra of all bounded linear operators on X. Let G be a locally compact group and let $\mathbf{T} = \{T_g : g \in G\}$ be a bounded and continuous representation of G on X. For an arbitrary finite regular Borel measure μ on G, we can define an operator $\mathbf{T}_\mu$ in $B(X)$ associated with μ, which integrates T_g with respect to μ. In case of probability measure μ, the papers [3], [4], [5], [6], [7], [11] studied the norm and almost everywhere behavior of iterates of $\mathbf{T}_\mu$. Recall that μ is said to be *adapted* if $\operatorname{supp}\mu$ generates a dense subgroup of G and *strictly aperiodic* if $\operatorname{supp}\mu$ is not contained in a proper closed left coset of G. Assume that X is uniformly convex and μ is an adapted, strictly aperiodic probability measure such that for some $n \in \mathbb{N}$, μ^n is not singular with respect to the Haar measure on G. In [7], it was proved that under the above conditions the sequence $\{\mathbf{T}_\mu^n x\}$ converges strongly for every $x \in X$. In [11], norm and almost everywhere convergence of the iterates of $\mathbf{T}_\mu$ in $L^p(\Omega, \Sigma, m)$ spaces was studied, where $\mathbf{T}$ is a continuous action of G in the positive measure space (Ω, Σ, m). For related results see also [9], [10], [16], [17], [21], [22].

In this paper, we study norm and almost everywhere convergence of the sequences $\{\mathbf{T}_\mu^n x\}$ in Banach spaces. We treat the case that G is a locally compact

Received 22 November 2018, in final form 23 June 2019. Accepted 30 July 2019.
DOI: https://doi.org/10.7146/math.scand.a-119601

abelian group and μ is an arbitrary power bounded measure on G. For locally compact abelian groups the most comprehensive work on power bounded measures is due to Schreiber [20].

Throughout this paper, G will denote a locally compact abelian group with the Haar measure and with the dual group Γ. As usual, $L^1(G)$ and $M(G)$ will denote the group algebra and the convolution measure algebra of G, respectively. As is well known, equipped with the involution $\widetilde{}$ given by $\widetilde{\mu}(B) = \overline{\mu(-B)}$, the algebra $M(G)$ becomes a Banach $*$-algebra. A measure $\mu \in M(G)$ is said to be *symmetric* if $\mu = \widetilde{\mu}$. For $n \in \mathbb{N} \cup \{0\}$, by μ^n we will denote n-th convolution power of $\mu \in M(G)$, where $\mu^0 := \delta_0$ is the Dirac measure concentrated at $\{0\}$. By $\widehat{f}$ and $\widehat{\mu}$ we denote the Fourier and the Fourier-Stieltjes transforms of $f \in L^1(G)$ and $\mu \in M(G)$, respectively. $C_0(G)$ will denote the space of all complex valued continuous functions on G vanishing at infinity.

Recall that an element a of a unital Banach algebra is said to be *power bounded* if $\sup_{n\geq 0} \|a^n\| < \infty$. For $\mu \in M(G)$, we put

$$C_\mu = \sup_{n\geq 0} \|\mu^n\|_1,$$

where $\|\cdot\|_1$ is the total variation norm. If S is any set, the characteristic function of S will be denoted by $\mathbf{1}_S$. As usual, $\sigma(T)$ and $R_\lambda(T)$ ($\lambda \notin \sigma(T)$) will denote the spectrum and the resolvent of $T \in B(X)$.

2. Hilbert space operators

In this section, we study strong and almost everywhere convergence of iterates of convolution operators in Hilbert spaces.

Notice that for any $\mu \in M(G)$,

$$F_\mu := \overline{(\delta_0 - \mu) * L^1(G)}$$

is a closed ideal of $L^1(G)$ associated with μ and $\mathrm{hull}(F_\mu) = \mathscr{F}_\mu$, where

$$\mathscr{F}_\mu = \{\gamma \in \Gamma : \widehat{\mu}(\gamma) = 1\}.$$

Assume that $\mu \in M(G)$ is power bounded. Then clearly, $|\widehat{\mu}(\gamma)| \leq 1$ for all $\gamma \in \Gamma$. Moreover, it is easy to check that

$$F_\mu = \left\{ f \in L^1(G) : \lim_{n\to\infty} \left\| \frac{1}{n} \sum_{i=0}^{n-1} \mu^i * f \right\|_1 = 0 \right\}. \tag{2.1}$$

Notice also that

$$E_\mu := \left\{ f \in L^1(G) : \underset{n}{\mathrm{l.i.m.}} \|\mu^n * f\|_1 = 0 \right\}$$

is another closed ideal of $L^1(G)$ associated with μ, where l.i.m. is a fixed Banach limit. We claim that $\text{l.i.m.}_n \|\mu^n * f\|_1 = 0$ implies $\lim_{n\to\infty} \|\mu^n * f\|_1 = 0$. Indeed, if $\text{l.i.m.}_n \|\mu^n * f\|_1 = 0$, then as $\underline{\lim}_{n\to\infty} \|\mu^n * f\|_1 = 0$, we have $\|\mu^{n_k} * f\|_1 \to 0$ $(k \to \infty)$, for some subsequence $\{n_k\}$. It follows from the relations

$$\|\mu^n * f\|_1 \le \|\mu^{n-n_k}\|_1 \|\mu^{n_k} * f\|_1 \le C_\mu \|\mu^{n_k} * f\|_1$$

that $\|\mu^n * f\|_1 \to 0$. This shows that E_μ does not depend on the choice of the Banach limit and therefore,

$$E_\mu = \left\{ f \in L^1(G) : \lim_{n\to\infty} \|\mu^n * f\|_1 = 0 \right\}.$$

By (2.1) we have $E_\mu \subseteq F_\mu$. Moreover, $\text{hull}(E_\mu) = \mathscr{E}_\mu$ ([12, Theorem 2.6] and [16, Proposition 2.1]), where

$$\mathscr{E}_\mu = \{\gamma \in \Gamma : |\widehat{\mu}(\gamma)| = 1\}.$$

As usual, to any closed subset S of Γ, the following two closed ideals of $L^1(G)$ are associated:

$$I_S := \{f \in L^1(G) : \widehat{f}(S) = \{0\}\}$$

and $J_S := \overline{J_S^0}$, where

$$J_S^0 = \{f \in L^1(G) : \text{supp}\, \widehat{f} \text{ is compact and } \text{supp}\, \widehat{f} \cap S = \emptyset\}.$$

The ideals J_S and I_S are respectively, the smallest and the largest closed ideals in $L^1(G)$ with hull S. When these two ideals coincide, the set S is said to be a *set of synthesis* (for instance, see [14, §8.3]).

We know that if $\mu \in M(G)$ is power bounded, then $\mathscr{E}_\mu$ is a set of synthesis (for instance, see [10] and references therein). Further if $\nu := \frac{\delta_0+\mu}{2}$, then ν is power bounded and as $\mathscr{F}_\mu = \mathscr{E}_\nu$, the set $\mathscr{F}_\mu$ is also a set of synthesis. It follows that $\mathscr{F}_\mu = \mathscr{E}_\mu$ if and only if $F_\mu = E_\mu$ if and only if

$$\lim_{n\to\infty} \|\mu^n * f - \mu^{n+1} * f\|_1 = 0, \quad \forall f \in L^1(G).$$

Moreover, we can write

$$\mathscr{F}_\mu = \mathscr{E}_\mu \iff \widehat{\mu}(\mathscr{E}_\mu) = \{1\} \iff \lim_{n\to\infty} \left|\widehat{\mu}(\gamma)^n - \widehat{\mu}(\gamma)^{n+1}\right| = 0, \ \forall \gamma \in \Gamma.$$

It can be seen that if $\mu \in M(G)$ is a probability measure, then μ is adapted (resp. aperiodic) if and only if $\mathscr{F}_\mu = \{0\}$ (resp. $\mathscr{E}_\mu = \{0\}$). In the sequel, the sets $\mathscr{F}_\mu$ and $\mathscr{E}_\mu$ turn out to be very important (see [9], [10], [12], [16], [17]).

Let $\mu \in M(G)$ be power bounded. The classical Foguel's theorem [9] asserts that $\lim_{n\to\infty} \|\mu^n * f\|_1 = 0$ for all $f \in L^1(G)$ with $\widehat{f}(0) = 0$ if and only if $\mathscr{E}_\mu \subseteq \{0\}$. Granirer [10, Theorem 2] proved that $\lim_{n\to\infty} \|\mu^n * f\|_1 = 0$ if and only if $\widehat{f}$ vanishes on $\mathscr{E}_\mu$. In [16, Corollary 2.5], it was proved that if $\|\mu\|_1 \leq 1$ and if $\mathscr{E}_\mu$ is a scattered compact (a locally compact Hausdorff space is said to be *scattered* if it contains no non-empty perfect subset), then

$$\lim_{n\to\infty} \|\mu^n * f\|_1 = \operatorname{dist}(f, I_{\mathscr{E}_\mu}), \quad \forall f \in L^1(G).$$

Let $\mathbf{U} = \{U_g : g \in G\}$ be a (strongly) continuous unitary representation of G on a complex Hilbert space H. For any $\mu \in M(G)$, we can define a bounded linear operator $\mathbf{U}_\mu$ on H by

$$\mathbf{U}_\mu x = \int_G U_g^{-1} x \, d\mu(g), \quad x \in H. \tag{2.2}$$

The map $\mu \mapsto \mathbf{U}_\mu$ is a contractive algebra $*$-homomorphism. Moreover, as $\mathbf{U}_\mu^* = \mathbf{U}_{\widetilde{\mu}}$, $\mathbf{U}_\mu$ is a normal operator and $\mathbf{U}_\mu^n = \mathbf{U}_{\mu^n}$ for all $n \in \mathbb{N}$. It follows that if μ is power bounded, so is $\mathbf{U}_\mu$ and therefore $\mathbf{U}_\mu$ is a contraction (a normal operator on a Hilbert space is power bounded if and only if it is a contraction).

By the general Stone's theorem [1], there exists a spectral measure P on Γ such that

$$U_g = \int_\Gamma \gamma(g) \, dP(\gamma), \quad \forall g \in G. \tag{2.3}$$

The spectral measure P obtained in Stone's theorem will be called the *spectral measure* for $\mathbf{U}$. Taking into account (2.3) in (2.2), we have

$$\mathbf{U}_\mu x = \int_\Gamma \widehat{\mu}(\gamma) \, dP(\gamma) x, \quad x \in H. \tag{2.4}$$

Let N be a normal contraction operator on H with spectral measure Q. It is easy to check that

$$\frac{1}{n} \sum_{i=0}^{n-1} N^i x \to Q(\{1\})x \quad \text{in norm, for every } x \in H.$$

Now, let $\mu \in M(G)$ and let Q be the spectral measure for $\mathbf{U}_\mu$. Then,

$$Q(B) = P(\widehat{\mu}^{-1}(B)),$$

for each Borel subset B of complex plane, where P is the spectral measure for $\mathbf{U}$. It follows that if μ is power bounded, then as

$$Q(\{1\}) = P(\widehat{\mu}^{-1}(1)) = P(\mathscr{F}_\mu),$$

we have

$$\frac{1}{n}\sum_{i=0}^{n-1}\mathbf{U}_\mu^i x \to P(\mathscr{F}_\mu)x \quad \text{in norm, for every } x \in H. \tag{2.5}$$

Notice also that if $\mathscr{F}_\mu$ is a clopen subset of Γ, then there exists an idempotent measure $\nu \in M(G)$ such that

$$\frac{1}{n}\sum_{i=0}^{n-1}\mathbf{U}_\mu^i x \to \mathbf{U}_\nu x \quad \text{in norm, for every } x \in H.$$

Indeed, since $\mathbf{1}_{\mathscr{F}_\mu}$ is a continuous function on Γ and

$$\frac{1}{n}\sum_{i=0}^{n-1}\widehat{\mu}(\gamma)^i \to \mathbf{1}_{\mathscr{F}_\mu}(\gamma) \quad (\forall\gamma \in \Gamma),$$

by [19, Theorem 1.9.2], $\mathbf{1}_{\mathscr{F}_\mu} = \widehat{\nu}$ for some $\nu \in M(G)$. Clearly, ν is an idempotent measure and by (2.4),

$$\mathbf{U}_\nu x = \int_\Gamma \mathbf{1}_{\mathscr{F}_\mu}(\gamma)\, dP(\gamma)x = P(\mathscr{F}_\mu)x.$$

If $\mu \in M(G)$ is power bounded, then by the mean ergodic theorem,

$$H = \ker(I - \mathbf{U}_\mu) \oplus \overline{(I - \mathbf{U}_\mu)H}, \tag{2.6}$$

where $P(\mathscr{F}_\mu)$ is the orthogonal projection (often called *mean ergodic projection*) onto $\ker(I - \mathbf{U}_\mu)$.

The following theorem improves [17, Proposition 3.1].

THEOREM 2.1. *Let $\mu \in M(G)$ be power bounded and assume that $\mathscr{F}_\mu = \mathscr{E}_\mu$. Then, there exists a (not necessarily closed) linear subspace E of H with the properties:*

(i) $H = \ker(I - \mathbf{U}_\mu) \oplus \overline{E}$;

(ii) $\sum_{n=0}^{\infty} \|\mathbf{U}_\mu^n x\| < \infty$, *for all $x \in E$;*

(iii) *the sequence $\{\mathbf{U}_\mu^n x\}$ converges for every $x \in H$, that is, $\mathbf{U}_\mu^n x \to P(\mathscr{F}_\mu)x$ strongly.*

Given $x \in H$, let λ_x be the measure on Γ defined by

$$\lambda_x(B) = \langle P(B)x, x\rangle = \|P(B)x\|^2, \tag{2.7}$$

where P is the spectral measure for $\mathbf{U}$.

LEMMA 2.2. *Under the above notations, we have:*

(a) $\operatorname{supp}\lambda_{x+y} \subseteq \operatorname{supp}\lambda_x \cup \operatorname{supp}\lambda_y$, $\forall x, y \in H$;

(b) $\operatorname{supp}\lambda_{\mathbf{U}_f x} \subseteq \operatorname{supp}\widehat{f} \cap \operatorname{supp}\lambda_x$, $\forall f \in L^1(G)$, $\forall x \in H$;

(c) *if S is a closed subset of Γ, then $\{x \in H : \operatorname{supp}\lambda_x \subseteq S\}$ is a closed subspace of H.*

PROOF. (a) Let $x, y \in H$ and assume that $\gamma \notin \operatorname{supp}\lambda_x \cup \operatorname{supp}\lambda_y$. Then, there is a neighborhood V of γ such that

$$\|P(V)x\|^2 = \lambda_x(V) = 0 \quad \text{and} \quad \|P(V)y\|^2 = \lambda_y(V) = 0.$$

Consequently, we have

$$\|P(V)(x+y)\| \leq \|P(V)x\| + \|P(V)y\| = 0$$

and so $\lambda_{x+y}(V) = 0$. This shows that $\gamma \notin \operatorname{supp}\lambda_{x+y}$.

(b) Let $f \in L^1(G)$, $x \in H$, and assume that $\gamma \notin \operatorname{supp}\widehat{f} \cap \operatorname{supp}\lambda_x$. Then, there is a neighborhood V of γ such that either $\widehat{f}(V) = \{0\}$ or $\lambda_x(V) = 0$. It follows from the identity

$$\lambda_{\mathbf{U}_f x}(V) = \|P(V)\mathbf{U}_f x\|^2 = \int_V |\widehat{f}(\gamma)|^2 \, d\lambda_x(\gamma)$$

that in both cases $\lambda_{\mathbf{U}_f x}(V) = 0$. Hence, $\gamma \notin \operatorname{supp}\lambda_{\mathbf{U}_f x}$.

(c) By (a), the set $\{x \in H : \operatorname{supp}\lambda_x \subseteq S\}$ is linear. Let $\{x_n\}$ be a sequence in H such that $\operatorname{supp}\lambda_{x_n} \subseteq S$ for all n and $x_n \to x$. We must show that $\operatorname{supp}\lambda_x \subseteq S$. Assume that the Fourier transform of $f \in L^1(G)$ vanishes on S. It suffices to show that $\widehat{f}$ vanishes on $\operatorname{supp}\lambda_x$. Since $\operatorname{supp}\lambda_{x_n} \subseteq S$, the function $\widehat{f}$ vanishes on $\operatorname{supp}\lambda_{x_n}$ for all n. It follows from the identity

$$\|\mathbf{U}_f x\|^2 = \int_\Gamma |\widehat{f}(\gamma)|^2 \, d\lambda_x(\gamma), \quad \forall f \in L^1(G), \ \forall x \in H, \tag{2.8}$$

that $\mathbf{U}_f x_n = 0$ for all n. As $x_n \to x$, we have $\mathbf{U}_f x = 0$. By (2.8), $\widehat{f}$ vanishes on $\operatorname{supp}\lambda_x$.

Now, we can prove Theorem 2.1.

PROOF OF THEOREM 2.1. By (2.4),

$$\mathbf{U}_\mu x = \int_\Gamma \widehat{\mu}(\gamma) \, dP(\gamma)x \quad (x \in H),$$

where P is the spectral measure for $\mathbf{U}$. We put $S := \mathscr{F}_\mu = \mathscr{E}_\mu$. Given $x \in H$, let λ_x be the measure on Γ defined by (2.7) and

$$E := \{x \in H : \operatorname{supp} \lambda_x \text{ is compact and } \operatorname{supp} \lambda_x \cap S = \emptyset\}.$$

By Lemma 2.2, E is linear. If $x \in E$, then as $\operatorname{supp} \lambda_x \cap S = \emptyset$, we have

$$\sup_{\gamma \in \operatorname{supp} \lambda_x} |\widehat{\mu}(\gamma)| := \delta < 1.$$

It follows from the identity

$$\|\mathbf{U}_\mu^n x\|^2 = \int_{\operatorname{supp} \lambda_x} |\widehat{\mu}(\gamma)|^{2n} \, d\lambda_x(\gamma) \tag{2.9}$$

that

$$\|\mathbf{U}_\mu^n x\|^2 \leq \delta^{2n} \|x\|^2, \quad \text{for all } n \in \mathbb{N},$$

and so

$$\sum_{n=0}^{\infty} \|\mathbf{U}_\mu^n x\| < \infty.$$

It remains to show that $\mathbf{U}_\mu x = x$ for all $x \in E^\perp$. Firstly, let us show that $\operatorname{supp} \lambda_x \subseteq S$ for all $x \in E^\perp$. To see this, let $x \in E^\perp$ and assume that the Fourier transform of $f \in L^1(G)$ vanishes on S. We must show that $\widehat{f}$ vanishes on $\operatorname{supp} \lambda_x$. Since S is a set of synthesis, there exists a sequence $\{f_n\}$ in $L^1(G)$ such that $\operatorname{supp} \widehat{f_n}$ is compact, $\widehat{f_n}$ vanishes in a neighborhood O_n of S, and $\|f_n - f\|_1 \to 0$. Let an arbitrary $y \in H$ be given. By Lemma 2.2,

$$\operatorname{supp} \lambda_{\mathbf{U}_{f_n} y} \subseteq \operatorname{supp} \widehat{f_n} \cap \operatorname{supp} \lambda_y$$

and therefore $\operatorname{supp} \lambda_{U_{f_n} y}$ is compact. On the other hand, as $\operatorname{supp} \widehat{f_n} \cap S = \emptyset$, we have

$$\operatorname{supp} \lambda_{\mathbf{U}_{f_n} y} \cap S = \emptyset.$$

Hence, $\mathbf{U}_{f_n} y \in E$, so that $\langle \mathbf{U}_{f_n} y, x\rangle = 0$ or $\langle y, \mathbf{U}_{f_n}^* x\rangle = 0$ for all n and for all $y \in H$. Consequently, $\mathbf{U}_{f_n}^* x = 0$. Since $\mathbf{U}_{f_n}$ is a normal operator, $\mathbf{U}_{f_n} x = 0$. It follows from (2.8) that $\widehat{f_n}$ vanishes on $\operatorname{supp} \lambda_x$ for all n. Since $\widehat{f_n} \to \widehat{f}$ uniformly on Γ, $\widehat{f}$ vanishes on $\operatorname{supp} \lambda_x$. Now since $\operatorname{supp} \lambda_x \subseteq S$, we have

$$\|\mathbf{U}_\mu x - x\|^2 = \int_S |\widehat{\mu}(\gamma) - 1|^2 \, d\lambda_x(\gamma) = 0$$

and so $\mathbf{U}_\mu x = x$.

(iii) is an immediate consequence of (i), (ii), and (2.5).

EXAMPLE 2.3. (a) There exists a power bounded measure $\mu \in M(G)$ with norm > 1. To see this, let λ, ν be two probability measures on G such that $\lambda * \nu = 0$ and $\mu := \lambda + \nu$. Then, $\|\mu\|_1 = 2$ and as $\mu^n = \lambda^n + \nu^n$, we have $\|\mu^n\|_1 \leq 2$ for all $n \in \mathbb{N}$.

(b) Let δ_n be the Dirac measure concentrated at $n \in \mathbb{Z}$ and let

$$\mu = \frac{1}{2i}\delta_{-1} - \frac{1}{2i}\delta_1.$$

Then, $\|\mu\|_1 = 1$ and as $\widehat{\mu}(\lambda) = \sin\lambda$ we have $\mathscr{F}_\mu = \left\{\frac{\pi}{2} + 2k\pi : k \in \mathbb{Z}\right\}$ and $\mathscr{E}_\mu = \left\{\frac{\pi}{2} + k\pi : k \in \mathbb{Z}\right\}$.

(c) If $\nu \in M(G)$ is power bounded and $\mu := \frac{1}{n}\sum_{i=0}^{n-1} \nu^i$ $(n > 1)$, then μ is power bounded and $\mathscr{F}_\mu = \mathscr{E}_\mu$ (see the proof of Corollary 3.2 in [17]).

As a consequence of Theorem 2.1 and Example 2.3(c), we have the following:

COROLLARY 2.4. *Let $\nu \in M(G)$ be power bounded and $\mu := \frac{1}{k}\sum_{i=0}^{k-1} \nu^i$, where $k > 1$ is a fixed integer. Then, the sequence $\{\mathbf{U}_\mu^n x\}$ converges strongly for all $x \in H$.*

We will always denote by (Ω, Σ, m) a σ-finite positive measure space (the Haar measure on G is σ-finite if and only if G is σ-compact). In the case when Ω is a locally compact Hausdorff space, m will denote a regular Borel measure on Ω. By $L(\Omega)$ we will denote the space of all measurable simple functions on Ω that vanish outside of a set of finite measure.

Recall that an *action* $\mathbf{\Theta}$ of G in (Ω, Σ, m) is a family $\mathbf{\Theta} = \{\theta_g : g \in G\}$ of invertible measure preserving transformations of (Ω, Σ, m) satisfying:

(1) $\theta_0 = \mathrm{id}$;

(2) $\theta_{g+s} = \theta_g\theta_s$, for all $g, s \in G$;

(3) $\mathbf{\Theta}$ is jointly measurable in the sense that the mapping $G \times \Omega \to \Omega$ defined by $(g, \omega) \to \theta_g\omega$ is measurable with respect to the product σ-algebra $\Sigma_G \times \Sigma$ in $G \times \Omega$.

If

$$\lim_{g\to 0} \|f \circ \theta_g - f\|_p = 0 \quad \text{for any } 1 < p < \infty \text{ and } f \in L^p(\Omega),$$

then the action $\mathbf{\Theta}$ is called *continuous*. For example, if G is σ-compact and $L^p(\Omega)$ $(1 < p < \infty)$ is separable (this is the case if Σ is countably generated), then the assumption of joint measurability of $\mathbf{\Theta}$ implies that $\mathbf{\Theta}$ is continuous (see [11] and references therein). We will assume the continuity of $\mathbf{\Theta}$ throughout in what follows.

A continuous action $\mathbf{\Theta}$ induces a continuous representation $\mathbf{T} = \{T_g : g \in G\}$ of G on $L^p(\Omega)$ $(1 < p < \infty)$ by invertible isometries defined by

$$(T_g f)(\omega) = f(\theta_g \omega) \quad (\omega \in \Omega).$$

Consequently, for any $\mu \in M(G)$, we can define a bounded linear operator $\mathbf{T}_\mu$ on $L^p(\Omega)$ by

$$(\mathbf{T}_\mu f)(\omega) = \int_G f(\theta_g^{-1}\omega)\, d\mu(g). \tag{2.10}$$

The map $\mu \mapsto \mathbf{T}_\mu$ is an algebra homomorphism and

$$\|\mathbf{T}_\mu f\|_p \le \|\mu\|_1 \|f\|_p, \quad \forall f \in L^p(\Omega).$$

It follows that if μ is power bounded, then so is $\mathbf{T}_\mu$;

$$\sup_{n\ge 0} \|\mathbf{T}_\mu^n\|_p \le C_\mu.$$

Definition 2.5. Let Ω be a locally compact Hausdorff space. We say that an action $\mathbf{\Theta}$ of G in (Ω, Σ, m) *has the separation property* if for any two compact subsets K_1, K_2 of Ω, there exists a compact subset K of G such that $\theta_g K_1 \cap K_2 = \emptyset$ for all $g \in G \setminus K$.

Notice that the regular action in G has the separation property. Indeed, if K_1, K_2 are two compact subsets of G, then $(g + K_1) \cap K_2 = \emptyset$ for all $g \in G \setminus (-K_1 + K_2)$.

Proposition 2.6. *Let Ω be a locally compact Hausdorff space and let $\mathbf{\Theta}$ be a continuous action of G in (Ω, Σ, m) with the separation property. Then, the function*

$$k(g) := \int_\Omega f(\theta_g^{-1}\omega) h(\omega)\, dm(\omega)$$

is in $C_0(G)$ for every $f \in L^p(\Omega)$ $(1 < p < \infty)$ and $h \in L^q(\Omega)$ $(1/p + 1/q = 1)$.

Proof. Clearly, k is a bounded continuous function. Let A, B be two sets in Σ with finite measure. If $f = \mathbf{1}_A$ and $h = \mathbf{1}_B$, then

$$\int_\Omega f(\theta_g^{-1}\omega) h(\omega)\, dm(\omega) = m(\theta_g A \cap B).$$

Firstly, let us show that the function $g \to m(\theta_g A \cap B)$ is in $C_0(G)$. Let $\varepsilon > 0$ be given. Since m is regular, there is a compact $K_1 \subset A$ such that $m(A) - m(K_1) < \varepsilon/2$ which implies

$$m(\theta_g A) - m(\theta_g K_1) < \varepsilon/2, \quad \forall g \in G.$$

Similarly, there is a compact $K_2 \subset B$ such that $m(B) - m(K_2) < \varepsilon/2$. Since

$$(\theta_g A \cap B) \setminus (\theta_g K_1 \cap K_2) \subseteq (\theta_g A \setminus \theta_g K_1) \cup (B \setminus K_2),$$

we have

$$m(\theta_g A \cap B) - m(\theta_g K_1 \cap K_2) \leq m(\theta_g A) - m(\theta_g K_1) + m(B) - m(K_2) < \varepsilon.$$

Since $\boldsymbol{\Theta}$ has the separation property, there exists a compact subset K of G such that $\theta_g K_1 \cap K_2 = \emptyset$ for all $g \in G \setminus K$. So we have

$$m(\theta_g A \cap B) < \varepsilon, \quad \forall g \in G \setminus K.$$

This shows that the function $g \to m(\theta_g A \cap B)$ is in $C_0(G)$. Consequently, if f and h is in $L(\Omega)$, then the corresponding function k is in $C_0(G)$. Now, let an arbitrary $f \in L^p(\Omega)$ and $h \in L^q(\Omega)$ be given. Since m is σ-finite, there exist sequences $\{f_n\}$ and $\{h_n\}$ in $L(\Omega)$ such that $\|f_n - f\|_p \to 0$ and $\|h_n - h\|_q \to 0$. Since

$$\int_\Omega f_n(\theta_g^{-1}\omega) h_n(\omega)\, dm(\omega) \to k(g) \quad \text{uniformly in } G,$$

we have that $k \in C_0(G)$.

The following result was proved in [17, Theorem 4.1].

THEOREM 2.7. *If $\mu \in M(G)$ is power bounded, then the limit*

$$\nu := \text{w}^*\text{-}\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu^i$$

exists in the weak *-topology of $M(G)$.

The measure ν obtained in this theorem will be called *limit measure associated with μ.*

PROPOSITION 2.8. *Let Ω be a locally compact Hausdorff space and let $\boldsymbol{\Theta}$ be a continuous action of G in (Ω, Σ, m) with the separation property. If $\mu \in M(G)$ is power bounded and $1 < p < \infty$, then*

$$\frac{1}{n} \sum_{i=0}^{n-1} \mathbf{T}_\mu^i f \to \mathbf{T}_\nu f \quad \text{in } L^p\text{-norm, for every } f \in L^p(\Omega),$$

where ν is the limit measure associated with μ.

PROOF. If $f \in L^p(\Omega)$, then by the mean ergodic theorem,

$$\frac{1}{n}\sum_{i=0}^{n-1} \mathbf{T}_\mu^i f \to k \quad \text{in } L^p\text{-norm, for some } k \in L^p(\Omega).$$

On the other hand, by Theorem 2.7,

$$\nu = \text{w}^*\text{-}\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \mu^i.$$

If $h \in L^q(\Omega)$ $(1/p + 1/q = 1)$, then by Proposition 2.6, the function

$$g \to \int_\Omega f(\theta_g^{-1}\omega)h(\omega)\, dm(\omega)$$

is in $C_0(G)$. Consequently, we can write

$$\begin{aligned}\langle k, h\rangle &= \lim_{n\to\infty}\left\langle \frac{1}{n}\sum_{i=0}^{n-1} \mathbf{T}_\mu^i f, h\right\rangle \\ &= \lim_{n\to\infty}\left\langle \frac{1}{n}\sum_{i=0}^{n-1} \mu^i, \int_\Omega f(\theta_g^{-1}\omega)h(\omega)\, dm(\omega)\right\rangle \\ &= \left\langle \nu, \int_\Omega f(\theta_g^{-1}\omega)h(\omega)\, dm(\omega)\right\rangle \\ &= \langle \mathbf{T}_\nu f, h\rangle.\end{aligned}$$

So we have $k = \mathbf{T}_\nu f$.

Next, we have the following:

THEOREM 2.9. *Let $\mathbf{\Theta}$ be a continuous action of G in (Ω, Σ, m) and let $\mu \in M(G)$ be power bounded. If $\mathscr{E}_\mu = \mathscr{F}_\mu$, then the sequence $\{\mathbf{T}_\mu^n f\}$ converges in L^p-norm for every $f \in L^p(\Omega)$ $(1 < p < \infty)$. Moreover, if $\mathbf{\Theta}$ has the separation property, then*

$$\mathbf{T}_\mu^n f \to \mathbf{T}_\nu f \quad \textit{in } L^p\textit{-norm,}$$

where ν is the limit measure associated with μ.

PROOF. If $\mathscr{E}_\mu = \mathscr{F}_\mu$, then by Theorem 2.1 the sequence $\{\mathbf{T}_\mu^n f\}$ converges in L^2-norm for every $f \in L^2(\Omega)$. Hence, we may assume that $p \neq 2$. Let $f \in L(\Omega)$ be given. If $\nu \in M(G)$, then $\mathbf{T}_\nu f \in L^p(\Omega)$ for all $1 \leq p \leq \infty$. By the Riesz-Thorin convexity theorem [8, Chapter VI, §10], $\alpha \to \log \|\mathbf{T}_\nu f\|_{\frac{1}{\alpha}}$ is a convex function on [0, 1]. Choose q such that $q > p$ if $p > 2$ and $1 < q < p$ if $1 < p < 2$. If $\lambda := \frac{2q-2p}{pq-2p}$, then $0 < \lambda < 1$ and $\frac{1}{p} = \frac{1-\lambda}{q} + \frac{\lambda}{2}$. Consequently, we have

$$\|\mathbf{T}_\nu f\|_p \leq \|\mathbf{T}_\nu f\|_q^{1-\lambda} \|\mathbf{T}_\nu f\|_2^{\lambda}, \quad \forall \nu \in M(G).$$

Replacing ν by $\mu^n - \mu^{n+1}$ ($n \in \mathbb{N}$) and taking into account that $\sup_{n\geq 0} \|\mathbf{T}_\mu^n\|_p \leq C_\mu$, we can write

$$\begin{aligned}\|\mathbf{T}_\mu^n f - \mathbf{T}_\mu^{n+1} f\|_p &\leq \|\mathbf{T}_\mu^n f - \mathbf{T}_\mu^{n+1} f\|_q^{1-\lambda} \|\mathbf{T}_\mu^n f - \mathbf{T}_\mu^{n+1} f\|_2^{\lambda} \\ &\leq (2C_\mu \|f\|_q)^{1-\lambda} \|\mathbf{T}_\mu^n f - \mathbf{T}_\mu^{n+1} f\|_2^{\lambda}.\end{aligned}$$

Since $\|\mathbf{T}_\mu^n f - \mathbf{T}_\mu^{n+1} f\|_2 \to 0$, it follows that

$$\lim_{n\to\infty} \|\mathbf{T}_\mu^n f - \mathbf{T}_\mu^{n+1} f\|_p = 0, \quad \forall f \in L(\Omega).$$

Also since $L(\Omega)$ is dense in $L^p(\Omega)$, we get

$$\lim_{n\to\infty} \|\mathbf{T}_\mu^n (I - \mathbf{T}_\mu) f\|_p = 0, \ \forall f \in L^p(\Omega)$$

or

$$\lim_{n\to\infty} \|\mathbf{T}_\mu^n f\|_p = 0, \quad \forall f \in \overline{(I - \mathbf{T}_\mu) L^p(\Omega)}.$$

On the other hand, by the mean ergodic theorem,

$$L^p = \ker(I - \mathbf{T}_\mu) \oplus \overline{(I - \mathbf{T}_\mu) L^p}.$$

It follows that the sequence $\{\mathbf{T}_\mu^n f\}$ converges in L^p-norm for every $f \in L^p(\Omega)$.

If $\mathbf{\Theta}$ has the separation property, then by Proposition 2.8,

$$\lim_{n\to\infty} \mathbf{T}_\mu^n f = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{T}_\mu^i f = \mathbf{T}_\nu f.$$

In $L^2(\Omega)$, the representation $\mathbf{T}$ and the operator $\mathbf{T}_\mu$ will be denoted by $\mathbf{U}$ and $\mathbf{U}_\mu$, respectively.

PROPOSITION 2.10. *Let $\mu \in M(G)$ be power bounded and assume that $\mathscr{F}_\mu = \mathscr{E}_\mu$. Then, the limit $\lim_{n\to\infty} (\mathbf{U}_\mu^n f)(\omega)$ exists a.e. for every f in a dense subspace of $L^2(\Omega)$.*

PROOF. By Theorem 2.1, there exists a linear subspace E of $L^2(\Omega)$ such that

$$L^2(\Omega) = \ker(I - \mathbf{U}_\mu) \oplus \overline{E} \quad \text{and} \quad \sum_{n=0}^{\infty} \|\mathbf{U}_\mu^n f\|_2 < \infty, \ \forall f \in E.$$

Since $\ker(I - \mathbf{U}_\mu) \oplus E$ is dense in $L^2(\Omega)$, it suffices to show that

$$(\mathbf{U}_\mu^n f)(\omega) \to 0 \text{ a.e.} \quad \forall f \in E.$$

Indeed, if $f \in E$ then as

$$\sum_{n=0}^{\infty} \|\mathbf{U}_\mu^n f\|_2^2 < \infty,$$

we have

$$\sum_{n=0}^{\infty} \int_\Omega |(\mathbf{U}_\mu^n f)(\omega)|^2 \, dm(\omega) < \infty.$$

By Beppo-Levi's theorem, the series

$$\sum_{n=0}^{\infty} |(\mathbf{U}_\mu^n f)(\omega)|^2$$

converges almost everywhere. It follows that $(\mathbf{U}_\mu^n f)(\omega) \to 0$ a.e.

As a consequence of Proposition 2.10 and Example 2.3(c), we have the following:

COROLLARY 2.11. *Let $\nu \in M(G)$ be power bounded and $\mu := \frac{1}{k}\sum_{i=0}^{k-1} \nu^i$, where k is a fixed integer > 1. Then, the limit $\lim_{n\to\infty}(\mathbf{U}_\mu^n f)(\omega)$ exists a.e. for every f in a dense subspace of $L^2(\Omega)$.*

Let T be a linear operator which is simultaneously defined and bounded on $L^1(\Omega)$ to itself and $L^\infty(\Omega)$ to itself. Moreover, if

$$\|Tf\|_1 \le \|f\|_1, \ \forall f \in L^1(\Omega), \quad \text{and} \quad \|Tf\|_\infty \le \|f\|_\infty, \ \forall f \in L^\infty(\Omega),$$

then T is called *Dunford-Schwartz operator*. By the Riesz-Thorin convexity theorem, Dunford-Schwartz operator can be extended to a contraction on $L^p(\Omega)$ $(1 < p < \infty)$. Notice that if $\|\mu\|_1 \le 1$, then the operator $\mathbf{T}_\mu$ defined by (2.10) is a Dunford-Schwartz operator. The Dunford-Schwartz theorem [8, Chapter VIII, §6] states that if T is a Dunford-Schwartz operator, $f \in L^p(\Omega)$ $(1 < p < \infty)$, and

$$f^*(\omega) := \sup_{n>1} \left| \frac{1}{n} \sum_{k=0}^{n-1} (T^k f)(\omega) \right|,$$

then there exists a constant $C_p > 0$ such that

$$\|f^*\|_p \le C_p \|f\|_p, \quad \forall f \in L^p(\Omega). \tag{2.11}$$

It follows that the sequence $\left\{\frac{1}{n}\sum_{k=0}^{n-1}(T^k f)(\omega)\right\}$ converges a.e. for every $f \in L^p(\Omega)$.

COROLLARY 2.12. *Let $\mu \in M(G)$ be a symmetric measure with $\|\mu\|_1 \leq 1$. If*

$$\{\gamma \in \Gamma : \widehat{\mu}(\gamma) = -1\} = \emptyset,$$

then the limit $\lim_{n\to\infty}(\mathbf{U}_\mu^n f)(\omega)$ exists a.e. for every $f \in L^2(\Omega)$.

PROOF. As we have noted above, $\mathbf{U}_\mu$ is a Dunford-Schwartz operator. Since $\mathbf{U}_\mu$ is a self-adjoint contraction, by the maximal ergodic theorem of Stein [22], there exists a constant $C > 0$ such that

$$\Big\|\sup_{n\geq 1}|\mathbf{U}_\mu^n f|\Big\|_2 \leq C\|f\|_2, \quad \forall f \in L^2(\Omega).$$

It follows that

$$\sup_{n\geq 1}|(\mathbf{U}_\mu^n f)(\omega)| < \infty \text{ a.e.} \quad \forall f \in L^2(\Omega).$$

Since the function $\gamma \to \widehat{\mu}(\gamma)$ is real valued, the condition $\{\gamma \in \Gamma : \widehat{\mu}(\gamma) = -1\} = \emptyset$ implies $\mathscr{F}_\mu = \mathscr{E}_\mu$. By Proposition 2.10, the limit $\lim_{n\to\infty}(\mathbf{U}_\mu^n f)(\omega)$ exists a.e. for every f in a dense subspace of $L^2(\Omega)$. By the Banach principle [13, Chapter 1, Theorem 7.2], the limit $\lim_{n\to\infty}(\mathbf{U}_\mu^n f)(\omega)$ exists a.e. for every $f \in L^2(\Omega)$.

Let $\mu \in M(G)$ be power bounded and assume that

$$|1 - \widehat{\mu}(\gamma)| \leq C(1 - |\widehat{\mu}(\gamma)|), \quad \text{for some } C > 0 \text{ and for all } \gamma \in \Gamma.$$

Notice that this is a quantitative generalization of the condition $\mathscr{F}_\mu = \mathscr{E}_\mu$. Next, we will show that under this condition the limit $\lim_{n\to\infty}(\mathbf{U}_\mu^n f)(\omega)$ exists a.e. for every $f \in L^2(\Omega)$.

Let $\mathbf{\Theta}$ be a continuous action of G in (Ω, Σ, m) and let $\mathbf{U}$ be the induced continuous unitary representation of G on $L^2(\Omega)$. Recall that the *Arveson spectrum* sp($\mathbf{U}$) of $\mathbf{U}$ [2] is defined as the hull in $L^1(G)$ of the ideal

$$I_{\mathbf{U}} := \{f \in L^1(G) : \mathbf{U}_f = 0\}.$$

It is easy to check that if U is a unitary operator on H, then $\sigma(U)$ is the Arveson spectrum of the representation $n \mapsto U^n$ $(n \in \mathbb{Z})$.

PROPOSITION 2.13. *Let $\mu \in M(G)$ be such that $\|\mu\|_1 \leq 1$. If $S := \mathscr{F}_\mu = \mathscr{E}_\mu$ and*

$$K_\mu := \sup_{\gamma\in \mathrm{sp}(\mathbf{U})\setminus S}\frac{|1-\widehat{\mu}(\gamma)|}{1-|\widehat{\mu}(\gamma)|} < \infty,$$

then the limit $\lim_{n\to\infty}(\mathbf{U}_\mu^n f)(\omega)$ exists a.e. for every $f \in L^2(\Omega)$.

PROOF. We basically follow the proof by Bellow-Jones-Rosenblatt [4]. For $f \in L^2(\Omega)$ we put

$$f^{**}(\omega) := \sup_{n \geq 1} |(T^n f)(\omega)|,$$

where $T = \mathbf{U}_\mu$. Since T is a Dunford-Schwartz operator, by (2.11) there exists a constant $L > 0$ such that

$$\|f^*\|_2 \leq L\|f\|_2, \quad \forall f \in L^2(\Omega).$$

We refer to [4] for an argument showing the inequality

$$\|f^{**}\|_2 \leq \|f^*\|_2 + \left(\sum_{k=0}^{\infty} k\left\|(T^{k+1} - T^k)f\right\|_2^2\right)^{1/2}.$$

If P is the spectral measure for $\mathbf{U}$, then it follows from (2.8) that supp $P =$ sp($\mathbf{U}$). Since

$$T = \mathbf{U}_\mu = \int_{\mathrm{sp}(\mathbf{U})} \widehat{\mu}(\gamma)\, dP(\gamma),$$

we can write

$$\begin{aligned}
\sum_{k=0}^{\infty} k\left\|(T^{k+1} - T^k)f\right\|_2^2 &= \int_{\mathrm{sp}(\mathbf{U})\setminus S} \left(\sum_{k=0}^{\infty} k|\widehat{\mu}(\gamma)|^{2k}\right)|1 - \widehat{\mu}(\gamma)|^2\, d\lambda_f(\gamma) \\
&= \int_{\mathrm{sp}(\mathbf{U})\setminus S} \frac{|\widehat{\mu}(\gamma)|^2}{(1 + |\widehat{\mu}(\gamma)|)^2}\left(\frac{|1 - \widehat{\mu}(\gamma)|}{1 - |\widehat{\mu}(\gamma)|}\right)^2 d\lambda_f(\gamma) \\
&\leq K_\mu^2\|f\|_2^2,
\end{aligned}$$

where λ_f is the measure on Γ defined by (2.7). So we have

$$\|f^{**}\|_2 \leq (L + K_\mu)\|f\|_2$$

which implies

$$\sup_{n \geq 1} |(\mathbf{U}_\mu^n f)(\omega)| < \infty \text{ a.e.} \quad \forall f \in L^2(\Omega).$$

By Proposition 2.10, the limit $\lim_{n\to\infty}(\mathbf{U}_\mu^n f)(\omega)$ exists a.e. for every f in a dense subspace of $L^2(\Omega)$. Now, it follows from the Banach principle [13, Chapter 1, Theorem 7.2] that the limit $\lim_{n\to\infty}(\mathbf{U}_\mu^n f)(\omega)$ exists a.e. for every $f \in L^2(\Omega)$.

Below, we give an example of a measure which satisfies the hypotheses of Proposition 2.13.

If $\nu \in M(G)$ is power bounded, then $|1 \pm \widehat{\nu}(\gamma)| \leq 2$ for all $\gamma \in \Gamma$. Assume that

$$|1 + \widehat{\nu}(\gamma)| \leq \frac{2C - 2}{C}, \quad \text{for some } C > 1 \text{ and for all } \gamma \in \Gamma.$$

If $\mu := \frac{\delta_0 + \nu}{2}$, then μ is power bounded and $\mathscr{F}_\mu = \mathscr{E}_\mu$. Since $2 - |1 + \widehat{\nu}(\gamma)| \geq \frac{2}{C}$, we have

$$\frac{|1 - \widehat{\mu}(\gamma)|}{1 - |\widehat{\mu}(\gamma)|} = \frac{\left|1 - \frac{1+\widehat{\nu}(\gamma)}{2}\right|}{1 - \left|\frac{1+\widehat{\nu}(\gamma)}{2}\right|} = \frac{|1 - \widehat{\nu}(\gamma)|}{2 - |1 + \widehat{\nu}(\gamma)|} \leq \frac{2}{2/C} = C.$$

Recall that a bounded linear operator T on a Banach space satisfies Ritt's condition if

$$\sup_{|\lambda|>1} |\lambda - 1| \|R_\lambda(T)\| < \infty.$$

By the Nagy-Zemanek result [18], T satisfies Ritt's condition if and only if T is power bounded with

$$\sup_{n \in \mathbb{N}} n \|T^n - T^{n+1}\| < \infty.$$

Proposition 2.14. *Assume that $\mu \in M(G)$ is power bounded and $S := \mathscr{E}_\mu = \mathscr{F}_\mu$. If*

$$K_\mu := \sup_{\gamma \in \mathrm{sp}(\mathbf{U}) \setminus S} \frac{|1 - \widehat{\mu}(\gamma)|}{1 - |\widehat{\mu}(\gamma)|} < \infty,$$

then

$$\overline{\lim_{n \to \infty}}\, n \|\mathbf{U}_\mu^n - \mathbf{U}_\mu^{n+1}\| \leq \frac{K_\mu}{e}.$$

Proof. We can write

$$\begin{aligned} |\widehat{\mu}(\gamma)^n - \widehat{\mu}(\gamma)^{n+1}| &= |\widehat{\mu}(\gamma)|^n |1 - \widehat{\mu}(\gamma)| \\ &\leq K_\mu (|\widehat{\mu}(\gamma)|^n - |\widehat{\mu}(\gamma)|^{n+1}), \quad \forall \gamma \in \mathrm{sp}(\mathbf{U}) \setminus S. \end{aligned}$$

Since $0 \leq |\widehat{\mu}(\gamma)| \leq 1$ and

$$\max_{x \in [0,1]} (x^n - x^{n+1}) = \frac{n^n}{(n+1)^{n+1}},$$

we have

$$n |\widehat{\mu}(\gamma)^n - \widehat{\mu}(\gamma)^{n+1}| \leq K_\mu \frac{n^{n+1}}{(n+1)^{n+1}}, \quad \forall \gamma \in \mathrm{sp}(\mathbf{U}) \setminus S.$$

On the other hand, we know [15, p. 450] that

$$\sigma(\mathbf{U}_\mu) = \overline{\widehat{\mu}(\mathrm{sp}(\mathbf{U}))}.$$

Since $\mathbf{U}_\mu$ is a normal operator, we get

$$\begin{aligned} n\|\mathbf{U}_\mu^n - \mathbf{U}_\mu^{n+1}\| &= n \sup_{\gamma\in\mathrm{sp}(\mathbf{U})\setminus S} |\widehat{\mu}(\gamma)^n - \widehat{\mu}(\gamma)^{n+1}| \\ &\leq K_\mu \frac{n^{n+1}}{(n+1)^{n+1}} = K_\mu \frac{1}{\left(1+\frac{1}{n}\right)^n} \frac{n}{n+1}. \end{aligned}$$

It follows that

$$\overline{\lim_{n\to\infty}}\, n\|\mathbf{U}_\mu^n - \mathbf{U}_\mu^{n+1}\| \leq \frac{K_\mu}{e}.$$

Let $M_0(G)$ be the set of all $\mu \in M(G)$ such that $\widehat{\mu}(\infty) = 0$. Then, $M_0(G)$ is a closed ideal of $M(G)$. Notice that if $\mu \in M_0(G)$, then both $\mathscr{F}_\mu$ and $\mathscr{E}_\mu$ are compact. If G is compact and $\mu \in M_0(G)$, then both $\mathscr{F}_\mu$ and $\mathscr{E}_\mu$ are finite.

PROPOSITION 2.15. *Let $\mu \in M_0(G)$ be power bounded and assume that $S := \mathscr{F}_\mu = \mathscr{E}_\mu$. If S is a clopen subset of Γ, then there exists a closed subspace E of H with the properties:*

(i) $H = \ker(I - \mathbf{U}_\mu) \oplus E$;

(ii) $\sum_{n=0}^\infty \|\mathbf{U}_\mu^n x\| < \infty$, *for all* $x \in E$;

(iii) $E = (I - \mathbf{U}_\mu)H$ *and consequently* $(I - \mathbf{U}_\mu)H$ *is closed.*

PROOF. Let

$$E := \{x \in H : \mathrm{supp}\,\lambda_x \subseteq \Gamma \setminus S\},$$

where λ_x is the measure on Γ defined by (2.7). Since $\Gamma \setminus S$ is closed, by Lemma 2.2, E is a closed subspace of H. Let $x \in E$ be given. Let us show that $\sum_{n=0}^\infty \|\mathbf{U}_\mu^n x\| < \infty$. Since $\mu \in M_0(G)$, there is a compact subset K of Γ such that

$$\sup\{|\widehat{\mu}(\gamma)| : \gamma \in \Gamma \setminus K\} := \delta_1 < 1$$

which implies

$$\sup\{|\widehat{\mu}(\gamma)| : \gamma \in \mathrm{supp}\,\lambda_x \cap \Gamma \setminus K\} \leq \delta_1.$$

Also since $|\widehat{\mu}(\gamma)| < 1$ for all $\gamma \in \mathrm{supp}\,\lambda_x$, we have

$$\sup\{|\widehat{\mu}(\gamma)| : \gamma \in \mathrm{supp}\,\lambda_x \cap K\} := \delta_2 < 1.$$

Hence,

$$\sup\{|\widehat{\mu}(\gamma)| : \gamma \in \mathrm{supp}\,\lambda_x\} \leq \max\{\delta_1, \delta_2\} := \delta < 1.$$

It follows from (2.9) that

$$\|\mathbf{U}_\mu^n x\| \leq \delta^n \|x\|, \quad \text{for all } n \in \mathbb{N}$$

and so

$$\sum_{n=0}^{\infty} \|\mathbf{U}_\mu^n x\| < \infty.$$

If $x \in E^\perp$, then as in the proof of Theorem 2.1, we can see that $\operatorname{supp} \lambda_x \subseteq S$ and therefore,

$$\|\mathbf{U}_\mu x - x\|^2 = \int_S |\widehat{\mu}(\gamma) - 1|^2 \, d\lambda_x(\gamma) = 0.$$

To show (iii), let $x \in E$ and

$$y = \sum_{n=0}^{\infty} \mathbf{U}_\mu^n x.$$

Then as $(I - \mathbf{U}_\mu)y = x$, we have $E \subseteq (I - \mathbf{U}_\mu)H$. On the other hand, since E is closed, by (2.6) we get

$$\overline{(I - \mathbf{U}_\mu)H} = E \subseteq (I - \mathbf{U}_\mu)H.$$

Notice that under the hypotheses of Proposition 2.15, the operator $\mathbf{U}_\mu$ is uniformly mean ergodic. Consequently by (2.5),

$$\frac{1}{n} \sum_{i=0}^{n-1} \mathbf{U}_\mu^i \to P(\mathscr{F}_\mu) \quad \text{in operator norm.}$$

We have the following two corollaries.

COROLLARY 2.16. *Let $\mu \in M_0(G)$ be power bounded and assume that $S := \mathscr{F}_\mu = \mathscr{E}_\mu$. If S is a clopen subset of Γ, then the limit* $\lim_{n\to\infty}(\mathbf{U}_\mu^n f)(\omega)$ *exists a.e. for every $f \in L^2(\Omega)$.*

COROLLARY 2.17. *Let G be a compact and let $\mu \in M_0(G)$ be power bounded. If $\mathscr{F}_\mu = \mathscr{E}_\mu$, then the limit* $\lim_{n\to\infty}(\mathbf{U}_\mu^n f)(\omega)$ *exists a.e. for every $f \in L^2(\Omega)$.*

3. Banach space operators

In this section, we study strong and almost everywhere convergence of iterates of convolution operators in Banach spaces.

Let X be a complex Banach space. Recall that an operator $T \in B(X)$ is called *mean ergodic* if the

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} T^i x \quad \text{exists in norm for all } x \in X.$$

It can be seen that the condition $\|T^n x\|/n \to 0$ ($\forall x \in X$) is necessary for the mean ergodicity of T (it is satisfied when T is power bounded). Now, assume that T is power bounded. Then, T is mean ergodic if and only if we have the decomposition

$$X = \ker(I - T) \oplus \overline{(I - T)X} \tag{3.1}$$

[13, Chapter 2, Theorem 1.2]. On the other hand, it is easy to check that

$$\overline{(I - T)X} = \left\{ x \in X : \lim_{n\to\infty} \left\| \frac{1}{n}\sum_{i=0}^{n-1} T^i x \right\| = 0 \right\}. \tag{3.2}$$

If X is reflexive, then T is mean ergodic [13, Chapter 2, Theorem 1.3].

Let $\mathbf{T} = \{T_g : g \in G\}$ be a bounded and (strongly) continuous representation of G on X (by passing to an equivalent norm $\mathbf{T}$ becomes representation by invertible isometries). For each $\mu \in M(G)$, we can define a bounded linear operator $\mathbf{T}_\mu$ on X by

$$\mathbf{T}_\mu x = \int_G T_g^{-1} x \, d\mu(g), \quad x \in X.$$

The map $\mu \mapsto T_\mu$ is a continuous algebra homomorphism. It follows that if μ is power bounded, then so is $\mathbf{T}_\mu$;

$$\sup_{n\geq 0} \|\mathbf{T}_\mu^n\| \leq C_\mu \sup_{g\in G} \|T_g\|.$$

Furthermore, it is easy to verify that

$$\overline{\text{span}}\{\mathbf{T}_f x : f \in L^1(G),\ x \in X\} = X. \tag{3.3}$$

PROPOSITION 3.1. *Let G be a compact abelian group. If $\mu \in M(G)$ is power bounded, then the operator $\mathbf{T}_\mu$ is mean ergodic, that is,*

$$\frac{1}{n}\sum_{i=0}^{n-1} \mathbf{T}_\mu^i x \to \mathbf{T}_\nu x \quad \textit{strongly for every } x \in X,$$

where ν is the limit measure associated with μ.

PROOF. By the mean ergodic theorem, it suffices to show that

$$\frac{1}{n}\sum_{i=0}^{n-1}\mathbf{T}_{\mu}^{i}x \to \mathbf{T}_{\nu}x \quad \text{weakly for every } x \in X.$$

Let $x \in X$ and $\varphi \in X^*$. Since

$$\text{w}^*\text{-}\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mu^i = \nu,$$

we can write

$$\left\langle \varphi, \frac{1}{n}\sum_{i=0}^{n-1}\mathbf{T}_{\mu}^{i}x\right\rangle = \left\langle \frac{1}{n}\sum_{i=0}^{n-1}\mu^i, \varphi(T_g^{-1}x)\right\rangle \to \langle \nu, \varphi(T_g^{-1}x)\rangle = \langle \varphi, \mathbf{T}_{\nu}x\rangle.$$

This shows that $\frac{1}{n}\sum_{i=0}^{n-1}T_{\mu}^{i}x \to \mathbf{T}_{\nu}x$ weakly.

LEMMA 3.2. *Let $\mu \in M(G)$ be power bounded and assume that $\mathscr{F}_{\mu} = \mathscr{E}_{\mu}$. Then, $\nu := \text{w}^*\text{-}\lim_{n\to\infty}\mu^n$ exists and ν is the limit measure associated with μ.*

PROOF. Let ν_1 be a w*-cluster point of the sequence $\{\mu^n\}$;

$$\nu_1 = \text{w}^*\text{-}\lim_{i}\mu^{n_i},$$

where $\{\mu^{n_i}\}_i$ is a subnet of $\{\mu^n\}$. Using the identity

$$\langle \nu, \widehat{f}\rangle = \int_{\Gamma}\widehat{\nu}(\gamma)f(\gamma)\,d\gamma$$

which is valid for an arbitrary $\nu \in M(G)$ and $f \in L^1(\Gamma)$, we can write

$$\begin{aligned}\langle \nu_1, \widehat{f}\rangle &= \lim_i\langle \mu^{n_i}, \widehat{f}\rangle = \lim_i\int_{\Gamma}\widehat{\mu}(\gamma)^{n_i}f(\gamma)\,d\gamma \\ &= \lim_i\int_{\mathscr{F}_{\mu}}\widehat{\mu}(\gamma)^{n_i}f(\gamma)\,d\gamma + \lim_i\int_{\Gamma\setminus\mathscr{E}_{\mu}}\widehat{\mu}(\gamma)^{n_i}f(\gamma)\,d\gamma \\ &= \int_{\mathscr{F}_{\mu}}f(\gamma)\,d\gamma, \quad \forall f \in L^1(\Gamma).\end{aligned}$$

If ν_2 is another w*-cluster point of the sequence $\{\mu^n\}$, similarly we have

$$\langle \nu_2, \widehat{f}\rangle = \int_{\mathscr{F}_{\mu}}f(\gamma)\,d\gamma, \quad \forall f \in L^1(\Gamma).$$

Hence,

$$\langle \nu_1, \widehat{f} \rangle = \langle \nu_2, \widehat{f} \rangle, \quad \forall f \in L^1(\Gamma).$$

Since $\{\widehat{f} : f \in L^1(\Gamma)\}$ is dense in $C_0(G)$, we obtain $\nu_1 = \nu_2$. This shows that the sequence $\{\mu^n\}$ has only one w*-cluster point and therefore, $\nu := \text{w}^*\text{-}\lim_{n\to\infty} \mu^n$ exists. Further, we have

$$\nu = \text{w}^*\text{-}\lim_{n\to\infty} \mu^n = \text{w}^*\text{-}\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu^i.$$

Next, we have the following:

PROPOSITION 3.3. *Let G be a compact abelian group and let $\mu \in M(G)$ be power bounded. If $\mathscr{F}_\mu = \mathscr{E}_\mu$, then $\mathbf{T}_\mu^n x \to \mathbf{T}_\nu x$ strongly for every $x \in X$, where ν is the limit measure associated with μ.*

PROOF. By Lemma 3.2, $\nu = \text{w}^*\text{-}\lim_{n\to\infty} \mu^n$. If $\varphi \in X^*$ and $x \in X$, then we can write

$$\langle \varphi, \mathbf{T}_\mu^n x \rangle = \langle \mu^n, \varphi(T_g^{-1} x) \rangle \to \langle \nu, \varphi(T_g^{-1} x) \rangle = \langle \varphi, \mathbf{T}_\nu x \rangle.$$

This shows that $\mathbf{T}_\mu^n x \to \mathbf{T}_\nu x$ weakly. Let K be the norm closure of the absolute convex hull of $\{T_g x : g \in G\}$. Since $\{T_g x : g \in G\}$ is compact, so is K. On the other hand, $\{(1/C_\mu)\mathbf{T}_\mu^n x : n \in \mathbb{N}\}$ is contained in K and therefore the sequence $\{\mathbf{T}_\mu^n x\}$ is relatively compact. This clearly implies that $\mathbf{T}_\mu^n x \to \mathbf{T}_\nu x$ strongly.

Let A be a complex commutative Banach algebra and let Σ_A be its Gelfand space equipped with the weak* topology. The Gelfand transform of $a \in A$ will be denoted by $\widehat{a}$. Recall that the algebra A is said to be *regular* if given a closed subset S of Σ_A and $\phi \in \Sigma_A \setminus S$, there exists an element $a \in A$ such that $\widehat{a}(S) = \{0\}$ and $\widehat{a}(\phi) \neq 0$. It is well known that if G is a locally compact abelian group, then the measure algebra $M(G)$ is a commutative semisimple Banach algebra with identity, but $M(G)$ fails to be regular, in general. However, there exists a largest closed regular subalgebra of $M(G)$ which we will denote by $M_{\text{reg}}(G)$. Since the algebra $L^1(G)$ and the discrete measure algebra $M_d(G)$ are regular subalgebras of $M(G)$, we have $L^1(G) \oplus M_d(G) \subseteq M_{\text{reg}}(G)$, but in general, $L^1(G) \oplus M_d(G) \neq M_{\text{reg}}(G)$ [15, Example 4.3.11]. This shows that the algebra $M_{\text{reg}}(G)$ is remarkably large.

The proof of the following lemma is based on the standard Banach algebra techniques and therefore is omitted.

LEMMA 3.4. *Let A be a commutative, semisimple, and regular Banach algebra and let $\{I_\lambda\}_{\lambda\in\Lambda}$ be a collection of the closed ideals of A. Then,*

$$\operatorname{hull}\left(\bigcap_{\lambda\in\Lambda} I_\lambda\right) = \overline{\bigcup_{\lambda\in\Lambda} \operatorname{hull}(I_\lambda)}^{w^*}.$$

We have the following:

THEOREM 3.5. *Let $\mathbf{T}$ be a bounded and continuous representation of G on a Banach space X and let $\mu \in M_{\mathrm{reg}}(G)$ be power bounded. If $\mathscr{F}_\mu = \mathscr{E}_\mu$, then there exists a (not necessarily closed) linear subspace E of X such that:*

(i) *$\overline{E} = \overline{(I-\mathbf{T}_\mu)X}$ and $\sum_{n=0}^{\infty} \|\mathbf{T}_\mu^n x\| < \infty$, for all $x \in E$;*

(ii) *if $\mathbf{T}_\mu$ is mean ergodic (or if X is reflexive), then $X = \ker(I-\mathbf{T}_\mu) \oplus \overline{E}$;*

(iii) *if $\mathbf{T}_\mu$ is mean ergodic (or if X is reflexive), then the sequence $\{\mathbf{T}_\mu^n x\}$ converges strongly for every $x \in X$.*

For the proof, we need some preliminary results.

Let $\mathbf{T}$ be a bounded and continuous representation of G on a Banach space X. The *Arveson spectrum* $\mathrm{sp}(\mathbf{T})$ of $\mathbf{T}$ [2] is defined as the hull in $L^1(G)$ of the ideal

$$I_{\mathbf{T}} := \{f \in L^1(G) : \mathbf{T}_f = 0\}.$$

It is easy to check that if $T \in B(X)$ is doubly power bounded, that is,

$$\sup_{n\in\mathbb{Z}} \|T^n\| < \infty,$$

then $\sigma(T)$ is the Arveson spectrum of the representation $n \mapsto T^n$ $(n \in \mathbb{Z})$.

By [15, Proposition 4.12.12], every measure $\mu \in M_{\mathrm{reg}}(G)$ has the *spectral mapping property*, that is,

$$\sigma(\mathbf{T}_\mu) = \overline{\widehat{\mu}(\mathrm{sp}(\mathbf{T}))}.$$

For $T \in B(X)$ and $x \in X$, we define $\rho_T(x)$ to be the set of all $\lambda \in \mathbb{C}$ for which there exists a neighborhood U_λ of λ with $u(z)$ analytic on U_λ having values in X such that $(zI-T)u(z) = x$ for all $z \in U_\lambda$. This set is open and contains the resolvent set $\rho(T)$ of T. The *local spectrum* of T at $x \in X$, denoted by $\sigma_T(x)$ is the complement of $\rho_T(x)$, so it is a compact subset of $\sigma(T)$. This object is most tractable if the operator T has the *single-valued extension property* (SVEP), i.e., for every open set U in $\mathbb{C}$, the only analytic function $u\colon U \to X$ for which the equation $(zI-T)u(z) = 0$ holds is the constant function $u \equiv 0$. If T has SVEP, then $\sigma_T(x) \neq \emptyset$, whenever $x \in X \setminus \{0\}$ [15, Proposition 1.2.16]. For example, if $\mu \in M_{\mathrm{reg}}(G)$, then the operator $\mathbf{T}_\mu$ is decomposable [15, Proposition 4.12.3] and therefore it has SVEP [15, Chapter 1].

Given an operator $T \in B(X)$ and $x \in X$, the quantity

$$r_T(x) := \overline{\lim_{n\to\infty}} \|T^n x\|^{\frac{1}{n}}$$

is called the *local spectral radius* of T at x. If T has SVEP, then

$$r_T(x) = \sup\{|\lambda| : \lambda \in \sigma_T(x)\}$$

[15, Proposition 3.3.13]. The *local Arveson spectrum* $\mathrm{sp}_{\mathbf{T}}(x)$ of $x \in X$ [2] is defined as the hull in $L^1(G)$ of the ideal

$$I_{\mathbf{T}}(x) := \{f \in L^1(G) : \mathbf{T}_f x = 0\}.$$

Clearly, $\mathrm{sp}_{\mathbf{T}}(x) \subseteq \mathrm{sp}(\mathbf{T})$ for all $x \in X$. Since $I_{\mathbf{T}} = \bigcap_{x\in X} I_{\mathbf{T}}(x)$, by Lemma 3.4,

$$\mathrm{sp}(\mathbf{T}) = \overline{\bigcup_{x\in X} \mathrm{sp}_{\mathbf{T}}(x)}^{w^*}.$$

By [15, Proposition 4.12.12], every measure $\mu \in M_{\mathrm{reg}}(G)$ has the *local spectral mapping property*, that is,

$$\sigma_{\mathbf{T}_\mu}(x) = \overline{\widehat{\mu}(\mathrm{sp}_{\mathbf{T}}(x))}, \quad \forall x \in X.$$

LEMMA 3.6. *Under the above notations we have:*

(a) $\mathrm{sp}_{\mathbf{T}}(x + y) \subseteq \mathrm{sp}_{\mathbf{T}}(x) \cup \mathrm{sp}_{\mathbf{T}}(y)$, $\forall x, y \in X$;

(b) $\mathrm{sp}_{\mathbf{T}}(\mathbf{T}_f x) \subseteq \mathrm{supp}\, \widehat{f} \cap \mathrm{sp}_{\mathbf{T}}(x)$, $\forall f \in L^1(G)$, $\forall x \in X$;

(c) *if S is a closed subset of Γ, then $\{x \in X : \mathrm{sp}_{\mathbf{T}}(x) \subseteq S\}$ is a closed subspace of X.*

PROOF. (a) Since $I_{\mathbf{T}}(x) \cap I_{\mathbf{T}}(y) \subseteq I_{\mathbf{T}}(x + y)$, by Lemma 3.4,

$$\begin{aligned}\mathrm{sp}_{\mathbf{T}}(x + y) &= \mathrm{hull}\, I_{\mathbf{T}}(x + y) \subseteq \mathrm{hull}[I_{\mathbf{T}}(x) \cap I_{\mathbf{T}}(y)]\\ &= \mathrm{hull}\, I_{\mathbf{T}}(x) \cup \mathrm{hull}\, I_{\mathbf{T}}(y) = \mathrm{sp}_{\mathbf{T}}(x) \cup \mathrm{sp}_{\mathbf{T}}(y).\end{aligned}$$

(b) Clearly, $I_{\mathbf{T}}(x) \subseteq I_{\mathbf{T}}(\mathbf{T}_f x)$ which implies $\mathrm{sp}_{\mathbf{T}}(\mathbf{T}_f x) \subseteq \mathrm{sp}_{\mathbf{T}}(x)$. It remains to show that $\mathrm{sp}_{\mathbf{T}}(\mathbf{T}_f x) \subseteq \mathrm{supp}\, \widehat{f}$. If $h \in I_{\mathrm{supp}\, \widehat{f}}$, then as $\widehat{h}\widehat{f} = 0$ we have $h * f = 0$ and so $\mathbf{T}_h \mathbf{T}_f x = 0$. Hence, $h \in I_{\mathbf{T}}(\mathbf{T}_f x)$. So we have

$$I_{\mathrm{supp}\, \widehat{f}} \subseteq I_{\mathbf{T}}(\mathbf{T}_f x)$$

which implies

$$\mathrm{sp}_{\mathbf{T}}(\mathbf{T}_f x) = \mathrm{hull}\, I_{\mathbf{T}}(\mathbf{T}_f x) \subseteq \mathrm{hull}(I_{\mathrm{supp}\, \widehat{f}}) = \mathrm{supp}\, \widehat{f}.$$

(c) By (a), the set $\{x \in X : \mathrm{sp}_{\mathbf{T}}(x) \subseteq S\}$ is linear. Let $\{x_n\}$ be a sequence in X such that $\mathrm{sp}_{\mathbf{T}}(x_n) \subseteq S$ for all n and $x_n \to x$. We must show that $\mathrm{sp}_{\mathbf{T}}(x) \subseteq S$. Since

$$\bigcap_{n=1}^{\infty} I_{\mathbf{T}}(x_n) \subseteq I_{\mathbf{T}}(x),$$

by Lemma 3.4,

$$\mathrm{sp}_{\mathbf{T}}(x) = \text{hull } I_{\mathbf{T}}(x) \subseteq \overline{\bigcup_{n=1}^{\infty} \text{hull } I_{\mathbf{T}}(x_n)}^{w^*} = \overline{\bigcup_{n=1}^{\infty} \mathrm{sp}_{\mathbf{T}}(x_n)}^{w^*} \subseteq S.$$

Now, we are in a position to prove Theorem 3.5.

Proof of Theorem 3.5. (i) Let $S := \mathscr{F}_\mu = \mathscr{E}_\mu$ and

$$E := \left\{x \in X : \mathrm{sp}_{\mathbf{T}}(x) \text{ is compact and } \mathrm{sp}_{\mathbf{T}}(x) \cap S = \emptyset\right\}.$$

By Lemma 3.6, E is linear. As we have noted above, the operator $\mathbf{T}_\mu$ has SVEP and therefore,

$$\overline{\lim_{n\to\infty}} \|\mathbf{T}_\mu^n x\|^{\frac{1}{n}} = \sup\{|\lambda| : \lambda \in \sigma_{\mathbf{T}_\mu}(x)\}, \quad \forall x \in X.$$

On the other hand, the local spectral mapping property holds, that is,

$$\sigma_{\mathbf{T}_\mu}(x) = \overline{\widehat{\mu}(\mathrm{sp}_{\mathbf{T}}(x))}.$$

Hence, we have

$$\overline{\lim_{n\to\infty}} \|\mathbf{T}_\mu^n x\|^{\frac{1}{n}} = \sup\{|\widehat{\mu}(\lambda)| : \lambda \in \mathrm{sp}_{\mathbf{T}}(x)\}, \quad \forall x \in X.$$

Let $x \in E$ be given. Since $\mathrm{sp}_{\mathbf{T}}(x)$ is compact and $\mathrm{sp}_{\mathbf{T}}(x) \cap S = \emptyset$, we have

$$\sup\{|\widehat{\mu}(\lambda)| : \lambda \in \mathrm{sp}_{\mathbf{T}}(x)\} < 1.$$

Now, since

$$\overline{\lim_{n\to\infty}} \|\mathbf{T}_\mu^n x\|^{\frac{1}{n}} < 1,$$

there is $0 < \delta < 1$ such that for sufficiently large n, $\|\mathbf{T}_\mu^n x\| \leq \delta^n$. So we have

$$\sum_{n=0}^{\infty} \|\mathbf{T}_\mu^n x\| < \infty, \quad \forall x \in E.$$

It remains to show that $\overline{E} = \overline{(I - \mathbf{T}_\mu)X}$. If $x \in E$, then as $\|\mathbf{T}_\mu^n x\| \to 0$, by (3.2), $x \in \overline{(I - \mathbf{T}_\mu)X}$ and therefore $\overline{E} \subseteq \overline{(I - \mathbf{T}_\mu)X}$. For the reverse inclusion, let $\varphi \in E^\perp$ be given. Since

$$\left[\overline{(I - \mathbf{T}_\mu)X}\right]^\perp = \{\varphi \in X^* : \mathbf{T}_\mu^* \varphi = \varphi\},$$

it suffices to show that $\mathbf{T}_\mu^* \varphi = \varphi$.

Assume that the Fourier transform of $f \in L^1(G)$ vanishes on S. Since S is a set of synthesis, there is a sequence $\{f_n\}$ in $L^1(G)$ such that supp $\widehat{f_n}$ is compact, $\widehat{f_n}$ vanishes in a neighborhood O_n of S, and $\|f_n - f\|_1 \to 0$. Let an arbitrary $x \in X$ be given. By Lemma 3.6,

$$\mathrm{sp}_{\mathbf{T}}(\mathbf{T}_{f_n} x) \subseteq \mathrm{supp}\, \widehat{f_n} \cap \mathrm{sp}_{\mathbf{T}}(x)$$

and therefore $\mathrm{sp}_{\mathbf{T}}(\mathbf{T}_{f_n} x)$ is compact. On the other hand, as supp $\widehat{f_n} \cap S = \emptyset$, we have

$$\mathrm{sp}_{\mathbf{T}}(\mathbf{T}_{f_n} x) \cap S = \emptyset.$$

Hence, $\mathbf{T}_{f_n} x \in E$ for all n. Since $\mathbf{T}_{f_n} x \to \mathbf{T}_f x$ in norm, $\mathbf{T}_f x \in \overline{E}$ and therefore,

$$\langle \mathbf{T}_f^* \varphi, x\rangle = \langle \varphi, \mathbf{T}_f x\rangle = 0.$$

Thus, we have shown that if the Fourier transform of $f \in L^1(G)$ vanishes on S, then $\langle \mathbf{T}_f^* \varphi, x\rangle = 0$ for all $x \in X$. Further, since $\widehat{\mu} = 1$ on S, the Fourier transform of $(\mu - \delta_0) * f$ vanishes on S for all $f \in L^1(G)$. Hence, $\langle (\mathbf{T}_\mu^* - I)\mathbf{T}_f^* \varphi, x\rangle = 0$ or $\langle (\mathbf{T}_\mu^* - I)\varphi, \mathbf{T}_f x\rangle = 0$ for all $x \in X$ and $f \in L^1(G)$. By (3.3) we have $\mathbf{T}_\mu^* \varphi = \varphi$.

(ii) follows from (i) and (3.1).

(iii) is an immediate consequence of (i) and (ii).

Let us show that the condition "$\mathscr{F}_\mu = \mathscr{E}_\mu$" in Theorem 3.5 is the best possible, in general. To see this, let G be a compact abelian group, $\mathbf{T}$ be the regular representation of G on $L^1(G)$, and let $\mathbf{T}_\mu f = \mu * f$ be the corresponding convolution operator. If $\mu \in M(G)$ is power bounded, then by Proposition 3.1, $\mathbf{T}_\mu$ is mean ergodic. Now, assume that the sequence $\{\mu^n * f\}$ converges strongly for every $f \in L^1(G)$. Then,

$$\lim_{n\to\infty} \|\mu^n * f - \mu^{n+1} * f\|_1 = 0, \quad \forall f \in L^1(G).$$

As we have seen above, this is the case if and only if $\mathscr{F}_\mu = \mathscr{E}_\mu$.

Recall that a representation $\mathbf{T} = \{T_g : g \in G\}$ of G on a Banach space is called *uniformly continuous* if

$$\lim_{g\to 0} \|T_g - I\| = 0.$$

A bounded representation $\mathbf{T}$ is uniformly continuous if and only if $\mathrm{sp}(\mathbf{T})$ is compact [2, Theorem 2.13]. If $\mathbf{T}$ is bounded and uniformly continuous, then the spectral mapping property $\sigma(\mathbf{T}_\mu) = \widehat{\mu}(\mathrm{sp}(\mathbf{T}))$ and the local spectral mapping property $\sigma_{\mathbf{T}_\mu}(x) = \widehat{\mu}(\mathrm{sp}_{\mathbf{T}}(x))$ hold for all $\mu \in M(G)$ and $x \in X$ [15, Proposition 4.12.12].

The proof of the following theorem is similar to the proof of Theorem 3.5.

THEOREM 3.7. *Let* $\mathbf{T}$ *be a bounded and uniformly continuous representation of* G *on a Banach space* X *and let* $\mu \in M(G)$ *be power bounded. If* $\mathscr{F}_\mu = \mathscr{E}_\mu$, *then there exists a (not necessarily closed) linear subspace* E *of* X *with the properties:*

(i) $\overline{E} = \overline{(I - \mathbf{T}_\mu)X}$ *and* $\sum_{n=0}^{\infty} \|\mathbf{T}_\mu^n x\| < \infty$, *for all* $x \in E$;

(ii) *if* $\mathbf{T}_\mu$ *is mean ergodic (or if* X *is reflexive), then* $X = \ker(I - \mathbf{T}_\mu) \oplus \overline{E}$;

(iii) *if* $\mathbf{T}_\mu$ *is mean ergodic (or if* X *is reflexive), then the sequence* $\{\mathbf{T}_\mu^n x\}$ *converges strongly for every* $x \in X$.

Given $\mu \in M(G)$, let $\mathbf{T}_\mu$ be the corresponding operator defined by (2.10).

PROPOSITION 3.8. *Let* $\mu \in M_{\mathrm{reg}}(G)$ *be power bounded and assume that* $\mathscr{F}_\mu = \mathscr{E}_\mu$. *Then, the limit* $\lim_{n\to\infty}(\mathbf{T}_\mu^n f)(\omega)$ *exists a.e. for every* f *in a dense subspace of* $L^p(\Omega)$ $(1 < p < \infty)$.

PROOF. By Theorem 3.5, there exists a subspace E of $L^p(\Omega)$ such that

$$L^p(\Omega) = \ker(I - \mathbf{T}_\mu) \oplus \overline{E} \quad \text{and} \quad \sum_{n=0}^{\infty} \|\mathbf{T}_\mu^n f\|_p < \infty, \quad \forall f \in E.$$

Since $\ker(I - \mathbf{T}_\mu) \oplus E$ is dense in $L^p(\Omega)$, it suffices to show that

$$(\mathbf{T}_\mu^n f)(\omega) \to 0 \quad \text{a.e. } \forall f \in E.$$

Indeed, if $f \in E$ then as

$$\sum_{n=0}^{\infty} \|\mathbf{T}_\mu^n f\|_p^p < \infty,$$

we have

$$\sum_{n=0}^{\infty} \int_\Omega |(\mathbf{T}_\mu^n f)(\omega)|^p \, dm(\omega) < \infty.$$

By Beppo-Levi's theorem, the series

$$\sum_{n=0}^{\infty} |(\mathbf{T}_\mu^n f)(\omega)|^p$$

converges almost everywhere. It follows that $(\mathbf{T}_\mu^n f)(\omega) \to 0$ a.e.

As a consequence of Proposition 3.8 and Example 2.3(c), we have the following:

COROLLARY 3.9. *Let $\nu \in M_{\mathrm{reg}}(G)$ be power bounded and $\mu := \frac{1}{k}\sum_{i=0}^{k-1} \nu^i$, where k is a fixed integer > 1. Then, the limit $\lim_{n\to\infty}(\mathbf{T}_\mu^n f)(\omega)$ exists a.e. for every f in a dense subspace of $L^p(\Omega)$ $(1 < p < \infty)$.*

COROLLARY 3.10. *Let $\mu \in M_{\mathrm{reg}}(G)$ be a symmetric measure with $\|\mu\|_1 \le 1$. If $\{\gamma \in \Gamma : \widehat{\mu}(\gamma) = -1\} = \emptyset$, then the limit $\lim_{n\to\infty}(\mathbf{T}_\mu^n f)(\omega)$ exists a.e. for every $f \in L^p(\Omega)$ $(1 < p < \infty)$.*

PROOF. Since $\mathbf{T}_\mu$ is a self-adjoint contraction on $L^2(\Omega)$, by the maximal ergodic theorem of Stein [22], there exists a constant $C_p > 0$ such that

$$\|\sup_{n\ge 1} |\mathbf{T}_\mu^n f|\|_p \le C_p \|f\|_p, \quad \forall f \in L^p(\Omega).$$

It follows that

$$\sup_{n\ge 1} |(\mathbf{T}_\mu^n f)(\omega)| < \infty \quad \text{a.e. } \forall f \in L^p(\Omega).$$

On the other hand, the condition $\{\gamma \in \Gamma : \widehat{\mu}(\gamma) = -1\} = \emptyset$ implies $\mathscr{F}_\mu = \mathscr{E}_\mu$. By Proposition 3.8, the limit $\lim_{n\to\infty}(\mathbf{T}_\mu^n f)(\omega)$ exists a.e. for every f in a dense subspace of $L^p(\Omega)$. By the Banach principle [13, Chapter 1, Theorem 7.2], the limit $\lim_{n\to\infty}(\mathbf{T}_\mu^n f)(\omega)$ exists a.e. for every $f \in L^p(\Omega)$.

ACKNOWLEDGEMENTS. I am grateful to the referee for his many helpful remarks, suggestions and substantial improvements to the paper.

The author was supported by the TUBITAK 1001 Project MFAG No. 118F410.

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VAN YUZUNCU YIL UNIVERSITY
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS
VAN
TURKEY
E-mail: hsmustafayev@yahoo.com