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VOLTERRA INTEGRAL OPERATORS AND LOGARITHMIC CONVEXITY

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1. Introduction and statement of results.

The purpose of this paper is to find out under what assumptions on Volterra integral operators of the form $u \to \int_0^t a(t-s)u(s)\,ds$ it follows that the function $t \to \log\big(a(t)\big)$ is convex. The related integral operators $u \to \int_0^t a(t-s)h(s)u(s)$ and $u \to \int_0^t a(t-s)g\big(u(s)\big)\,ds$ are also studied and certain properties of these operators are explored. The assumption that $\log\big(a(t)\big)$ is convex has several very desirable consequences from the point of view of Volterra integral equations, see e.g. [2], [3] and [5] and the references mentioned there. Hence it is reasonable to ask what relationships there are between the logarithmic convexity of the kernel and other properties of the integral operators under consideration.

The crucial assumption concerning these integral operators is that their inverses should be accretive in $L^1(\mathsf{R}^+;\mathsf{R})$ or $L^\infty(\mathsf{R}^+;\mathsf{R})$. Let X be a Banach space with norm $\|\cdot\|$ and let $A \subset X \times X$ be a relation on X, (for example a function $X \supset D(A) \to X$). Then A is said to be accretive if $[x_i, y_i] \in A$, (i.e. $y_i \in A(x_i)$), $i=1,2$, implies that

$$\|x_1 - x_2\| \leqq \|x_1 - x_2 + \lambda(y_1 - y_2)\| \qquad \text{for all} \quad \lambda > 0\,.$$

One says that A is m-accretive if for every $\lambda > 0$ and every $v \in X$, there exists $[x,y] \in A$ such that $x + \lambda y = v$. Note that $A - \omega I$ is m-accretive if and only if $(\omega\lambda + 1)(I + \lambda A)^{-1}$, $\lambda > 0$, is a nonexpansive function: $X \to X$. For further results on accretivity, see e.g. [1].

The following definitions will be needed. If $a \in C(\mathsf{R}^+;\, \mathsf{R}^+)$, $(\mathsf{R}^+ = [0,\infty))$, $\mu > 0$, $h \in C(\mathsf{R}^+;\, \mathsf{R})$ and $g \in C(\mathsf{R};\, \mathsf{R})$, then the integral operators K_μ, L_h, and M_g are defined as follows:

$$(1.1) \qquad (K_\mu u)(t) = \int_0^t a(t-s)^\mu u(s)\,ds, \qquad u \in L^1_{\mathrm{loc}}(\mathsf{R}^+;\, \mathsf{R}),\ t \in \mathsf{R}^+\,,$$

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$$(1.2)\qquad (L_h u)(t) = \int_0^t a(t-s)h(s)u(s)\,ds, \qquad u \in L^1_{\mathrm{loc}}(\mathsf{R}^+;\mathsf{R}),\ t \in \mathsf{R}^+ ,$$

$$(1.3)\qquad (M_g u)(t) = \int_0^t a(t-s)g(u(s))\,ds, \qquad u \in C(\mathsf{R}^+;\mathsf{R}),\ t \in \mathsf{R}^+ .$$

The main result of this paper is the following:

THEOREM 1. *Assume that*

$$(1.4)\qquad a \in C(\mathsf{R}^+;\mathsf{R}), \quad a \text{ is nonnegative and } a \neq 0 .$$

Then

$$(1.5)\qquad a \text{ is positive, nonincreasing and } \log(a) \text{ is convex}$$

if and only if

$$(1.6)\qquad K_\mu^{-1} - \|a^\mu\|_{L^1(\mathsf{R}^+)}^{-1} I \quad \text{is } m\text{-accretive in} \quad L^1(\mathsf{R}^+;\mathsf{R})$$
$$\text{or} \quad L^\infty(\mathsf{R}^+;\mathsf{R}) \quad \text{for every} \quad \mu \in (0,1] .$$

Here "I" denotes the identity operator, K_μ^{-1} is the inverse of K_μ and $\infty^{-1} = 0$. It is easily seen from the proof that if (1.5) holds, then $K_\mu^{-1} - \|a^\mu\|_{L^1(\mathsf{R}^+)}^{-1} I$ is m-accretive in $L^p(\mathsf{R}^+;\mathsf{R})$ for all p, $1 \leqq p \leqq \infty$, and all $\mu \in (0,1]$. It will also be seen that for the operators considered in Theorem 1, m-accretivity in one of the spaces $L^1(\mathsf{R}^+;\mathsf{R})$ or $L^\infty(\mathsf{R}^+;\mathsf{R})$ implies m-accretivity in the other one.

The main emphasis in Theorem 1 is on the fact that (1.6) implies (1.5) since the opposite implication follows almost directly from [5, Theorem 2, Corollary 1].

It is reasonable to ask whether (1.6) could be replaced by another, (perhaps simpler), condition involving for example the operators L_h or M_g. The next theorem shows that at least certain assumptions involving these operators are not sufficient to imply (1.5). Furthermore, we need some of the results below in the proof of Theorem 1.

THEOREM 2. *Assume that* (1.4) *holds. Then the following statements are equivalent*:

(1.7) $K_1^{-1} - \|a\|_{L^1(\mathsf{R}^+)}^{-1} I$ *is m-accretive in* $L^1(\mathsf{R}^+;\mathsf{R})$ *or* $L^\infty(\mathsf{R}^+;\mathsf{R})$,

(1.8) $L_h^{-1} - \|h\|_{L^\infty(\mathsf{R}^+)}^{-1} \|a\|_{L^1(\mathsf{R}^+)}^{-1} I$ *is m-accretive in* $L^\infty(\mathsf{R}^+;\mathsf{R})$ *for every nonnegative function* $h \in C(\mathsf{R}^+;\mathsf{R})$, $h \neq 0$ *and in* $L^1(\mathsf{R}^+;\mathsf{R})$ *for every nonnegative and nonincreasing function* $h \in C(\mathsf{R}^+;\mathsf{R})$, $h \neq 0$, (1.9) *$M_g^{-1}-\gamma^{-1}\|a\|_{L^1(\mathsf{R}^+)}^{-1}I$ is m-accretive in $C(\mathsf{R}^+;\mathsf{R})$ (with L^∞-norm) for every nondecreasing and nonconstant function $g\in C(\mathsf{R};\mathsf{R})$ such that $g(x_0)=0$ for some $x_0\in\mathsf{R}$ if*

$$\gamma = \sup\{|g(x)-g(y)|/|x-y| \mid x,y\in\mathsf{R},\ x\neq y\}\ ,$$

(1.10) *$a(0)>0$ and there exists a nonnegative and nonincreasing function b on $(0,\infty)$ such that $b\in L^1_{\mathrm{loc}}(\mathsf{R}^+;\mathsf{R})$ and*

$$a(t)/a(0)+\int_0^t b(t-s)a(s)\,ds = 1, \qquad t\geqq 0\,.$$

It follows from results in e.g. [4, Lemma 13] that if $b\in L^1_{\mathrm{loc}}(\mathsf{R}^+)$ is nonnegative and nonincreasing and $a(0)>0$, then the equation in (1.10) has a unique solution that satisfies (1.4). Thus if one takes for example $b(t)=1$, $t\in[0,1]$, $b(t)=0$, $t>1$, $a(0)=1$, then one sees that a is not even nonincreasing, so that (1.5) cannot be satisfied although (1.7)–(1.10) hold, in other words, (1.7) does not imply (1.6).

Unfortunately the condition (1.10) is not very informative, that is, in general it is not too hard to check if (1.5) is satisfied but given a function a, it can be very difficult to see if (1.10) holds.

Several of the assumptions above can be formulated in terms of resolvents of Volterra equations. Thus we will for example see that (1.7)–(1.10) are equivalent to the statement that if r_λ, $\lambda>0$, is the solution of the equation

$$\lambda r_\lambda(t)+\int_0^t a(t-s)r_\lambda(s)\,ds = a(t), \qquad t\geqq 0\,, \tag{1.11}$$

then

$$\|r_\lambda\|_{L^1(\mathsf{R}^+)} \leqq (\lambda\|a\|_{L^1(\mathsf{R}^+)}^{-1}+1)^{-1}\,.$$

This fact implies in turn that $r_\lambda(t)\geqq 0$ for all $t\geqq 0$ and all $\lambda\geqq 0$. For more information on nonnegative resolvents, see e.g. [2] or [5].

2. Proof of Theorem 2.

Assume that (1.7) holds. We are going to show that (1.10) follows. From (1.7) one sees that if $u\in L^1(\mathsf{R}^+;\mathsf{R})$ or $L^\infty(\mathsf{R}^+;\mathsf{R})$ and $\lambda>0$, then

$$((I+\lambda K_1^{-1})^{-1}u)(t) = \int_0^t r_\lambda(t-s)u(s)\,ds, \qquad t\in\mathsf{R}^+\,, \tag{2.1}$$

where r_λ is the solution of the equation (1.11).

Thus one sees that (1.7) is equivalent to the statement that

$$\|r_\lambda\|_{L^1(\mathbf{R}^+)} \leqq (\lambda\|a\|_{L^1(\mathbf{R}^+)}^{-1}+1)^{-1}, \qquad \lambda>0 . \tag{2.2}$$

It follows that

$$\int_0^\infty e^{-\sigma t}|r_\lambda(t)|\,dt < 1 \quad \text{if} \quad \sigma>0 ,$$

and then $e^{-\sigma t}a(t) \in L^1(\mathbf{R}^+)$ by (1.11) and

$$\int_0^\infty e^{-\sigma t} r_\lambda(t)\,dt = \left(\lambda\left(\int_0^\infty e^{-\sigma t}a(t)\,dt\right)^{-1}+1\right)^{-1}, \qquad \sigma>0,\ \lambda>0 .$$

From this equation combined with (2.2) we see that

$$\|r_\lambda\|_{L^1(\mathbf{R}^+)} \leqq \int_0^\infty r_\lambda(t)\,dt ,$$

so that

$$r_\lambda(t) \geqq 0, \qquad t \in \mathbf{R}^+, \quad \lambda > 0 . \tag{2.3}$$

Since $\|r_\lambda\|_{L^1(\mathbf{R}^+)} \leqq 1$ it is a consequence of (1.11) that

$$\operatorname{Re}\int_0^\infty e^{-(\sigma+i\tau)t}a(t)\,dt \geqq 0, \qquad \sigma > 0, \quad \tau \in \mathbf{R}$$

so that a is a positive definite function (in the Bochner sense), see [7]. This implies that $a(0)>0$. In fact it is true that

$$a(t) > 0, \qquad t \in \mathbf{R}^+ , \tag{2.4}$$

because if (2.4) is false, then there exists a number $t_0>0$ such that $a(t_0)=0 < a(t)$, $t \in [0, t_0)$. But then one gets a contradiction from (1.11) and (2.3).

Now (1.10) follows from the proof of [3, Lemma 2.1], (see [3, p. 202] and note that the assumption (1.5) is only used there to derive (2.2) and (2.3)), where it is shown that

$$\lambda^{-1}\int_0^t \left(1-\int_0^s r_\lambda(\tau)\,d\tau\right)ds \to a(0)^{-1}+\int_0^t b(s)\,ds \qquad \text{as } \lambda \to 0 , \tag{2.5}$$

and b has the desired properties. If one integrates both sides of the equation (1.11) twice one gets

$$\lambda^{-1}\int_0^t a(t-s)\int_0^s \left(1-\int_0^\tau r_\lambda(\sigma)\,d\sigma\right)d\tau\,ds$$
$$= \int_0^t \left(1-\int_0^s \left(1-\int_0^\tau r_\lambda(\sigma)\,d\sigma\right)d\tau\right)ds$$

and therefore (2.5) gives the equation in (1.10).

Next we assume that (1.10) holds and first we prove that $L_h^{-1} - \|h\|_{L^\infty(\mathsf{R}^+)}^{-1}\|a\|_{L^1(\mathsf{R}^+)}^{-1}I$ is m-accretive in $L^\infty(\mathsf{R}^+;\mathsf{R})$ if h is continuous, nonnegative and $h \not\equiv 0$. Thus we must show that if $\lambda > 0$ and $v \in L^\infty(\mathsf{R}^+;\mathsf{R})$, then there exists a solution u of the equation

$$\lambda u(t) + \int_0^t a(t-s)h(s)u(s)\,ds = \int_0^t a(t-s)h(s)v(s)\,ds, \qquad t \in \mathsf{R}^+, \tag{2.6}$$

such that

$$\|u\|_{L^\infty(\mathsf{R}^+)} \leqq (\lambda\|h\|_{L^\infty(\mathsf{R}^+)}^{-1}\|a\|_{L^1(\mathsf{R}^+)}^{-1}+1)^{-1}\|v\|_{L^\infty(\mathsf{R}^+)}\,. \tag{2.7}$$

Suppose that this can be done under the additional assumptions that

$$h, v \in C^1(R^+;R), \qquad h(t) > 0,\ t \in \mathsf{R}^+\,. \tag{2.8}$$

Since a is continuous if follows from Gronwall's lemma that the solution u of (2.6) depends continuously, (uniformly on compact intervals), on the functions h and v (if h converges uniformly on R^+ and v in $L^1(0,T;\mathsf{R})$ for all $T>0$). Therefore it is sufficient to consider the case when (2.8) holds and it is easily seen that then the equation (2.6) has a unique continuously differentiable solution.

We conclude from the equation in (1.10) and (2.6) that

$$h(t)(v(t)-u(t)) = \lambda u'(t)/a(0) + \lambda d/dt\left(\int_0^t b(t-s)u(s)\,ds\right), \qquad t \in \mathsf{R}^+\,. \tag{2.9}$$

Let $T>0$ be arbitrary and assume that $t_0 \in [0,T]$ is such that

$$u(t_0) = \max_{t\in[0,T]} u(t) > 0\,.$$

Then, $(u(0)=0)$,

$$u'(t_0) \geqq 0 \tag{2.10}$$

and since b is nonincreasing,

$$\left(b(t) = -\int_{(t,\infty)} db(s) + b(\infty)\right)$$

$$\frac{d}{dt}\left(\int_0^t b(t-s)u(s)\,ds\right)\Bigg|_{t=t_0} \tag{2.11}$$

$$= b(1/n)u(t_0) + \int_{(1/n,t_0)} u(t_0-s)\,db(s) + \int_0^{1/n} u'(t_0-s)\big(b(s)-b(1/n)\big)\,ds$$

$$\geqq b(t_0)u(t_0)+\int_0^{1/n} u'(t_0-s)(b(s)-b(1/n))\,ds$$

$$\to b(t_0)u(t_0) \quad \text{as} \quad n\to\infty\,.$$

It follows from (1.10) that

$$\lim_{t\to\infty} b(t) = \|a\|_{L^1(\mathbf{R}^+)}^{-1}$$

and hence we deduce from (2.9)–(2.11) that

$$\max_{t\in[0,T]} u(t) \leqq (\lambda\|h\|_{L^\infty(\mathbf{R}^+)}^{-1}\|a\|_{L^1(\mathbf{R}^+)}^{-1}+1)^{-1}\max\left\{0,\ \max_{t\in[0,T]} v(t)\right\}. \tag{2.12}$$

A similar argument gives this inequality with max replaced by min and as $T>0$ was arbitrary we get (2.7).

To establish the second part of (1.8), where we assume that h is in addition nondecreasing, we have to show that if $v\in L^1(\mathbf{R}^+;\mathbf{R})$, then there exists a solution u of (2.6) such that

$$\|u\|_{L^1(\mathbf{R}^+)} \leqq (\|h\|_{L^\infty(\mathbf{R}^+)}^{-1}\|a\|_{L^1(\mathbf{R}^+)}^{-1}+1)^{-1}\|v\|_{L^1(\mathbf{R}^+)}\,. \tag{2.13}$$

For the same reasons as above we may furthermore assume that (2.8) holds and hence (2.9) and the first equality in (2.11), (for arbitrary $t_0>0$, $n>1/t_0$), hold. Therefore, if $T>0$ is arbitrary, it follows since b is nomincreasing and h nondecreasing that

$$\begin{aligned} &\int_{1/n}^{T}\left(u(t)+\lambda h(t)^{-1}\frac{d}{dt}\int_0^t b(t-s)u(s)\,ds\right)\operatorname{sign}(u(t))\,dt \\ &\geqq (1+\lambda h(T)^{-1}b(1/n))\int_0^T |u(t)|\,dt \\ &- \lambda h(T)^{-1}(b(1/n)-b(T))\int_0^T |u(t)|\,dt \\ &- \int_{1/n}^T\left|\int_0^{1/n} u'(t-s)(b(s)-b(1/n))\,ds\right|dt \to \\ &(1+\lambda h(T)^{-1}b(T))\int_0^T |u(t)|\,dt \quad \text{as} \quad n\to\infty\,. \end{aligned} \tag{2.14}$$

On the other hand, one obtains from an integration by parts that

$$\int_0^T (v(t)-\lambda h(t)^{-1}u'(t))\operatorname{sign} u(t)\,dt \tag{2.15}$$

$$\leqq \int_0^T |v(t)|\,dt - \lambda h(T)^{-1}|u(T)| - \lambda \int_0^T h'(t)h(t)^{-2}|u(t)|\,dt$$

$$\leqq \int_0^T |v(t)|\,dt$$

where the facts that h is positive and nondecreasing were again used. Since $T>0$ was arbitrary we get (2.13) from (2.9), (2.14), and (2.15). Thus we have shown that (1.10) implies (1.8).

Next we prove that (1.8) implies (1.9). First we have to establish that if $\lambda>0$ and $v \in L^\infty(\mathrm{R}^+;\mathrm{R}) \cap C(\mathrm{R}^+;\mathrm{R})$, then there exists a function $u \in L^\infty(\mathrm{R}^+;\mathrm{R}) \cap C(\mathrm{R}^+;\mathrm{R})$ such that $u+\lambda M_g^{-1}u=v$ or by (1.3) equivalently that

$$u(t) = \int_0^t a(t-s)g\big((v(s)-u(s))/\lambda\big)\,ds, \qquad t \in \mathrm{R}^+ . \tag{2.16}$$

It follows from standard arguments, see e.g. [6, pp. 87, 95] that there exists a local, continuous solution of (2.16) and if this solution is bounded, then it can be continued to R^+. For as long as the solution exists we can define h by

$$g\big((v(s)-u(s))/\lambda\big) = \lambda^{-1}h(s)\big(v(s)-u(s)+f(s)\big)$$

where

$$f(s) = \lambda\big(-x_0+\operatorname{sign}\big(v(s)-u(s)-\lambda x_0\big)\big)$$

so that h is then a continuous, nonnegative function, (since g is nondecreasing, $g(x_0)=0$). But then an application of (1.8) and the definition of accretivity to (2.16), shows that u remains bounded, (by $\|v\|_{L^\infty(\mathrm{R}^+)}+\lambda|x_0|+\lambda$), and hence there exists a global bounded solution.

The second result that must be established is that

$$\|u_1-u_2\|_{L^\infty(\mathrm{R}^+)} \leqq (\lambda\gamma^{-1}\|a\|_{L^1(\mathrm{R}^+)}^{-1}+1)^{-1}\|v_1-v_2\|_{L^\infty(\mathrm{R}^+)} \tag{2.17}$$

if $\lambda>0$, $u_i, v_i \in L^\infty(\mathrm{R}^+;\mathrm{R}) \cap C(\mathrm{R}^+;\mathrm{R})$ and $(I+\lambda L_g^{-1})^{-1}v_i=u_i$, $i=1,2$. But then

$$\begin{aligned} u_1(t)-u_2(t) = \int_0^t a(t-s)\big(g\big((v_1(s)-u_1(s))/\lambda\big) \\ -g\big((v_2(s)-u_2(s))/\lambda\big)\,ds, \qquad t \in \mathrm{R}^+ . \end{aligned} \tag{2.18}$$

If $\varepsilon>0$ is arbitrary, then

$$\begin{aligned} &g\big((v_1(s)-u_1(s))/\lambda\big)-g\big((v_2(s)-u_2(s))/\lambda\big) \\ &\quad= \lambda^{-1}h(s)\big(v_1(s)-v_2(s)+\varepsilon\operatorname{sign}\big(v_1(s)-v_2(s)-u_1(s)+u_2(s)\big)\big) \\ &\quad- \lambda^{-1}h(s)\big(u_1(s)-u_2(s)\big), \qquad s \in \mathrm{R}^+ , \end{aligned} \tag{2.19}$$

where h is a continuous nonnegative function such that $|h(s)| \leqq \gamma$, $s \in \mathbf{R}^+$, (since g is nondecreasing). If we combine (1.8), (2.18), and (2.19) then we get the desired conclusion as ε was arbitrary.

Finally assume that (1.9) holds. If $g(x) \equiv x$, then it follows that $K_1^{-1} - \|a\|_{L^1(\mathbf{R}^+)}^{-1} I$ is m-accretive in $C(\mathbf{R}^+; \mathbf{R})$ (with sup-norm) and therefore (2.1) shows that (2.2) must hold. But it was observed above that (2.2) implies (1.7); in fact, one gets the m-accretivity in both $L^1(\mathbf{R}^+; \mathbf{R})$ and $L^\infty(\mathbf{R}^+; \mathbf{R})$. This completes the proof of Theorem 2.

3. Proof of Theorem 1.

From the results in the previous section, see (2.1) and (2.2), we conclude that (1.6) follows from (1.5) provided that

$$\|r_{\lambda,\mu}\|_{L^1(\mathbf{R}^+)} \leqq (\lambda \|a^\mu\|_{L^1(\mathbf{R}^+)} + 1)^{-1},$$

where $r_{\lambda,\mu}$ is the solution of the equation

$$\lambda r_{\lambda,\mu}(t) + \int_0^t a(t-s)^\mu r_{\lambda,\mu}(s)\, ds = a(t)^\mu, \; t \in \mathbf{R}^+, \quad \mu \in (0,1], \quad \lambda > 0.$$

But this assertion is a direct consequence of (1.5) and [5, Theorem 2, Corollary 1].

Assume that (1.6) holds. Taking $\mu = 1$ we have (1.7) and hence we obtain (2.4). We will show that for every $\mu \in (0,1]$ and for all choices of points $0 = t_0 < t_1 < t_2 < t_3$ we have

$$\sum_{m=0}^{2} (-1)^m \sum_{0=j_0<\dots<j_{m+1}=3} \prod_{k=0}^{m} \left(a(t_{j_{k+1}} - t_{j_k})/a(0)\right)^\mu \geqq 0 \tag{3.1}$$

If we take $\mu = 0$ in the expression on the left hand side in (3.1), then we get 0. Hence, if we divide both sides of the inequality in (3.1) by μ, then we get when $\mu \to 0$, (differentiate the left hand side in (3.1) with respect to μ),

$$\begin{aligned} 0 &\leqq \sum_{m=0}^{2} (-1)^m \sum_{0=j_0<\dots<j_{m+1}=3} \sum_{k=0}^{m} \log\left(a(t_{j_{k+1}} - t_{j_k})/a(0)\right) \\ &= \log(a(t_3)) - \log(a(t_3 - t_1)) - \log(a(t_2)) + \log(a(t_2 - t_1)). \end{aligned}$$

Since $0 < t_1 < t_2 < t_3$ are arbitrary, this inequality shows that $\log(a(t))$ is convex. Since (1.7) holds we have by Theorem 2 also (1.10) so that $a(t) \leqq a(0)$, $t \in \mathbf{R}^+$ and hence $a(t)$ must be nonincreasing since it is convex.

Thus it remains to establish (3.1). We will show that (1.8) implies (3.1) with

$\mu=1$ and since (1.6) implies, (by Theorem 2), that (1.8) holds with $a(t)$ replaced by $a(t)^\mu$, we obtain (3.1).

Let h be a continuously differentiable and positive function on $\mathbf{R}^+$. It is easy to check that

$$((I+L_h^{-1})^{-1}v)(t) = \int_0^t r_h(t,s)v(s)\,ds, \qquad t \in \mathbf{R}^+, \tag{3.2}$$

where r_h is the solution of the equation

$$r_h(t,s) = a(t-s)h(s) - \int_s^t a(t-u)h(u)r_h(u,s)\,du, \qquad 0 \leqq s \leqq t. \tag{3.3}$$

(It follows from [6, Theorem 3.1, p. 202] that the equation (3.3) has a continuous solution). Let v be a continuously differentiable nonpositive function. If u is the solution of equation (2.6), then u is nonpositive by (2.12) and hence we get from (1.2) and (2.6) that

$$u(t) = \int_0^t r_h(t,s)v(s)\,ds \leqq 0, \qquad t \in \mathbf{R}^+.$$

This shows that $r_h(t,s) \geqq 0$, $0 \leqq s \leqq t$. Therefore we obtain from (3.3), when $p_h(t) = r_h(t,0)/h(0)$,

$$0 \leqq p_h(t) = a(t) - \int_0^t a(t-s)h(s)p_h(s)\,ds, \qquad t \in \mathbf{R}^+. \tag{3.4}$$

We want to show that p_h depends continuously on h and by (3.4) we have

$$\begin{aligned} |p_{h_1}(t) - p_{h_2}(t)| \leqq& \int_0^t |a(t-s)|\,|h_1(s)-h_2(s)|\,|p_{h_1}(s)|\,ds \\ &+ \int_0^t a(t-s)|h_2(s)|\,|p_{h_1}(s)-p_{h_2}(s)|\,ds, \qquad t \in \mathbf{R}^+. \end{aligned}$$

Applying Gronwall's inequality we deduce, since

$0 \leqq a(t) \leqq a(0), \quad 0 \leqq p_{h_1}(t) \leqq a(t)$

and $h_2(t) \geqq 0$, that

$$|p_{h_1}(t) - p_{h_2}(t)| \leqq a(0)^2 \int_0^t |h_1(s)-h_2(s)|\,ds \exp\left(a(0)\int_0^t h_2(s)\,ds\right), \qquad t \in \mathbf{R}^+.$$

This shows that we can take the function h in (3.4) to be for example

$$h(t) = (\lambda_1\sigma_1)^{-1}\chi_{[t_1,t_1+\sigma_1]}(t) + (\lambda_2\sigma_2)^{-1}\chi_{[t_2,t_2+\sigma_2]}(t), \qquad t \in \mathbf{R}^+ \tag{3.5}$$

where $0<t_1<t_2$ are arbitrary $\lambda_i, \sigma_i>0$, $i=1,2$, and $\chi_{[a,b]}$ denotes the characteristic function of the interval $[a,b]$.

We will let $\lambda_i, \sigma_i \to 0$, $i=1,2$ and for this reason we need the following auxiliary result.

LEMMA 1. *Let the assumptions of* Theorem 1 *hold. If f is a bounded measurable function on* $\mathbf{R}^+$, *f is continuous at the point $T>0$, $\lambda, \sigma>0$ and if $q_{\lambda,\sigma}$ satisfies*

$$(3.6)\quad q_{\lambda,\sigma}(t) = f(t) - \int_0^t a(t-s)(\lambda\sigma)^{-1}\chi_{[T,T+\sigma]}(s)q_{\lambda,\sigma}(s)\,ds \geqq 0, \qquad t \in \mathbf{R}^+,$$

then

$$0 \leqq \lim_{\sigma\to 0}\lim_{\lambda\to 0} q_{\lambda,\sigma}(t) = \begin{cases} f(t), & t \leqq T \\ f(t) - a(t-T)f(T)/a(0), & t>T \end{cases}$$

with uniform convergence on compact subsets of $[0,T)\cup(T,\infty]$.

PROOF. When $t \leqq T$ we have $q_{\lambda,\sigma}(t) = f(t)$ and if $t \in (T, T+\sigma]$, then

$$q_{\lambda,\sigma}(t) = f(t) - (\lambda\sigma)^{-1}\int_T^t a(t-s)q_{\lambda,\sigma}(s)\,ds.$$

Applying (1.11) we obtain the solution in the form

$$q_{\lambda,\sigma}(t) = f(t) - \int_T^t r_{\lambda\sigma}(t-s)f(s)\,ds, \qquad t \in (T, T+\sigma].$$

Therefore we get when $t > T+\sigma$

$$(3.7)\quad \int_0^t a(t-s)(\lambda\sigma)^{-1}\chi_{[T,T+\sigma]}(s)q_{\lambda,\sigma}(s)\,ds$$

$$= a(t-T)(\lambda\sigma)^{-1}\int_T^{T+\sigma}\left(f(s) - \int_T^s r_{\lambda\sigma}(s-\tau)f(\tau)\,d\tau\right)ds$$

$$+ (\lambda\sigma)^{-1}\int_T^{T+\sigma}(a(t-s)-a(t-T))q_{\lambda,\sigma}(s)\,ds.$$

With the aid of (2.4) and (3.6) we deduce that

$$(3.8)\quad \left|(\lambda\sigma)^{-1}\int_T^{T+\sigma}(a(t-s)-a(t-T))q_{\lambda,\sigma}(s)\,ds\right|$$

$$\leqq \sup_{s\in[T,T+\sigma]}(|a(t-s)-a(t-T)|/a(t-s))\int_T^{T+\sigma}a(t-s)(\lambda\sigma)^{-1}q_{\lambda,\sigma}(s)\,ds$$

$$\leqq \sup_{s\in[T,T+\sigma]}(|a(t-s)-a(t-T)|/a(t-s))f(t).$$

On the other hand we have by (2.2) and (3.6) since

$$f(s)-\int_T^s r_{\lambda\sigma}(s-\tau)f(\tau)\,d\tau = d/ds\int_T^s\left(1-\int_0^{s-\tau} r_{\lambda\sigma}(v)\,dv\right)f(\tau)\,d\tau\,,$$

$$\left|\int_T^{T+\sigma}(\lambda\sigma)^{-1}\left(f(s)-\int_T^s r_{\lambda\sigma}(s-\tau)f(\tau)\,d\tau\right)ds-f(T)/a(0)\right|$$

$$\leqq (\lambda\sigma)^{-1}\left|\int_T^{T+\sigma}\left(1-\int_0^{T+\sigma-s} r_{\lambda\sigma}(\tau)\,d\tau\right)(f(s)-f(T))\,ds\right|$$

$$+f(T)\left|(\lambda\sigma)^{-1}\int_T^{T+\sigma}\left(1-\int_0^{T+\sigma-s} r_{\lambda\sigma}(\tau)\,d\tau\right)ds-1/a(0)\right|$$

$$\leqq \sup_{s\in[T,T+\sigma]}|f(s)-f(T)|(\lambda\sigma)^{-1}\int_0^{\sigma}\left(1-\int_0^{s} r_{\lambda\sigma}(\tau)\,d\tau\right)ds$$

$$+f(T)\left|(\lambda\sigma)^{-1}\int_0^{\sigma}\left(1-\int_0^{s} r_{\lambda\sigma}(\tau)\,d\tau\right)ds-1/a(0)\right|\,.$$

Using this inequality, (1.4), (2.4), (2.5), (3.7), and (3.8) we conclude that the assertion of Lemma 1 holds.

To derive (3.1) proceed as follows: Let h be the function given in (3.5) and let first $\lambda_1 \to 0$, then $\sigma_1 \to 0$ and apply Lemma 1 with $f(t)=a(t)$ and $T=t_1$ to equation (3.4). It is easy to check that $p_h(t)$ converges, for $t>t_1$, to a nonnegative solution $q_{\lambda_2,\mu_2}(t)$ of the equation

$$x(t) = a(t)-a(t-t_1)a(t_1)/a(0)$$
$$-\int_0^t a(t-s)(\lambda_2\sigma_2)^{-1}\chi_{[t_2,t_2+\sigma_2]}(s)x(s)\,ds, \qquad t>t_1\,.$$

If we now let $\lambda_2 \to 0$ then $\mu_2 \to 0$ and again apply Lemma 1, (this time with $f(t)=a(t)-a(t-t_1)a(t_1)/a(0)$, $t>t_1$, and $T=t_2$), then we obtain (3.1) with $\mu=1$ when we evaluate $\lim_{\sigma_2\to 0}\lim_{\lambda_2\to 0} q_{\lambda_2,\sigma_2}(t_3)$. This completes the proof of Theorem 1.

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