SOME INFINITE FAMILIES
OF U-HYPERSONFACES

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Dedicated to the memory of Adrienne Hand

0. Introduction.

Since Milnor first calculated the complex bordism ring $\text{MU}_\ast$ [6] and thus
initiated the study of the bordism functor $\text{MU}_\ast(\cdot)$, its algebraic properties have
been exhaustively investigated. However, many geometric questions concerning the ring remain unanswered, and, with current techniques, unanswerable.

We consider here one such question, inspired by work in [4] and [11]
connected with finding the least embedding codimension of a representative
for a given bordism class. More precisely, we investigate those bordism classes
carried by hypersurfaces, i.e. submanifolds of $\mathbb{R}^{n+1}$ with codimension 1.

The structure defined by these classes seems interestingly rich, and this is a
natural consequence of our observations in section 1 relating the problem to
computing $\pi_\ast(\Omega(S^1 \wedge \text{SO}/U))$. Of course, this also tells us that we cannot
expect a complete solution!

However, the methods we use, and the small gains we make, are entirely
inspired by the geometry of $\text{MU}_\ast$, and might not otherwise suggest themselves
when attacking the homotopy problem by more conventional means.

In section 1 we give a precise formulation of the hypersurface question, and
relate it to $\Omega(S^1 \wedge \text{SO}/U)$ via a Pontrjagin–Thom construction. In section 2
we discuss in detail the topology of our hypersurfaces, and explain how to
describe their $U$ structures.

We develop our calculational machinery in section 3, showing in particular
how the problem collapses over the rationals, and proving an interesting result
(3.8) on hypersurfaces which are stably a wedge of spheres.

Our main section is section 4, where we investigate hypersurfaces which arise
as the boundaries of regular neighbourhoods of embedded 2 cell complexes. It
is here that we uncover the infinite families noted in our title. An interesting
relationship emerges with the well-known summands $\text{Im} J$ of the stable
homotopy groups of spheres.

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In the concluding section we tie up some loose ends, and describe work currently in progress to improve our understanding of the hypersurface ring.

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1. Preliminary constructions.

Consider, by analogy with [4] and [11], the bordism group $\Omega^U_{n,k}$ of $n$-manifolds $M$ embedded in $S^{n+k}$ for some fixed $k$, and with a $U$ structure (i.e. a lift to $BU$) on their stable normal bundle. The bordisms are required to be of codimension $k$ also, in $S^{n+k} \times I$.

The images of $\Omega^U_{*,k}$ in $\text{MU}_{*}$ we shall denote by $F^k_*$. For stability reasons $F^2_{2n} = \text{MU}_{2n}$, so $\{F^k_* : k = 0, 1, 2, \ldots \}$ gives a finite filtration of $\text{MU}_{*}$ by increasing subgroups. Moreover, with respect to the cartesian product structure, $F^k_* \cdot F^l_* \subseteq F^{k+l}$.

Note that $F^k_*$ arises from $\Omega^U_{*,k}$ merely by allowing the bordisms to be arbitrary $U$ manifolds.

We are interested here in the groups $F^1_*$. For a detailed analysis of the higher filtration groups, see [2]. It suffices here to note

(1.1) **Proposition.** $\Omega^U_{*,1}$ is a ring, and each $\Omega^U_{*,k}$ is a module over this ring for $k > 1$.

**Proof.** Each $x \in \Omega^U_{n,1}$ is represented by some hypersurface $M^n \subset S^{n+1}$ carrying a $U$ structure (U-hypersurface, for brevity). Similarly, $y \in \Omega^U_{m,k}$ is represented by some $N^m \subset S^{n+k}$.

So $xy \in \Omega^U_{n+m,k+1}$ is represented by the product $U$-structure on $M^n \times N^m$. But $M^n$ is oriented by its $U$ structure, so there is a compatible embedding $M^n \times R \subset S^{n+1}$, since any hypersurface has trivial normal bundle. So we have

$$M^n \times N^m \subset M^n \times R^{m+k} = M^n \times R \times R^{m+k-1} \subset S^{n+1} \times R^{m+k-1}.$$  

This yields an embedding in $S^{n+m+k}$, which on suspension is isotopic to the product embedding $M^n \times N^m \subset S^{n+m+k+1}$. So $xy \in \Omega^U_{n+m,k}$.

Note that the construction also applies to bordisms, and therefore is independent of the representatives chosen for $x$ and $y$.

Restricting to $k=1$ gives the first part of the result: the second then follows by choosing arbitrary $k$.

(1.2) **Corollary.** $F^1_*$ is a graded subring of $\text{MU}_*$, and $F^k_*$ is a module over this ring for $k > 1$. 
PROOF. Apply (1.1) to images in $\text{MU}_\ast$.

Now consider $F_n^0$ awhile. The only closed codimension zero submanifold of $S^n$ is $S^n$ itself, so $\Omega^U_{n,0} \cong \pi_n(\text{SO/U})$ and $F_n^0$ is represented by bordism classes containing U-structures on $S^n$. This is precisely the group $\text{Im} J_n \subset \text{MU}_n$ determined in [8], being the image of $J : \pi_\ast(\text{SO/U}) \to \text{MU}_\ast$. The result is

\begin{equation}
\text{Im} J_{2n} \cong \mathbb{Z} \quad \text{if } n \text{ odd}
= 0 \quad \text{if } n \text{ even}.
\end{equation}

In addition, if $n = 1$ or $n \equiv 3(4)$, the group is a direct summand, whereas if $n > 1$ and $n \equiv 1(4)$ it is divisible by 2.

Also in $\text{MU}_n$, as explained in [8], there is the subring $J_n^S$ consisting of the image of the stable $J$ homomorphism $J^S : \pi_\ast^S(\text{SO/U}) \to \text{MU}_\ast$. This contains elements representable on frameable manifolds.

(1.4) NOTATION. Let $\text{Hyp}_\ast$ denote the subring $F_1^1 \subset \text{MU}_\ast$, and let $R \text{Im} J_\ast$ denote the polynomial subring generated by $\text{Im} J_\ast$.

(1.5) PROPOSITION. There are inclusions of subrings

\[ R \text{Im} J_\ast \subset \text{Hyp}_\ast \subset J^S_\ast. \]

PROOF. The first inclusion holds since a product of spheres is a hypersurface, and the second since a hypersurface is always frameable.

One of our main interests is the extent to which the above inclusions are strict.

Before proceeding, it is important to record the homotopy interpretation of the above geometry. We shall write $\Omega^\infty \text{MU}(\infty)$ for the space $\lim \Omega^{2n} \text{MU}(n)$, and $QX$ for the space $\lim \Omega^n(S^n \wedge X)$.

Now recall (e.g. from [8]) the commutative triangle

\[
\begin{array}{ccc}
\text{SO/U} & \xrightarrow{j} & \Omega^\infty \text{MU}(\infty) \\
Q(\text{SO/U}_+) & \xrightarrow{j^S} & \\
\end{array}
\]

where $j^S$ is the infinite loop extension of $j$. Then on homotopy groups $j_\ast = J$, whilst $j^S_\ast = j^S$.

We may restrict $j^S$ to a map $j'$ defined on $\Omega(S^1 \wedge (\text{SO/U}_+))$. 
(1.6) Proposition. There is an isomorphism

\[ \varphi : \Omega_{n,1}^U \to \pi_n(\Omega(S^1 \wedge SO/U)) \],

and the map \( \Omega_{n,1}^U \to MU_n \) is induced by \( j' \) (and so may be written as \( J'=j'_* \)).

Proof. To define \( \varphi \), suppose given a \( U \)-hypersurface \( M \subset S^{n+1} \). It has normal bundle trivialised by the orientation, so the \( U \)-structure corresponds to a map \( \tilde{v} : M \to SO/U \). Performing a Pontrjagin–Thom collapse \( c \) onto a tubular neighbourhood of \( M \) yields the composition

\[ S^{n+1} \xrightarrow{e} S^1 \wedge M_+ \xrightarrow{\tilde{v}} S^1 \wedge SO/U_+ . \]

This process sends bordisms to homotopies, so, after adjoining, \( \varphi \) is well defined.

To construct \( \varphi^{-1} \), start by adjoining a map \( S^n \to \Omega(S^1 \wedge SO/U_+) \) to a map \( S^{n+1} \to S^1 \wedge SO/U_+ \). Making this transverse to \( SO/U \) produces (up to bordism) an oriented hypersurface \( M \times R \subset S^{n+1} \). \( M \) is equipped with a map into \( SO/U \), applying which to the normal line yields a compatible \( U \)-structure. Such a \( \varphi^{-1} \) sends homotopies to bordisms, and so is well defined.

Clearly \( \varphi \) and \( \varphi^{-1} \) are mutually inverse, whilst \( j' \) forgets the restrictive nature of the bordism.

The other filtration groups \( F^k_* \) have a more complicated homotopy interpretation, fully documented in [2].

Once (1.6) is established, we should note that the product structure in \( \Omega_{n,1}^U \) is induced by smashing two homotopy classes together via the map

\[ \mu : \Omega(S^1 \wedge SO/U_+) \wedge \Omega(S^1 \wedge SO/U_+) \to \Omega(S^1 \wedge SO/U_+) . \]

\( \mu \) is defined on the pair \((l_1,l_2)\) to be the composite

\[ S^1 \xrightarrow{l_1} S^1 \wedge SO/U_+ \xrightarrow{l_2 \wedge 1} S^1 \wedge SO/U_+ \wedge SO/U_+ \xrightarrow{1 \wedge 0} S^1 \wedge SO/U_+ . \]

Under \( j' \), \( \mu \) maps to \( \wedge \) on \( \Omega^\infty MU(\infty) \) (since on spheres, \( \wedge \) and composition are stably homotopic).

(1.7) Notes. (i) So far as \( Hyp_* \) and \( J^S_* \) are concerned, it is sufficient to limit attention to

\[ \Omega(S^1 \wedge SO/U) \xrightarrow{j'} \Omega^\infty MU(\infty) \]

Under \( j' \), \( \mu \) maps to \( \wedge \) on \( \Omega^\infty MU(\infty) \) (since on spheres, \( \wedge \) and composition are stably homotopic).
and discard the disjoint base point, since $\pi_\ast(QS^0)$ is torsion (in positive dimensions) and so maps to zero in $\text{MU}_\ast$.

(ii) Since $(\text{SO}/U, \oplus)$ is an infinite loop space (e.g. see [5]) there are retractions

$$
\begin{align*}
\Omega(S^1 \wedge \text{SO}/U) & \xrightarrow{r'} \text{SO}/U \\
Q(\text{SO}/U) & \xrightarrow{r^S} \text{SO}/U
\end{align*}
$$

which exhibit $\pi_\ast(\text{SO}/U)$ as a direct summand of $\Omega^U_{\ast, 1}$ and of $\pi^S_\ast(\text{SO}/U)$. Thus $\text{Im} J_\ast$ is a direct summand of $\text{Hyp}_\ast$ and $J^S_\ast$.

The geometrical significance of these facts will become clear in section 3.

To conclude, we explain how the complementary summand to $\pi_\ast(\text{SO}/U)$ in $\Omega^U_{\ast, 1}$ (and hence to $\text{Im} J_\ast$ in $\text{Hyp}_\ast$) has an interesting interpretation.

For any topological group $G$ (and in fact for any loop space), $G$ acts on the join $G \ast G$ to give a principal $G$ bundle

$$G \to G \ast G \to S^1 \wedge G$$

(see [7]). The projection is the hopf construction $H(m)$ on the group multiplication $m: G \times G \to G$.

Extending the sequence, we obtain a fibration

$$\Omega(G \ast G) \to \Omega(S^1 \wedge G) \xrightarrow{r} G$$

with splitting map $G \to \Omega(S^1 \wedge G)$ the standard inclusion.

Applying this to $G = \text{SO}/U$, $r$ becomes $r'$ of (1.7). We deduce

\begin{equation}
(1.8) \text{ Proposition.} \quad \Omega^U_{\ast, 1} \cong \pi_\ast(\text{SO}/U) \oplus \pi_\ast(\Omega(\text{SO}/U \ast \text{SO}/U)).
\end{equation}

Hence, under $J'$, we obtain a splitting of $\text{Hyp}_\ast$. It is convenient to label the second summand as $J^\perp_\ast$. So we have

\begin{equation}
(1.9) \text{ Corollary.} \quad \text{Hyp}_\ast = \text{Im} J_\ast \oplus J^\perp_\ast.
\end{equation}

2. The topology of $U$-hypersurfaces.

We now investigate some simple properties of our hypersurfaces.

Suppose that $M \subset S^{n+1}$ is an oriented hypersurface, which we may assume connected when studying $\Omega(S^1 \wedge \text{SO}/U)$. $M$ determines a division of $S^{n+1}$ into
two compact submanifolds $A$ and $B$, such that $S^{n+1} = A \cup B$ and $M = A \cap B$, their common boundary. One of these, say $A$, is distinguished by the orientation of $M$.

As CW complexes $A$ and $B$ have dimension $< n$, and they are $n$-duals in the sense of Spanier, admitting duality maps

$$\varrho: S^{n+1} \to S^1 \wedge A \wedge B \quad \text{and} \quad \varrho': S^1 \wedge A \wedge B \to S^{n+1}.$$ 

Vice versa, given a reasonable embedding of a complex $A'$ in $S^{n+1}$, we may thicken it into a regular neighbourhood $A$ and take its boundary $M$, an oriented hypersurface. Then $B = S^{n+1} - \text{int}(A)$ is an $n$-dual to $A$, so we shall label it henceforth as $DA$.

Note that the hypersurfaces described above are oriented (and, of course, framed) boundaries by $A$.

Now regard the Mayer–Vietoris cofibration

$$A \vee DA \xrightarrow{i \vee i'} A \cup DA \xrightarrow{c} S^1 \wedge (A \cap DA) \to S^1 \wedge (A \vee DA)$$

$$\xrightarrow{\simeq} S^{n+1} \quad \xrightarrow{\simeq} S^1 \wedge M$$

where $c$ is the Pontrjagin–Thom collapse. Both $i$ and $i'$ have images within $n+1$ discs in $S^{n+1}$, and so create an explicit nul-homotopy for $i \vee i'$. So there is a homotopy equivalence

$$h: S^{n+1} \vee (S^1 \wedge A) \vee (S^1 \wedge DA) \to S^1 \wedge M.$$ 

Restricting $h$ to $S^{n+1}$ recovers $c$, i.e. a splitting for the suspension of the collapse $M \to S^n$ onto the top cell. Similarly, $h|S^1 \wedge A$ splits the suspension of the boundary inclusion $j: M \to A$, whilst $h|S^1 \wedge DA$ splits the suspension of $j': M \to DA$ (up to sign).

Returning to the U structure on $M$, it is given as a map $M \to \text{SO}/U$, which may be thought of as twisting the given framing. Homotopic maps give equivalent U structures. Since the group $[M, \text{SO}/U]$ is exactly $\text{KO}^{-2}(M)$, we have to calculate the KO groups of $M$. Using the above splittings, we obtain

(2.1) Lemma

$$\text{KO}^{-2}(M) \cong \text{KO}^{-2}(A) \oplus \text{KO}^{-2}(DA) \oplus \text{KO}^{-2}(S^n).$$

Of course, such a formula holds for any (co)homology functor.

So we can now investigate the contribution made by each summand of (3.1) to the bordism class of $M$, first in $\Omega^U_{*,1}$ and then in $\text{MU}_*$. For any twist $\delta \in \text{KO}^{-2}(M)$ we shall write these bordism classes as $\{M, \delta\} \in \Omega^U_{n,1}$ and $[M, \delta] = J'\{M, \delta\} \in \text{MU}_n$. 


(2.2) Proposition.

(i) \( \{ M, \alpha \} = \{ M, \alpha' \} = 0 \quad \forall \alpha \in \text{KO}^{-2}(A), \ \alpha' \in \text{KO}^{-2}(DA) \)

(ii) \( \{ M, \sigma \} \in \pi_n(\text{SO/U}) \quad \forall \sigma \in \text{KO}^{-2}(S^n) \).

Proof. (i) If we twist the framing on \( M \) by \( \alpha \), it bounds \( A \) with framing twisted by \( \alpha \). Similarly for \( DA \).

(ii) \( M \) is diffeomorphic to the connected sum \( M \neq S^n \), and \( \sigma \) may be displayed trivially on \( M^n \) and as \( \sigma \) on \( S^n \). Thus the new \( U \) structure is equivalent to retaining the original on \( M \) and twisting \( S^n \) by \( \sigma \). But the original framing on \( M \) bounds \( A \), so

\[ \{ M^n, \sigma \} = \{ S^n, \sigma \}. \]

(2.3) Corollary. With the data above, in \( \text{MU}_n \)

(i) \( [M, \alpha] = [M, \alpha'] = 0 \)

(ii) \( [M, \sigma] = J(\sigma) \).

The above observations give substance to the splittings of (1.7), for if \( M \) is initially a sphere then \( \text{KO}^{-2}(M) \cong \text{KO}^{-2}(S^n) \) in (2.1). Furthermore, the splitting is the same if \( M \) is merely framed, since the top cell is still stably trivially attached.

Thus when investigating \( \text{Hyp}_\ast \), our interest should centre on twistings \( \delta \) which are non-zero in both \( \text{KO}^{-2}(A) \) and \( \text{KO}^{-2}(DA) \), yet which vanish on the top cell. Such \( U \)-hypersurfaces constitute the summands \( \pi_\ast(\Omega(\text{SO/U} \ast \text{SO/U})) \) and \( J_\ast \) of (1.9). They can be obtained directly from a representative \( f: S^n \to \Omega(\text{SO/U} \ast \text{SO/U}) \) as follows.

Adjoint \( f \), and regard the join as the space

\[ (C(\text{SO/U}) \times \text{SO/U}) \cup (\text{SO/U} \times C(\text{SO/U})) \],

where the cones are thought of as attached to \( \text{SO/U} \times \text{SO/U} \) in opposite directions. Now make the map transverse to \( \text{SO/U} \times \text{SO/U} \), giving a hypersurface \( M \subset S^{n+1} \) equipped with a map

\[ \alpha \times \alpha': M \to \text{SO/U} \times \text{SO/U}. \]

By construction, \( \alpha \) extends over \( A \) and \( \alpha' \) extends over \( DA \).

Enthusiasts may verify that this process yields a Pontrjagin–Thom isomorphism between the geometry and the homotopy theory.

Our programme is to begin by restricting the defining complex \( A \), and hence the hypersurface \( M \), to be of a simple type. Note that if \( A \) is a point (i.e. a 0 cell complex), \( M \) will be a sphere and so represent an element of \( \text{Im} J_\ast \); whilst if \( A \)
is a wedge of spheres (i.e. 1 cell complexes), $M$ will be a connected sum of products of spheres, and so represent an element in $R \text{Im} J_*^s$.

Therefore we propose to study the situation when $A$ is either a (wedge of) 2 cell complexes, or else is stably a wedge of spheres. As we shall see, both these choices yield interesting, and apparently new information.

3. Sample calculations.

In this section we set up the calculational procedures we shall adopt in section 4. But first we note some general facts.

(3.1) **Proposition.** $R \text{Im} J_*^s \otimes Q \cong J_*^S \otimes Q$ as subrings of $MU_* \otimes Q$.

**Proof.** The following diagram of Hurewicz homomorphisms is commutative (e.g. see [8])

$$
\begin{array}{ccc}
\pi_*^S(\text{SO/U}) & \xrightarrow{J} & MU_* \\
\downarrow h_2 & & \downarrow h_3 \\
H_*^s(\text{SO/U}) & \xrightarrow{i_*} & H_*(\text{BU}) \\
\end{array}
$$

(and $i_*$ is a monomorphism of Pontrjagin rings). Thus $i_* h_2$ has image $h_3(\text{Im} J_*^s)$.

In addition, $h_2$ maps $\pi_*^s(\text{SO/U})$ onto multiples of ring generators for $H_*^s(\text{SO/U})$. Thus when we apply $\otimes Q$ to the above diagram, we observe that

$$(h_3 \otimes 1)(R \text{Im} J_*^s \otimes Q) = (i_* \otimes 1)(H_*^s(\text{SO/U}) \otimes Q)$$

$$= (i_* h_1 \otimes 1)(\pi_*^S(\text{SO/U}) \otimes Q).$$

But both $h_1 \otimes 1$ and $h_3 \otimes 1$ are isomorphisms.

(3.2) **Corollary.**

$$R \text{Im} J_*^s \otimes Q = \text{Hyp}_* \otimes Q = J_*^S \otimes Q.$$ 

**Proof.** Apply (3.1) to (1.5).

Thus the problem of relating the three subrings of $MU_*$ described in (1.5) is entirely concerned with subgroups of maximal rank in a free abelian group. In other words the issues are of divisibility, and as such may be investigated by taking quotients.

The “smallest” subring involved is $R \text{Im} J_*^s$, and $MU_*/R \text{Im} J_*^s$ is theoretically determinable by $KU$ theory. Passing to $BP$ may well be a fruitful method for organising the algebra. On the other hand we know of no sure method for computing $J_*^S$, the “largest” subring.
Note from (1.3) that none of $R \text{Im} J_{*}$, $\text{Hyp}_{*}$, $J_{*}^{S}$ are summands of $\text{MU}_{*}$. And we shall see below that $\text{Hyp}_{*}$ is not polynomial.

To put these matters into perspective, consider the stabilisation map $\text{st}: \Omega(S^{1} \wedge \text{SO}/U) \rightarrow Q(\text{SO}/U)$. It gives

$$\Omega_{*,1}^{U} \otimes Q \xrightarrow{\text{st}_{*} \otimes 1} \pi_{*,1}^{S}(\text{SO}/U) \otimes Q \xrightarrow{h_{*} \otimes 1} \approx \text{Hyp}_{*} \otimes Q \xrightarrow{\approx} H_{*}(\text{SO}/U) \otimes Q$$

Thus the free part of $\Omega_{*,1}^{U}$ consists of two summands. Firstly those elements which survive under $\text{st}_{*}$, say $\Phi$; and secondly $\text{Ker}(\text{st}_{*})$. Then $J'$ is a monomorphism on $\Phi$, so $\text{Hyp}_{*}$ is the image of faithful representation of $\Phi$. This alone suggests that our divisibility problems will be intractible!

To make more precise computations, we must recall some notation pertaining to $\text{MU}_{*}$. We may write $\text{MU}_{*}$ as a polynomial algebra $Z[x_{1}, x_{2}, \ldots, x_{k}, \ldots]$, where $x_{k}$ has real dimension $2k$. Several authors have given procedures for choosing the $x_{k}$'s, but no canonical choice has emerged.

Over $Q$, the requirement for a generator is simple, viz. $s_{k}(x_{k}) \neq 0$, where $s_{k} \in H^{2k}(BU)$ is the additive characteristic class corresponding to the symmetric polynomial $\sum t_{i}^{k}$ in $H^{2k}(\times CP^{\infty})$ (e.g. see [10]).

For example, amongst the results needed to prove (1.3) is the following. Let $[S^{4k+2}]$ be the U bordism class represented by a generator of $\pi_{4k+2}(\text{SO}/U) \cong Z$. Then

$$s_{2k+1}[S^{4k+2}] = \begin{cases} 2(2k+1)! & k \text{ even} \\ (2k+1)! & k \text{ odd} \end{cases} = \kappa(k), \text{ say}.$$ 

Thus $[S^{4k+2}]$ is an acceptable choice for $x_{4k+2}$ over $Q$. We deduce

(3.3) **NOTE.** All the rings of (3.2) may be described as

$$Q[[S^{2}], [S^{6}], \ldots, [S^{4k+2}], \ldots].$$

Thus our overall strategy is to describe our (integral) bordism classes in $\text{MU}_{*}$ as rational combinations of these sphere classes.

We start by recalling that

$$H^{*}(BU; Q) \cong Q[s_{1}, s_{2}, \ldots, s_{k}, \ldots]$$

and that in dimension $2n$ a basis is given by monomials

$$s_{w} = (s_{1})^{w_{1}}(s_{2})^{w_{2}} \ldots (s_{n})^{w_{n}} \text{ where } \sum_{i=1}^{n} iw_{i} = n.$$
We write $W(n)$ for the set of such sequences $(w_1, w_2, \ldots, w_n)$, so that a $\mathbb{U}$-bordism class $y_n$ is determined by the Chern numbers $\{s_w(y_n); \ w \in W(n)\}$.

Equivalently, and dually, we have to evaluate the image of $y_n$ under the Hurewicz isomorphism

$$h \otimes 1: \text{MU}_* \otimes \mathbb{Q} \to H_*(\text{MU}; \mathbb{Q}) \cong H_*(\text{BU}; \mathbb{Q})$$

But we may write

$$H_*(\text{BU}; \mathbb{Q}) \cong \mathbb{Q}\langle s^w; \ w \in W(n) \ \forall \ n \rangle$$

where $s^w$ is the basis element dual to $s_w$. Moreover $h \otimes 1$ identifies $[S^{4k+2}]$ with $\pi_*^k s^{4t(2k+1)}$, where $\Delta(t)$ is the sequence $(0, \ldots, 0, 1)$ with $i-1$ zeros. By (3.2) and (3.3), these elements are ring generators for the image of $H_*(\text{SO}/U; \mathbb{Q})$ in $H_*(\text{BU}; \mathbb{Q})$.

Now given a twisting $\delta: M \to \text{SO}/U$, we note that the new normal structure on the hypersurface $M$ is given by the composition

$$M \to \text{SO}/U \to \text{BU}$$

which we label $\delta$. Hence the $s$ classes of $[M, \delta]$ arise by evaluating monomials $s_w(\delta)$ on the fundamental class $\sigma \in H_*(M)$. Such calculations are aided by the fact that the diagram

$$\begin{array}{ccc}
KO^{-2}(M) & \xrightarrow{i^*} & KU^{-2}(M) \\
\downarrow & & \downarrow \zeta \\
KU^0(M) & \xrightarrow{\iota} &
\end{array}$$

is commutative, where $\zeta$ is the Bott isomorphism. So $s_w(\delta) = s_w(\zeta c \delta)$ may be read off from $\text{ch}(c \delta)$.

The most systematic way of organising this information is to utilise the scheme laid out in [9].

\begin{quote}
(3.4) **Lemma.** Let $M^n$ be framed, and $\delta \in KO^{-2}(M)$, and let $\Psi: KO^{-2}(M) \to \text{MU}_n$ be given by $\Psi(\delta) = [M, \delta]$. Then $\Psi$ may be factorised as

$$KO^{-2}(M) \to \text{MU}_n^0(M_+) \to \text{MU}_n,$$

where $J_u = 1 + \tilde{J}_u$ is induced by $j: \text{SO}/U \to \Omega^{\infty} \text{MU}(\infty)$ (and is exponential) whereas $D = \langle \cdot, \Sigma \rangle$ is the duality homomorphism induced by the fundamental class $\Sigma \in \text{MU}_n(M_+)$.
\end{quote}

Our plan is to use the characteristic class calculations discussed earlier in order to trace this factorisation of $\Psi$ in (co)homology, using the Hurewicz and Boardman maps
\[ h: \text{MU}_*(M) \to H_*(\text{BU}_+) \otimes H_*(M) \otimes \mathbb{Q} \]
\[ \bar{h}: \text{MU}^*(M) \to H^*(\text{BU}_+) \otimes H^*(M) \otimes \mathbb{Q} \]

To this end, we need the following useful formula:

(3.5) Formula. Let \( \delta \in \text{KO}^{-2}(M) \). Then
\[ \bar{h}J_u(\delta) = \sum_v \epsilon_v S^v \otimes \text{ch}_v(\exp \delta) \]

Here \( \exp \delta \) is to be interpreted as the usual power series in the (nilpotent) ring \( \text{KU}^0(M) \otimes \mathbb{Q} \). Also, \( v \) ranges over the non-decreasing elements in the set \( V \) consisting of all sequences of odd integers, so if \( v = (v(1), \ldots, v(t)) \), \( \text{ch}_v(\delta) = \text{ch}_{v(1)}(\delta) \ldots \text{ch}_{v(t)}(\delta) \). Similarly, \( S^v = [S^{2v(1)}] \ldots [S^{2v(t)}] \) whilst \( \epsilon_v = \epsilon(v(1)) \ldots \epsilon(v(t)) \), where
\[ \epsilon(v(i)) = \begin{cases} 1 & \text{as } v(i) \equiv 3(4) \\ \frac{1}{2} & \text{as } v(i) \equiv 1(4) \end{cases} \]

Proof. In \( H^*(\text{BU}_+) \otimes \mathbb{Q} \), \( \text{ch}_{2k+1} \) is dual to \( (2k+1)! S^{2(2k+1)} \). Thus the linear terms in \( \bar{h}J_u(\delta) \) have the form
\[ \sum_k \epsilon(2k+1)[S^{2k+1}] \otimes \text{ch}_{2k+1}(\delta) \]

Considering product terms in the same fashion, remark that \( \text{ch}_v(\delta) \) has value \( t! / \epsilon_v \) on \([S^{2v(1)}] \ldots [S^{2v(t)}]\), and zero on other monomials of the same dimension. Thus a typical non-linear term of \( \bar{h}J_u(\delta) \) is \( S^v \otimes (1/t!) \text{ch}_v(\delta) \). Now sum over \( t \).

This formula neatly captures the idea that \( J_u \) is exponential: applied to a sum it yields
\[ \bar{h}J_u(\delta_1 + \delta_2) = \sum_{v, w} \epsilon_{v, w} S^v \cdot w \text{ch}_v(\exp \delta_1) \text{ch}_w(\exp \delta_2) \]

To crystallise these ideas, we conclude with an interesting application: it requires two lemmas.

(3.6) Lemma. Suppose \( X \) is stably a finite wedge of spheres. Then in \( \text{KU}^0(X) \), \( x^t \) is divisible by \( t! \) \( \forall x \) or \( t \) (i.e. "exp \( x \) is integral").

Proof. Let \( \psi^p \) be the Adams operation for the prime \( p \), so that \( \psi^p(x) \equiv x^p(p) \) in the free abelian group \( \text{KU}^0(X) \). Using the stable structure of \( X \), \( \psi^p(x) \equiv 0(p) \), whence \( p \mid x^p \) for any prime. So \( t! \mid x^t \).
Note that replacing \( \text{KU}^0(X) \) by \( H^*(X; \mathbb{Z}) \), and \( \psi^p \) by the Steenrod power (or square) gives the same result for integral cohomology.

(3.7) **Lemma.** With \( X \) as in (3.6), and \( x \) in the image of \( c: \text{KO}^{-2}(X) \to \text{KU}^{-2}(X) \), then \( \text{ch}_v(x^i) \) is divisible by \( t!/\varepsilon_v \) \( \forall v \in V \).

**Proof.** We can rewrite \( x \) as \( 2x' + x'' \), where \( x' \) incorporates all the components of \( \text{ch}_v \) in dimensions \( \equiv 2(8) \). Thus

\[
x^i = 2^t(x')^t + \ldots + \binom{t}{k}2^{t-k}(x')^{t-k}(x'')^k + \ldots + (x'')^t
\]

and by (3.6) \( (t-k)! \)\( | (x')^{t-k} \), and \( k! \)\( | (x'')^k \).

The result now follows by a simple induction.

Hence we can prove

(3.8) **Theorem.** Suppose that the hypersurface \( M \subset \mathbb{S}^{n+1} \) is stably a wedge of spheres. Then

\[
\Psi(M) = R \text{Im} J_n.
\]

**Proof.** Let \( \delta \in \text{KO}^{-2}(M) \), so that

\[
\hbar \Psi(\delta) = \langle \hbar J_\nu(\delta), \hbar \Sigma \rangle \quad \text{in} \quad H_{2n}(\text{BU}).
\]

Now the splitting of \( M \) discussed in section 2 ensures that \( \hbar \Sigma = 1 \otimes \sigma \) in \( H_*(\text{BU}_+ \otimes H_*(M) \), so by (3.5),

\[
\hbar \Psi(\delta) = \sum \langle \text{ch}_v(\exp \delta), \sigma \rangle \varepsilon_v S^v,
\]

summing over all \( v \) with \( \sum v(i) = n \).

But from (3.7), \( \varepsilon_v \text{ch}_v(\exp \delta) \in H^{2n}(M; \mathbb{Z}) \), so \( \hbar \Psi(\delta) \) is an integral combination of \( S^v \)'s. So \( \Psi(\delta) \in R \text{Im} J_* \).

(3.9) **Notes.** (i) We have shown that \( R \text{Im} J_* \) consists precisely of those classes representable on hypersurfaces which are stably a wedge of spheres.

(ii) The given proof, with minimal modification, applies to a framed \( M \) of arbitrary codimension.

4. 2 cell complexes.

We now concentrate on hypersurfaces defined by 2 cell complexes.

Let \( x \in \text{Hyp}_{2n} \) be represented by \( M \subset \mathbb{S}^{2n+1} \), with defining complex \( A = \mathbb{S}^n \cup_0 e^b \). We shall see below that it suffices to restrict \( a \) and \( b \) by \( a \geq 2 \) and
$b \leq 2n - 2$. According to [3], we may then assume that $DA \cong S^{2n-b} \cup e^{2n-a}$, where $s^{2n-b-1}\theta \cong (-1)^{a+b} s^{a-1}\varphi$.

(4.1) **Lemma.** Let $X$ be one of $A$, $DA$ (or $M$) as above. Then $\text{MU}_\ast(X)$ and $\text{MU}_\ast(X)$ are free modules on 2 (or 5) generators over $\text{MU}_\ast$, if $b > a + 1$.

**Proof.** Since $\theta$ is stably torsion, the result it true for $A$ and $DA$ (for $\text{MU}_\ast$ is free). It follows for $M$ from (2.1).

As both an illustration and a general explanation of our method, we now examine a typical case;

(4.2) **Example.** Let $v \in \pi_7(S^4)$ be the Hopf map. Choose an embedding $A = S^6 \cup v^10 \subset S^{21}$ (say by suspending a smooth embedding $HP^2 \subset S^{19}$). Then $DA$ is given by $S^{10} \cup v e^{14}$. Thus

$$M = S^6 \cup e^{10} \cup e^{10} \cup e^{14} \cup e^{20}$$

$$S^1 \wedge M \cong (S^7 \cup v e^{11}) \vee (S^{11} \cup v e^{15}) \vee S^{21}.$$

It is advantageous to consider $A$ and $DA$ as Thom complexes of the “adjoint bundles” $v: S^4 \to \text{BSO}(6)$ and $v: S^4 \to \text{BSO}(10)$ respectively (for $v$ lies in the classical $\text{Im} J$). Being Spin bundles, these are KO orientable, so we have

$$\text{KO}^{-2}(A) \cong \text{KO}(S^4_+) \cong \mathbb{Z} \oplus \mathbb{Z}.$$  

$$\text{KO}^{-2}(DA) \cong \text{KO}^{-4}(S^4_+) \cong \mathbb{Z} \oplus \mathbb{Z}.$$  

We may choose generators $\alpha_1, \beta_1$ for $\text{KO}^{-2}(A)$ and $\alpha_2, \beta_2$ for $\text{KO}^{-2}(DA)$ related by

$$\alpha_2 = x\alpha_1$$ and $$y\beta_1 = x\beta_2,$$

where $x$ generates $\text{KO}_8 \cong \mathbb{Z}$, $y$ generates $\text{KO}_8 \cong \mathbb{Z}$, and $x^2 = 4y$.

To apply (3.5), we must evaluate $\text{ch}$ on the image of each element in $\text{KO}^{-2}(-)$. Since complexification preserves Thom isomorphisms, we deduce with the help of [1] that

$$\text{ch}_3(\tilde{\alpha}_1) = t_1,$$  

$$\text{ch}_5(\tilde{\alpha}_1) = \frac{1}{2}u_1$$

$$\text{ch}_3(\tilde{\beta}_1) = 0,$$  

$$\text{ch}_5(\tilde{\beta}_1) = 2u_1,$$

where $t_1, u_1$ are respective generators of $H^6(A), H^{10}(A)$ (with $\mathbb{Z}$ coefficients). The $1/12$ arises as the $e$ invariant of $e_C(v)$.

Substituting in (3.5) gives in $H_\ast(BU_+) \otimes H_\ast(M)$

$$f \tilde{\gamma}(\alpha_1 + \mu \beta_1) = 1 \otimes 1 + \lambda[S^6] \otimes t_1 + (\frac{1}{2}\mu + \mu)[S^{10}] \otimes u_1.$$
∀ λ, μ ∈ ℤ. Of course, products vanish in KO*(A).

Similarly, we can write

\[ hJ_{u}(\lambda'x_2 + \mu'b_2) = 1 \otimes 1 + \lambda'[S^{10}] \otimes t_2 + (\frac{1}{6}\lambda' + \mu')[S^{14}] \otimes u_2. \]

Now by duality, products in H*(M) are given by

\[ t_1u_2 = -u_1t_2 = g_{20} \quad (\text{e.g. see [3]}). \]

where \( g_{20} \in H^{20}(M) \cong ℤ \) is the dual of \( \sigma \in H_{20}(M) \).

So if \( \delta = \lambda x_1 + \mu \beta_1 + \lambda'x_2 + \mu'\beta_2 \in KO^{-2}(M) \), then

\[ hJ_{u}(\delta) = 1 \otimes 1 + [\lambda(\frac{1}{6}\lambda' + \mu')[S^6][S^{14}] - \lambda'(\frac{1}{24}\lambda + \mu)[S^{10}]^2] \otimes g_{20} \]

modulo the linear terms in \( t_1, u_1, t_2, \) and \( u_2 \).

Moreover from the splitting of (2.1),

\[ h\Sigma = 1 \otimes \sigma \quad \text{in } H_*(BU) \otimes H_*(M). \]

Thus ∀ λ, μ, λ', μ' ∈ ℤ

\[ h\Psi(\delta) = \langle hJ_{u}, h\Sigma \rangle \]

\[ = \lambda(\frac{1}{6}\lambda' + \mu')[S^6][S^{14}] - \lambda'(\frac{1}{24}\lambda + \mu)[S^{10}]^2. \]

Clearly we have unearthed elements of Hyp_{20} which do not lie in R Im J_*. For example

\[ \lambda = \lambda' = 1, \quad \mu = \mu' = 0 \quad \text{gives} \quad \frac{1}{24}(4[S^6][S^{14}] - [S^{10}]^2) \]

\[ \lambda = 1, \quad \lambda' = 6, \quad \mu = 0, \quad \mu' = 1 \quad \text{gives} \quad \frac{1}{4}[S^{10}]^2. \]

(4.3) COROLLARY. Hyp_ is not polynomial.

(4.4) COROLLARY. The quotients

\[ \text{Hyp}_{20}/R \text{Im } J_{20} \cong MU_{20}/R \text{Im } J_{20} \]

have \( Z_{24} \) as a subgroup.

PROOF. This is a simple algebraic consequence of the formulae arising in (4.2).

We now propose to let \( \theta \) range over all possible \( \pi_{b-1}(S^9) \). It is therefore convenient to write \( A_{\theta} \) for \( S^a \cup_{\theta} e^b \), and \( M_{\theta} \) for the hypersurface defined by some embedding \( A_{\theta} \subset S^n \). Before we begin calculating, it is also helpful to divide these complexes \( A \) into various types, according to how \( c \) and \( ch \) work.
(4.5) Definitions. We shall say that $A_{\theta}$ has type A if both cells are of dimension $\equiv 2\ (4)$, and write
\[ A_{\theta} = S^{2c} \cup_{\theta} e^{2d} \quad \text{with } c, d \equiv 1 \ (2). \]

We shall say that $A_{\theta}$ has type B if only the top cell has dimension $\equiv 2(4)$, and if $\theta$ is not stably $\mu_1 \in \pi_{8*+1}^S$ (where $d_R(\mu_1) = 1 \in \mathbb{Z}_2$; see [1]). If $\theta$ is stably $\mu_1$ we shall say that $A_{\theta}$ has type $B^u$. In either case, we write
\[ A_{\theta} = S^a \cup_{\theta} e^{2d} \quad \text{with } d \equiv 1(2). \]

Similarly, we shall say that $A_{\theta}$ has type C if only the bottom cell has dimension $\equiv 2(4)$, and $\theta$ is not stably $\mu_1$ or $\mu_2 \in \pi_{8*+2}^S$. Otherwise, $A_{\theta}$ has type $C^u$. In either case, we write
\[ A_{\theta} = S^{2c} \cup_{\theta} e^{b} \quad \text{with } c \equiv 1(2). \]

Next we note two lemmas, easily proved.

(4.6) Lemma. The composition
\[ KO^{-2}(A_{\theta}) \xrightarrow{c} KU^0(A_{\theta}) \xrightarrow{ch^*} H^*(A_{\theta}; \mathbb{Q}) \]
is trivial unless $A_{\theta}$ is of one of the above types. More, if of type B or $B^u$, then $ch\circ c$ is trivial on the bottom cell, and if of type C or $C^u$, then $ch\circ c$ is trivial on the top cell.

(4.7) Lemma. If $A_{\theta}$ has type A, B or C, then the cofibration of $\theta$ yields a splitting
\[ KO^{-2}(A_{\theta}) \cong KO^{-2}(S^a) \oplus KO^{-2}(S^b). \]

If $A_{\theta}$ has type $B^u$, then projection onto the top cell induces $2: \mathbb{Z} \rightarrow \mathbb{Z}$ in $KO^{-2}(\cdot)$; and if $A_{\theta}$ has type $C^u$, inclusion of the bottom cell also induces $2: \mathbb{Z} \rightarrow \mathbb{Z}$ in $KO^{-2}(\cdot)$.

Armed with these tools, we can now generalise (4.2).

(4.8) Theorem. Let $A_{\theta}$ be of type A, and suppose $A_{\theta} \subset S^{4n+1}$. Then we have that
\[ \Psi(M_{\theta}/R \text{Im} J_{4n}) \subset \text{Hyp}_{4n}/R \text{Im} J_{4n} \]
is a finite cyclic group. A generator is
\[ \frac{1}{2}e(4[S^2][S^{4n-2c}] - [S^2d][S^{4n-2d}]) \quad \text{if} \quad \begin{cases} d - c \equiv 2(4) \\ 2n \equiv c - 1 \equiv 2(4) \end{cases} \]
and
\[ qe([S^2][S^{4n-2c}] - [S^{2d}][S^{4n-2d}]) \quad \text{otherwise}, \]
where

\[ q = \frac{1}{2} \quad \text{if} \quad \begin{cases} d - c \equiv 2(4) \\ c \equiv 3(4), \ 2n \equiv 0(4) \end{cases} \quad q = 2 \quad \text{if} \quad \begin{cases} d - c \equiv 2(4) \\ 2n \equiv c - 1 \equiv 0(4) \end{cases} \]

and \( q = 1 \) in all other cases. Here \( e = e_\zeta(\theta) \).

Note that if \( a = 0 \), then \( e = 0 \).

(4.9) COROLLARY. (i) The quotients

\[ \text{Hyp}_{4n}/R\text{Im}\ J_{4n} \hookrightarrow \text{MU}_{4n}/R\text{Im}\ J_{4n} \]

all have \( \mathbb{Z}_\tau \) as a subgroup, where \( r = 1 \) or \( 4 \) and \( \sigma \) is the order of \( \text{Im}\ J \subset \pi_{2d-1}(S^{2c}) \), with \( c, d, n \) as above.

(ii) \( \frac{1}{2}[S^{8a+2}]^2 \in \text{Hyp}_* \quad \forall a > 0 \).

PROOF. For (ii), embed \( S^{8a-2} \cup e^{8a+2} \) in \( S^{16a+5} \).

To consider the other possible types for \( A_\theta \), we refer to (4.6) and (4.7). Firstly, let \( A \) have type \( B^a \) or \( C^a \): the following cases are of interest.

(4.10) LEMMA. Let \( \theta \in \pi_{8a+2b+1}(S^{2b}) \) have \( d_\zeta(\theta) = 1 \), and suppose \( A_\theta \subset S^{8n+5} \), where \( 2b \equiv 0 \) or \( 2(8) \). Then \( \Psi(M_\theta)/R\text{Im}\ J_{8n+4} \cong \mathbb{Z}_2 \), generated by

\[ \frac{1}{2}[S^{2b}][S^{8n+4-2b}] \quad \text{if} \ 2b \equiv 2(8) \]

\[ \frac{1}{2}[S^{8a+2b+2}][S^{8(n-a)-2b+2}] \quad \text{if} \ 2b \equiv 0(8) \].

Again referring to (4.6) and (4.7) we can conclude

(4.11) LEMMA. For all other types of \( A_\theta \), and all other dimensions excluded from (4.8) and (4.10), \( \Psi(M_\theta)/R\text{Im}\ J_* = 0 \).

Now (4.8) and (4.10) may be combined:

(4.12) THEOREM. The subset of \( \text{Hyp}_*/R\text{Im}\ J_* \) realisable on hypersurfaces defined by 2 cell complexes consists of the cyclic subgroups described in (4.8).

PROOF. It remains only to show that (4.10) actually describes a subgroup of

(4.8).

For this, suppose we have an embedding

\[ S^{8k+2} \cup e^{8l+4} \subset S^{8n+5} \].

Then \( n > k \) (even if \( \mu = \eta \), i.e. \( k = l \)). So \( \Psi(M_\mu) \) is generated by

\[ \frac{1}{2}[S^{8k+2}][S^{8n-8k+2}] \].
But from (4.2), we may embed $S^{8k+2} \cup e^{8k+6}$ in $S^{8n+5}$ \( (n > k) \) such that $\Psi(M_\bigvee) \ni \frac{1}{4}[S^{8k+2}][S^{8n-8k+2}]$.

As stated, (4.12) has several unsatisfactory aspects. For example, we can offer no general results as to the smallest values of $c$ and $d$ for which a given $e$ invariant is realisable on $A_\theta = S^{2c} \cup e^{2d}$. Moreover, even given such knowledge, we cannot specify the least $n$ for which $A_\theta$ embeds in $S^{4n+1}$.

So far as the first shortcoming is concerned, we may of course assume that $\theta$ is a generator of $\text{Im} J \subset \pi_{4k-1}^S$, and that $A_\theta$ is the mapping cone in the smallest stable dimension available, i.e. $A_\theta = S^{4k+2} \cup e^{8k+2}$. To deal with the second drawback we recall the following lemma, which is based on the join construction.

(4.13) Lemma. Any complex $S^a \cup e^b$ may always be embedded in $S^{a+b+1}$.

We may then assemble a weaker, but more systematic and memorable version of (4.12).

(4.14) Theorem. In dimensions of the form $12d + 4r \geq 20$, where $r = 1, 2$ or $3$ and $d = 1, 2, \ldots$, the quotients $\text{Hyp}_*/R \text{Im} J_\bigvee \hookrightarrow \text{MU}_*/R \text{Im} J_\bigvee$

all have

$$Z_{\sigma(1)} \oplus \ldots \oplus Z_{\sigma(i)} \oplus \ldots \oplus Z_{\sigma(d)}$$

as a subgroup, where $\sigma(i)$ is the order of $\text{Im} J \subset \pi_{4i-1}^S$.

Moreover, each summand may be realised on a hypersurface defined by a 2 cell complex; and each general element on a hypersurface defined by a wedge of 2 cell complexes.

5. Epilogue.

Although the main purpose of section 4 is to study $\text{Hyp}_*/R \text{Im} J_\bigvee$, we have deliberately included $\text{MU}_*/R \text{Im} J_\bigvee$ in the statement of our results, since we previously had little information concerning the latter quotient.

This may, in theory, be thoroughly investigated via $\text{KU}_*(\text{BU})$ and the Hattori–Stong theorem. Of course, a full scale analysis of the situation by these means is a formidable algebraic undertaking, involving a detailed study of the coaction primitives in $\text{KU}_*(\text{BU})$.

However, the formulae are clearly of interest, and we hope to examine them more closely in future, with the aid of recent calculations by Francis Clarke.
We conclude by noting that several simple questions on $\text{Hyp}_*$ remain unanswered. Here are a sample two which have persistently eluded us.

(5.1) Problems.

(i) Does $\frac{1}{2}[S^{8k+2}][S^{8l+6}]$ lie in $\text{Hyp}_*$ for any $k, l$?

(ii) Does $\frac{1}{4r}[S^{8k_1+2}] \ldots [S^{8k_r+2}]$ lie in $\text{Hyp}_*$ for all $k_1, \ldots, k_r$, and every $r$?

We can in fact partially answer (i) by proving that such classes are not representable on any hypersurface which is stably a wedge of 2 cell complexes. We can also partially answer (ii) by proving that such classes do indeed arise on hypersurfaces for a large range of integers $k_1, \ldots, k_r$, and $r$.

REFERENCES