STRONGLY ANNULAR FUNCTIONS
WITH GIVEN SINGULAR VALUES

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0. Introduction.

For a function $f$ holomorphic in the unit disk $D = \{z : |z| < 1\}$, we consider the closed subset $Z'(f, a)$ of the unit circle $C = \{z : |z| = 1\}$ consisting of limit points of the set $\{z : f(z) = a\}$ of $a$-points of $f$ in $D$. If $f$ is strongly annular, that is, if there is a sequence of positive numbers $r_n$ such that $r_n \uparrow 1$ and

$$\min \{|f(z)| : |z| = r_n\} \to \infty$$

as $n$ increases, then the set $Z'(f, a)$ cannot be empty for any finite complex number $a$, and in fact, as is guaranteed by the Koebe–Gross theorem concerning meromorphic functions omitting three values, $Z'(f, a)$ must coincide with the full circle $C$ for every value $a$, except possibly for at most countably many $a$'s. Therefore the set $S(f)$ consisting of those exceptional values mentioned above may be viewed as being singular for $f$. We are interested in the question raised by D. D. Bonar [3]: What cardinalities are possible for $S(f)$?

Partial answers to this question has been given by K. Barth, D. D. Bonar and F. W. Carroll [2], and also the present author [5]. Recently F. W. Carroll [4] constructed a strongly annular function $f$ whose $S(f)$ is an increasing sequence $0 = a_0 < a_1 < \ldots < a_n < \ldots$ on the real line. In connection with this result one is naturally led to ask whether there exists a strongly annular function $f$ whose $S(f)$ is an arbitrary prescribed set, for instance, a countable dense subset of the complex plane. The purpose of this paper is to remark that the procedure involved in the above construction has much wider applicability so that the above question can be positively answered. Namely, we shall prove the following

**Theorem.** Let $S$ be any non-empty subset of the complex plane which is at most countable. Then there exists a strongly annular function $f$ with $S(f) = S$. 

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1. Fundamental lemma.

Let \( I \) be an open and proper subarc of \( C \). For any \( z \) in \( D \), denote by \( G(I, z) \) the Jordan domain in \( D \) whose boundary consists of \( I \) and two segments connecting \( z \) with each of end points of \( I \). When \( z \) is a point in \( G(I, 0) \), the domain \( G(I, z) \) will be referred to as a sectorial neighborhood of \( I \). Now let \( \zeta_1 = 1, \ldots, \zeta_{N+1} \) be distinct points on \( C \) such that \( \arg \zeta_k < \arg \zeta_{k+1} \) and \( \arg \zeta_{N+1} < 2\pi (k = 1, \ldots, N) \). Further denote by \( I_k \) the open arc on \( C \) having \( \zeta_k \) and \( \zeta_{k+1} \) as end points and not containing any other \( \zeta_j \). We denote by \( C_j \) the circle \( \{ z : |z| = R_j \} \) (\( 0 < R_1 < R_2 < 1 \)) such that \( C_1 \) intersects any sectorial \( G(I_k, z_k) = G_k \) where \( z_k \) is a point in \( G(I_k, 0) \). Then \( C_2 \) also intersects any \( G_k \), and hence we can divide \( C_2 \) into non-overlapping closed subarcs \( A_k, B_k, B_{k,1}, \) and \( B_{k,2} \) as follows:

First we denote by \( A_k \) a subarc whose middle point is the point where \( C_2 \) meets the radius \((0, \zeta_k)\) if \( 2 \leq k \leq N \) where the argument of the middle point of \( A_1 \) is \((\arg \zeta_{N+1} - 2\pi)/2 \). Here we choose \( A_k \) so small that it does not meet the boundary of \( G_k \cup G_{k-1} \) \((G_0 = G_N) \). Next removing \( N \) arcs \( A_k \) \((k = 1, \ldots, N) \) from \( C_2 \), we obtain \( N \) remaining arcs \( \{ J_k \} \) each of which intersects only one of \( N \) sectorial neighborhoods \( \{ G_k \} \). Assume that \( J_k \) intersects \( G_k \). Then \( J_k \) is also divided into three subarcs \( B_k, B_{k,1}, \) and \( B_{k,2} \). Namely \( B_k \) is the intersection \( J_k \cap G_k \). Observing here that \( B_k \) protrudes outside \( G_k \) at each of its end points, we denote by \( B_{k,1} \) the subarc protruding towards \( A_k \), and \( B_{k,2} \) the subarc protruding towards \( A_{k+1} \), where \( A_{N+1} = A_1 \). Finally we denote by \( H(D) \) the set of functions holomorphic in \( D \). The following lemma will play a fundamental role in proving Theorem.

**Lemma.** Assume that for any given \( N \) complex numbers \( w_k \) \((k = 1, \ldots, N) \), there exists a function \( f \) in \( H(D) \) such that

\[
\begin{align*}
(1.1) & \quad f(z) - w_k \text{ is bounded away from 0 in } G_k \quad (k = 1, \ldots, N-1), \\
(1.2) & \quad f \text{ is bounded in } G_k \quad (k = 1, \ldots, N-1).
\end{align*}
\]

Then, for each pair of positive numbers \( M \) and \( q \), there exists a function \( g \) in \( H(D) \) such that

\[
\begin{align*}
(1.1)' & \quad g(z) - w_k \text{ is bounded away from 0 in } G_k \quad (k = 1, \ldots, N), \\
(1.2)' & \quad g \text{ is bounded in } G_k \quad (k = 1, \ldots, N), \\
(1.3) & \quad |g(z)| > M \text{ for every } z \text{ on } C_2, \\
(1.4) & \quad |g(z) - f(z)| < q \text{ for every } z \text{ in } \{ z : |z| \leq R_1 \}.
\end{align*}
\]

**Proof.** Let \( q \) be a function defined as follows:

\[
\begin{align*}
(1.5) & \quad q(z) = f(z) \text{ on } \{ z : |z| \leq R_1 \} \text{ and also for every } z \text{ in } \overline{G_1}, \ldots, \overline{G_{N-1}}, \\
(1.6) & \quad q(z) = f(z_k, j) \text{ on } B_{k, j} \quad (k = 1, \ldots, N-1; \ j = 1, 2), \\
(1.6)' & \quad q(z) = a \text{ constant, distinct from } w_N, \text{ on the set } G_N \cup B_{N,1} \cup B_{N,2}.
\end{align*}
\]
Here $\bar{G}_k$ denotes the closure of $G_k$ with respect to $D$ and $\{z_{k,1}, z_{k,2}\}$ the end points of $B_k$ with $\arg z_{k,1} < \arg z_{k,2}$. Then, by virtue of the Arakelian approximation theorem [1], we can find, for any $\varepsilon > 0$, a function $h$ in $H(D)$ such that

$$|h(z) - q(z)| < \varepsilon.$$  

(1.7)

If we choose a positive number $\varepsilon$ properly, then $h$ is easily seen to possess required properties of the lemma except for (1.3). In order to make $h$ satisfy (1.3) without losing the other requirements of the lemma, we first use the Runge approximation theorem. Recall here that the union of $B_k, B_{k,1},$ and $B_{k,2}$ form a single closed arc $J_k$ which is off the boundary of the annular sector

$$S_k = \{z : R_1 < |z| \leq 1, \arg \zeta_k < \arg z < \arg \zeta_{k+1}\},$$

using $(\arg \zeta_{N+1} - 2\pi)/2$ in place of $\arg \zeta_1$ for $S_1$ and $(\arg \zeta_{N+1} + 2\pi)/2$ in place of $\arg \zeta_{N+1}$ for $S_N$, and again consider a function $q_k$ defined as follows:

$$q_k(z) = a_k,$$  

(1.8) $q_k(z)$ = a positive number, on the arc $J_k$,

$$q_k(z) = 0$$  

(1.9) on the set $\{z : |z| \leq 1\} - S_k$.

For this function $q_k$ and a positive number $\varepsilon_k$, Runge's theorem assures the existence of a polynomial $p_k$ such that

$$|p_k(z) - q_k(z)| < \varepsilon_k.$$  

(1.10)

Using $h$ and $\{p_k\}$, we define a function in $H(D)$ by

$$H(z) = (h(z) - w_N)\exp (p_N(z) + \ldots + p_1(z))$$

$$+ (w_N - w_{N-1})\exp (p_{N-1}(z) + \ldots$$

$$+ p_1(z)) + \ldots + (w_2 - w_1)\exp (p_1(z)) + w_1.$$  

(1.11)

To see that $H$ satisfies all the requirements (1.1)'--(1.4) in the lemma, except for that $H$ is large on $A_k$ ($k = 1, \ldots, N$) in modulus, we have only to note, as well as (1.5)--(1.11), the equality

$$H(z) - w_k = \{u_k(z) + (h(z) - w_k)\exp (p_{k-1}(z) + \ldots$$

$$+ p_1(z))\} \exp (p_k(z)) + v_k(z),$$  

(1.12)

where

$$u_k(z) = \{(h(z) - w_N)[\exp (p_N(z) + \ldots + p_{k+1}(z)) - 1] + \ldots$$

$$+ (w_{k+2} - w_{k+1})[\exp (p_{k+1}(z)) - 1]\} \exp (p_{k-1}(z) + \ldots + p_1(z))$$

(1.13)

$$u_{N-1}(z) = (h(z) - w_N)(\exp (p_N(z) - 1) \exp (p_{N-2}(z) + \ldots + p_1(z)),$$  

(1.14)

$$u_{N-1}(z) = (h(z) - w_N)(\exp (p_N(z) - 1) \exp (p_{N-2}(z) + \ldots + p_1(z)),$$  

(1.14)
and
\[ u_N(z) = 0 \quad \text{for all } z \]
\[ v_k(z) = (w_k - w_{k-1})[\exp (p_{k-1}(z) + \ldots + p_1(z)) - 1] + \ldots + (w_2 - w_1)[\exp (p_1(z)) - 1]. \]

In fact, by virtue of (1.1), (1.5), (1.6), and (1.7), we can find a positive number \( K \), independent of \( \varepsilon \) and \( N \) numbers \( \varepsilon_k \) \((k = 1, \ldots, N)\), such that
\[
\inf \{|h(z) - w_k| : z \in G_k \cup B_{k,1} \cup B_{k,2}\} > 4K \text{ for any } k.
\]

Therefore, if we make each of \( \varepsilon_k \)'s sufficiently small, it follows from (1.9) and (1.10) together with (1.2), (1.5), and (1.7) that
\[
\inf \{|(h(z) - w_k) \exp (p_{k-1}(z) + \ldots + p_1(z))| : z \in G_k \cup B_{k,1} \cup B_{k,2}\} > 2K
\]
and
\[
\sup \{|u_k(z)| : z \in G_k \cup B_{k,1} \cup B_{k,2}\} < K
\]
for any \( k \). Consequently we obtain
\[
|H(z) - w_k| > K|\exp (p_k(z))| - |v_k(z)|
\]
for every \( z \) in \( G_k \cup B_{k,1} \cup B_{k,2} \) and for each \( k \) \((k = 1, \ldots, N)\). On the other hand, since \( v_k \) does not involve \( p_j \) \((j \geq k)\), we can make \( v_k \) arbitrarily and uniformly small in \( G_j \) \((j \geq k)\) if we choose \( k - 1 \) positive numbers \( \varepsilon_{k-1}, \ldots, \varepsilon_1 \) suitably. Therefore, first, using \( N \) as a value of \( k \) in each of the conditions (1.14), (1.10), (1.9), and (1.8), we can make \( H \) possess the required properties in the lemma, except for that \( H(z) - w_k \) is bounded away from 0 in \( G_k \) \((k = 1, \ldots, N - 1)\) and that \( H \) is large on \( A_k \) \((k = 1, \ldots, N)\) in modulus. Here we put emphasis on the fact that \( H(z) - w_N \) is really bounded away from 0 in \( G_N \). Subsequently letting \( \varepsilon_N \) be fixed and keeping in mind that \( v_{N-1} \) does not involve \( p_N \) and \( p_{N-1} \), we shall next use \( N - 1 \) as a value of \( k \) in each of (1.14), (1.10), (1.9), and (1.8). We continue this procedure until we use 1 as a value of \( k \) in each of the four conditions just mentioned above. Then, as a consequence of this process, we can conclude that \( H \) satisfies the requirements of the lemma, except for that \( |H(z)| \) is large for every \( z \) on \( A_k \) \((k = 1, \ldots, N)\). Moreover, adding a small vector if necessary, we may assume that \( H \) does not vanish on \( C_2 \). Thus the remaining task is to make \( |H(z)| \) large on \( A_k \) \((k = 1, \ldots, N)\). To this end, we need some more geometric definitions. Namely, let \( A_{k,1} \) and \( A_{k,2} \) denote small arcs encroaching on \( B_{k-1,2} \) and \( B_{k,1} \) at the end points of \( A_k \), and further \( T_k \) a small "triangular" region including \( A_k, A_{k,1}, \) and \( A_{k,2} \) in its interior, and pointing at \( \zeta_k \) if \( 2 \leq k \leq N \) where \( T_1 \) is pointing at the point \( \exp \{ (\arg \zeta_{N+1} - 2\pi)/2 \} i \) and \( B_{0,2} = B_{N,2} \). Now let \( H_k \) be a function "closing the gaps" [4, Lemma A], that is, a function in \( H(D) \) such that
(1.15) \[ |H_k(z)| > a \quad \text{for every } z \text{ on } A_k, \]
(1.16) \[ \text{Re } H_k(z) > -\sigma \quad \text{for every } z \text{ on } A_{k,1}, A_{k,2}. \]
(1.17) \[ |H_k(z)| < 2\sigma \quad \text{for every } z \text{ in } D - T_k. \]

Using \( H \) defined by (1.11) and \( \{H_k\} \), we consider
\[ g(z) = H(z)\{1 + H_1(z)\} \ldots \{1 + H_N(z)\}. \]

Then choosing a pair of positive numbers \( a \) and \( \sigma \) suitably, and making use of (1.15), (1.16), and (1.17), we can easily show that \( g \) satisfies all the requirements of the lemma. Thus the proof is complete.

2. Proof of Theorem.

Let \( \{r_n\} \) and \( \{\sigma_n\} \) be two sequences of positive numbers with \( r_n \uparrow 1 \) and \( \sum \sigma_n < \infty \). Let \( \{G_k\}_{1}^{\infty} \) be sectorial neighborhoods defined as in Section 1 such that the circle \( C_n = \{z : |z| = r_n\} \) intersects all sectorials \( \{G_k\}_{1}^{n+1} \), \( n = 1, 2, \ldots \) To prove the theorem, we have only to construct inductively a sequence \( \{f_n\} \) in \( H(D) \) such that, for any \( n \),

(2.1) \[ f_n(z) - w_k \text{ is bounded away from 0 in } G_k \quad (k = 1, \ldots, n), \]
(2.2) \[ f_n \text{ is bounded in } G_k \quad (k = 1, \ldots, n), \]
(2.3) \[ |f_n(z)| > j \text{ for every } z \text{ on } C_j \quad (j = 1, \ldots, n), \]
(2.4) \[ |f_n(z) - f_{n-1}(z)| < \sigma_{n-1} \text{ for every } z \text{ on } \{z : |z| \leq r_{n-1}\}. \]

In fact, suppose that we have constructed \( \{f_n\} \) satisfying (2.1)–(2.4). Then, by virtue of (2.4), \( \{f_n\} \) converges to a function \( f \) in \( H(D) \). Further, it follows from (2.3) that \( f \) is strongly annular. Moreover, the Hurwitz theorem together with (2.1) assures that \( f(z) \neq w_k \) for every \( z \) in \( G_k, k = 1, 2, \ldots \). To prove the existence of the sequence \( \{f_n\} \), let \( f_1 \) be a constant, distinct from each of the \( w_k \)'s, whose modulus is greater than 1, and suppose that we have obtained \( \{f_1, \ldots, f_{n-1}\} \) satisfying (2.1)–(2.4). To construct \( f_n \), we use the lemma, taking \( r_{n-1} = R_1, r_n = R_2, f_{n-1} = f \), and \( n + 1 = M. \) Consequently, we get a function \( g \) in \( H(D) \) which is denoted by \( f_n \). Here, in order that the inequality
\[ |f_n(z)| > j \]
holds for every \( z \) on \( C_j \) \( (j = 1, 2, \ldots n - 1) \), a sufficiently small number \( \sigma'_n \) \( (< \sigma_n) \) must be used as a value of \( g \). Thus, the proof of the theorem is complete.
REFERENCES


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