MATH. SCAND. 50 (1982), 55–65

A GENERAL "IN BETWEEN THEOREM"

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Consider the following general "in between problem": Let $\mathscr{C}$ be a class of real functions on a set X, and let $\psi \leqq \varphi$ be two given functions; when does there exist a function $g \in \mathscr{C}$ such that $\psi \leqq g \leqq \varphi$? We shall in this note give a necessary condition, which includes many of the known "in between theorems" and "extension theorems" in topology and analytic set theory.

The basis of an "in between theorem" is a separation property. Let $\mathscr{C}$ be a set of real functions on X and let α and β be real functions on X. If K_1 and K_2 are subsets of X we say that *K_1 and K_2 are (α,β)-separated by $\mathscr{C}$*, if there exist $g \in \mathscr{C}$, so that $\alpha \leqq g \leqq \beta$ and $g(x)=\alpha(x)$ for $x \in K_1$ and $g(x)=\beta(x)$ for $x \in K_2$. Note that this notion is assymetric in (K_1,K_2). If K_1 and K_2 are (α,β)-separated by $\mathscr{C}$, then clearly we have

$$\alpha \leqq \beta \quad \text{and} \quad K_1 \cap K_2 \subseteq \{x \mid \alpha(x)=\beta(x)\} . \tag{1}$$

Thus if $\alpha<\beta$, then $K_1 \cap K_2=\emptyset$.

If $\mathscr{C}$ is a set of real valued functions on X, we say that $\mathscr{C}$ is *σ-convex*, if $\sum_n \lambda_n g_n \in \mathscr{C}$ whenever $g_n \in \mathscr{C}\ \forall n$, $\lambda_n \in [0,\infty[$, $\forall n$ and $\sum_n \lambda_n=1$.

THEOREM 1. *Let α, β, φ and ψ be functions: $X \to \bar{\mathsf{R}}$, K and L two subsets of X and $\mathscr{C}$ a set of real functions on X. Suppose that α and β are finite and*

(1.1) $\mathscr{C}(\alpha,\beta) = \{f \in \mathscr{C} \mid \alpha \leqq f \leqq \beta\}$ *is σ-convex,*

(1.2) $\alpha(x) \leqq \varphi(x)\ \forall x \in K; \quad \psi(x) \leqq \beta(x),\ \forall x \in L,$

(1.3) $K \cap \{\varphi<u\}$ *and* $L \cap \{\psi>v\}$ *are* (α,β)*-separated by*

$\mathscr{C}$ *if* $u,v \in \mathscr{C}_0(\alpha,\beta)$ *and* $v-u=\lambda(\beta-\alpha)$ *for some* $\lambda>0$,

where $\mathscr{C}_0(\alpha,\beta)$ is the convex hull of $\mathscr{C}(\alpha,\beta)\cup\{\alpha,\beta\}$. Then there exist a function $g \in \mathscr{C}(\alpha,\beta)$, so that

$$g(x) \leqq \varphi(x)\ \forall x \in K; \quad \psi(x) \leqq g(x),\ \forall x \in L . \tag{1.4}$$

Received November 17, 1980; in revised form July 18, 1981.

Supported by the Danish National Science Research Council.

NOTE. Observe that (1.4) implies that $\psi \leqq g \leqq \varphi$ on $K \cap L$, so the result is certainly an "in between theorem". And by (1) we find

$$K \cap L \cap \{\varphi \leqq u\} \cap \{\psi \geqq v\} = K \cap L \cap \{\alpha = \beta = \varphi = \psi = u = v\} \tag{4}$$

whenever $\alpha \leqq u \leqq v \leqq \beta$ and $u < v$ on $\{\alpha < \beta\}$. In particular, we see that the two sets, $K \cap \{\varphi < u\}$ and $L \cap \{\psi > v\}$, in (1.3) are disjoint.

PROOF. Clearly we may assume that $\alpha \leqq g \leqq \beta$ for all $g \in \mathscr{C}$. Then by (1.1) we find that $\mathscr{C}$ is a convex set of functions. Now let us fix an $s \in]0, \frac{1}{2}[$. We shall then show that there exist functions $g_1, g_2, \ldots \in \mathscr{C}$ so that

$$\text{(i)} \quad \varphi(x) \geqq \sum_{j=1}^{n-1} s(1-s)^{j-1} g_j(x) + (1-s)^{n-1}\alpha(x), \qquad \forall x \in K, \forall n \geqq 1$$

$$\text{(ii)} \quad \psi(x) \leqq \sum_{j=1}^{n-1} s(1-s)^{j-1} g_j(x) + (1-s)^{n-1}\beta(x), \qquad \forall x \in L, \forall n \geqq 1.$$

Note that (i) and (ii) hold for $n=1$ by (1.2). So assume that $g_1 \ldots g_{n-1} \in \mathscr{C}$ satisfies (i) and (ii) for some $n \geqq 1$. Then

$$u = \sum_{j=1}^{n-1} s(1-s)^{j-1} g_j + (1-s)^n \alpha + s(1-s)^{n-1}\beta \in \mathscr{C}_0$$

$$v = \sum_{j=1}^{n-1} s(1-s)^{j-1} g_j + s(1-s)^{n-1}\alpha + (1-s)^n \beta \in \mathscr{C}_0 ,$$

since

$$\sum_{j=1}^{n} s(1-s)^{j-1} = 1 - (1-s)^n .$$

Moreover we have

$$v - u = (1-s)^{n-1}(1-2s)(\beta - \alpha) .$$

Since $0 < s < \frac{1}{2}$ we conclude from (1.3) that there exists $g_n \in \mathscr{C}$ with

$$g_n(x) = \begin{cases} \alpha(x) & \forall x \in K \cap \{\varphi < u\} \\ \beta(x) & \forall x \in L \cap \{\psi > v\} . \end{cases}$$

We shall then verify that $\{g_1, \ldots, g_n\}$ satisfies (i) and (ii). By induction hypothesis we have

$$\varphi - \sum_{1}^{n} s(1-s)^{j-1} g_j \geqq \begin{cases} (1-s)^{n-1}\alpha - s(1-s)^{n-1}\alpha & \text{on} \quad K \cap \{\varphi < u\} \\ u - \sum_{1}^{n} s(1-s)^{j-1} g_j & \text{on} \quad \{\varphi \geqq u\} , \end{cases}$$

and since $g_n \leqq \beta$ we find

$$(1-s)^{n-1}\alpha - s(1-s)^{n-1}\alpha = (1-s)^n\alpha$$

$$u - \sum_{j=1}^{n} s(1-s)^{j-1} g_j = (1-s)^n\alpha + s(1-s)^{n-1}(\beta - g_n) \geqq (1-s)^n\alpha .$$

Thus (i) holds since $K \subseteq K \cap \{\varphi < u\} \cup \{\varphi \geqq u\}$. Similarly

$$\psi - \sum_{1}^{n} s(1-s)^{j-1} g_j \leqq \begin{cases} (1-s)^{n-1}\beta - s(1-s)^{n-1}\beta & \text{on} \quad L \cap \{\psi > v\} \\ v - \sum_{1}^{n} s(1-s)^{j-1} g_j & \text{on} \quad \{\psi \leqq v\} , \end{cases}$$

and since $g_n \geqq \alpha$ we have

$$(1-s)^{n-1}\beta - s(1-s)^{n-1}\beta = (1-s)^n\beta$$

$$v - \sum_{1}^{n} s(1-s)^{j-1} g_j \leqq (1-s)^n\beta .$$

Thus (ii) holds since $L \subseteq \{\psi > v\} \cup \{\psi \leqq v\}$, and the induction is completed, i.e. there exists $\{g_n\} \subseteq \mathscr{C}$ satisfying (i) and (ii).

By assumption we have $\alpha \leqq g_n \leqq \beta$, $\forall n$, hence from (1.1) we conclude that

$$g = \sum_{1}^{\infty} s(1-s)^{j-1} g_j \in \mathscr{C}$$

and $\alpha \leqq g \leqq \beta$. Letting $n \to \infty$ in (i) and (ii) we obtain $\varphi \geqq g$ on K and $\psi \leqq g$ on L, which proves the theorem.

The crucial condition in Theorem 1 is of course the separation property (1.3). So let us discuss this condition in some detail.

Let $\alpha \leqq \beta$ be two real functions on X, then notice that

$$(f \vee \alpha) \wedge \beta = (f \wedge \beta) \vee \alpha = \begin{cases} \alpha & \text{if } f \leqq \alpha \\ f & \text{if } \alpha \leqq f \leqq \beta \\ \beta & \text{if } f \geqq \beta . \end{cases}$$

Thus if $\mathscr{C}$ contains $(f \vee \alpha) \wedge \beta$ for all $f \in \mathscr{C}$, then K and L are (α, β)-separated by $\mathscr{C}$ if there exists a function $f \in \mathscr{C}$ satisfying

$$(2) \qquad f(x) \leqq \alpha(x), \ \forall x \in K, \quad f(x) \geqq \beta(x), \ \forall x \in L .$$

It is wellknown that compactness makes separation possible in many cases. If $\mathscr{G}$ is a paving on X we say that $K \subseteq X$ is *$\mathscr{G}$-compact* if every covering of K with sets from $\mathscr{G}$ admits a finite subcovering. Note that by Alexandrow's subbasis theorem (see e.g. [1, p. 139]) this is equivalent to compactness in the topology generated by $\mathscr{G}$.

If $\mathscr{C}$ is a set of real functions on X, then $\mathscr{C}$ is said to be $(\wedge f)$-stable if

$$f \wedge g \in \mathscr{C}, \qquad \forall f, g \in \mathscr{C}\,.$$

Similarly $\mathscr{C}$ is $(\vee f)$-*stable* if

$$f \vee g \in \mathscr{C}, \qquad \forall f, g \in \mathscr{C}$$

and $\mathscr{C}$ is a *lattice* if $\mathscr{C}$ is $(\vee f, \wedge f)$-stable.

2. PROPOSITION. *Let $\alpha \leqq \beta$ be two real functions on X, $\mathscr{G}_1$ and $\mathscr{G}_2$ two pavings on X, K_1 and K_2 two subsets of X and $\mathscr{C}$ a set of functions: $X \to \mathrm{R}$. Then K_1 and K_2 are (α, β)-separated by $\mathscr{C}$ in each of the following cases:*

(2.1) *$\mathscr{C}$ is $(\wedge f)$-stable, K_1 is $\mathscr{G}_1$-compact, and $\forall x_1 \in K_1\ \exists G_1 \in \mathscr{G}_1$ so that $x_1 \in G_1$ and $K_1 \cap G_1$ and K_2 are (α, β)-separated by $\mathscr{C}$.*

(2.2) *$\mathscr{C}$ is $(\vee f)$-stable, K_2 is $\mathscr{G}_2$-compact, and $\forall x_2 \in K_2\ \exists G_2 \in \mathscr{G}_2$ so that $x_2 \in G_2$, and K_1 and $K_2 \cap G_2$ are (α, β)-separated by $\mathscr{C}$.*

(2.3) *$\mathscr{C}$ is a lattice, K_1 is $\mathscr{G}_1$-compact, K_2 is $\mathscr{G}_2$-compact and $\forall (x_1, x_2) \in K_1 \times K_2\ \exists (G_1, G_2) \in \mathscr{G}_1 \times \mathscr{G}_2$ so that $(x_1, x_2) \in G_1 \times G_2$ and $K_1 \cap G_1$ and $K_2 \cap G_2$ are (α, β)-separated by $\mathscr{C}$.*

PROOF. Suppose that (2.1) holds. Let $\mathscr{G}_1^*$ be the set of all $G_1 \in \mathscr{G}_1$ so that $K_1 \cap G_1$ and K_2 are (α, β)-separated by $\mathscr{C}$. Then $\mathscr{G}_1^*$ is a covering of K_1 by assumption. And since K_1 is $\mathscr{G}_1$-compact, there exists $G_1, \ldots, G_n \in \mathscr{G}_1$ and $f_1, \ldots, f_n \in \mathscr{C}$ so that

(i) $\qquad K_1 \subseteq \bigcup_{j=1}^{n} G_j$

(ii) $\qquad \alpha \leqq f_j \leqq \beta,\ f_j = \alpha$ on $K_1 \cap G_j$, $\ f_j = \beta$ on $K_2 \qquad \forall 1 \leqq j \leqq n\,.$

Let $f = \bigwedge_{j=1}^{n} f_j$, then $f \in \mathscr{C}$ by assumption, and $\alpha \leqq f \leqq \beta$. Moreover, by (i) and (ii) we have $f = \alpha$ on K_1 and $f = \beta$ on K_2.

The other two cases follow similarly.

3. EXAMPLE. (Continuous functions). Let X be a topological space and put $\alpha \equiv 0$, $\beta \equiv 1$ and $\mathscr{C}$ the set of all continuous functions: $X \to [0, 1]$. Then clearly $\mathscr{C}$ satisfies (1.1) and it is well-known (and easy to verify) that

(3.1) K_1 and K_2 are (0,1)-separated by $\mathscr{C}$ if and only if there exist disjoint zero-sets F_1 and F_2 with $K_j \subseteq F_j$ for $j = 1,2$.

($F \subseteq X$ is a *zero-set* if $F = f^{-1}(0)$ for some continuous function $f: X \to \mathbf{R}$.)

Now let K, L, φ and ψ satisfy (1.2) with $(\alpha, \beta) = (0, 1)$. Then (1.3) holds, if and only if we have

(3.2) $K \cap \{\varphi \leqq r\}$ and $L \cap \{\psi \geqq q\}$ are separated by disjoint zero-sets for all real numbers $r < q$.

The necessity of (3.2) is evident. To see that (3.2) is sufficient we choose zero-sets $F_1(r,q)$ and $F_2(r,q)$ so that

(i) $K_1 \cap \{\varphi \leqq r\} \subseteq F_1(r,q), \quad \forall r<q$

(ii) $K_2 \cap \{\psi \geqq q\} \subseteq F_2(r,q), \quad \forall r<q$

(iii) $F_1(r,q) \cap F_2(r,q) = \emptyset, \quad \forall r<q$.

Let D be a countable dense subset of $\mathbf{R}$ and let $u, v \in \mathscr{C}$ so that $u<v$. If

$$F_1 = \bigcap_{r \in D} [\{u \geqq r\} \cup \bigcap_{q \in D, q>r} F_1(r,q)]$$

$$F_2 = \bigcap_{q \in D} [\{v \leqq q\} \cup \bigcap_{r \in D, r<q} F_2(r,q)] ,$$

then F_1 and F_2 are zero-sets since the zero-sets evidently are closed under countable intersections and finite unions. Moreover, it is easily verified that

(iv) $$K_1 \cap \{\varphi < u\} \subseteq F_1$$

(v) $$K_2 \cap \{\psi > v\} \subseteq F_2 .$$

Now let $x \in X$. Since $u(x) < v(x)$ there exist $r, q \in D$ so that $u(x) < r < q < v(x)$. If $x \in F_1 \cap F_2$, then $x \in F_2(r,q)$ since $x \notin \{v \leqq q\}$, and $x \in F_1(r,q)$, since $x \notin \{u \geqq r\}$, but this contradicts (iii), and so $F_1 \cap F_2 = \emptyset$. Thus the sufficiency of (3.2) follows from (3.1).

Combining this with Proposition 2 gives the following corrolaries (cf. [2]):

4. Corollary. *Let X be a topological space, K and L two subsets of X and φ and ψ functions: $X \to \bar{\mathbf{R}}$ satisfying*

(4.1) $0 \leqq \varphi(x)\ \forall x \in K; \quad \psi(x) \leqq 1,\ \forall x \in L,$

(4.2) *$K \cap \{\varphi \leqq r\}$ and $L \cap \{\psi \geqq q\}$ are separated by disjoint zero-sets for all real numbers $r < q$.*

Then there exists a continuous function f: $X \to [0,1]$ so that $f \leqq \varphi$ on K and $f \geqq \psi$ on L.

5. Corollary. *Let K and L be closed subsets of X, and suppose that φ is lower semicontinuous on K and ψ is upper semicontinuous on L, and that*

(5.1) $\quad 0 \leqq \varphi(x) \;\forall x \in K; \quad \psi(x) \leqq 1 \;\forall x \in L,$

(5.2) $\quad \psi(x) \leqq \varphi(x), \quad \forall x \in K \cap L.$

Then there exists a continuous function $f\colon X \to [0,1]$, *so that* $f \leqq \varphi$ *on* K *and* $f \geqq \psi$ *on* L *in either of the following three cases,*

(5.3) *K and L are compact, and X is completely Hausdorff* (i.e. *points are separated by* $\mathscr{C}(X)$).

(5.4) *Either K or L is compact, and X is completely regular* (i.e. *points and closed sets are separated by* $\mathscr{C}(X)$).

(5.5) *X is normal* (i.e. *closed sets are separated by* $\mathscr{C}(X)$).

6. Example. (Uniformly continuous function). Let $(X,\mathscr{U})$ be a uniform space and put $\alpha \equiv 0$ and $\beta \equiv 1$ and $\mathscr{C}$ the set of all uniformly continuous functions: $X \to [0,1]$. Then $\mathscr{C}$ satisfies (1.1). Two sets, K_1 and K_2, are said to be *positively separated* if there exists $U \in \mathscr{U}$ so that

$$U(K_1) \cap U(K_2) = \emptyset$$

where $U(A) = \{x \mid \exists y \in A \text{ with } (x,y) \in U\}$. If K_1 and K_2 are positively separated and $U \in \mathscr{U}$ is as above, we can find a uniformly continuous pseudometric d, so that

$$\{(x,y) \mid d(x,y) < 1\} \subseteq U$$

and $0 \leqq d \leqq 1$. Then the function

$$f(x) = \inf_{y \in K_1} d(x,y)$$

clearly satisfies

(i) $\quad 0 \leqq f \leqq 1$

(ii) $\quad |f(x) - f(y)| \leqq d(x,y)$

(iii) $\quad f(x) = 0 \;\forall x \in K_1, \quad f(x) = 1 \;\forall x \in K_2.$

Thus K_1 and K_2 are (0,1)-separated by $\mathscr{C}$. The converse is evident, so we find

(6.1) K_1 and K_2 are (0,1)-separated by $\mathscr{C}$ if and only if K_1 and K_2 are positively separated.

Now let K, L, φ and ψ satisfy (1.2) with $\alpha \equiv 0$ and $\beta \equiv 1$. Then (1.3) holds, if and only if

(6.2) $\quad \forall \varepsilon > 0 \, \exists \, U \in \mathscr{U}$ with $\psi(y) - \varphi(x) \leqq \varepsilon, \forall (x,y) \in (K \times L) \cap U$.

The necessity of (6.2) follows from Theorem 1: Let $g \in \mathscr{C}$ be chosen so that $g \leqq \varphi$ on K and $g \geqq \psi$ on L. Then

$$\psi(y) - \varphi(x) \leqq |g(y) - g(x)|, \quad \forall (x,y) \in K \times L .$$

Thus (6.2) follows by uniform continuity of g.

Conversely suppose that (6.2) holds and let $u, v \in \mathscr{C}$ with $v(x) - u(x) \equiv \lambda$ for some $\lambda > 0$. Then we choose $U \in \mathscr{U}$ so that

(iv) $\quad |u(x) - u(y)| < \lambda/2 \quad \forall (x,y) \in U$

(v) $\quad \psi(y) - \varphi(x) < \lambda/2 \quad \forall (x,y) \in (K \times L) \cap U$

and let $W \in \mathscr{U}$ so that $W^{-1} \circ W \subseteqq U$. Now put

$$K_1 = K \cap \{\varphi < u\}, \qquad K_2 = L \cap \{\psi > v\}$$

and suppose that $x \in W(K_1) \cap W(K_2)$. Then there exist $x_1 \in K_1$ and $x_2 \in K_2$ with $(x_1, x) \in W^{-1}$ and $(x, x_2) \in W$. By definition of K_1 and K_2 we have

$$\begin{aligned} \tfrac{1}{2}\lambda &< \lambda - |u(x_1) - u(x_2)| \leqq \lambda + u(x_2) - u(x_1) \\ &= v(x_2) - u(x_1) \leqq \psi(x_2) - \varphi(x_1) . \end{aligned}$$

But this contradicts (v), since $(x_1, x_2) \in W^{-1} \circ W \subseteqq U$ and $(x_1, x_2) \in K_1 \times K_2 \subseteqq K \times L$. Thus $W(K_1) \cap W(K_2) = \varnothing$ and (1.3) is then a consequence of (6.1).

Thus we have the following in-between theorem for uniformly continuous functions (cf. [2]):

7. Corollary. *Let $(X, \mathscr{U})$ be a uniform space, K and L subsets of X, and φ and ψ functions: $X \to \bar{\mathsf{R}}$ satisfying*

(7.1) $\quad 0 \leqq \varphi(x) \, \forall x \in K, \quad \psi(x) \leqq 1, \forall x \in L$,

(7.2) $\quad \forall \varepsilon > 0 \, \exists \, U \in \mathscr{U}$ *with* $\psi(y) - \varphi(x) \leqq \varepsilon, \, \forall (x,y) \in (K \times L) \cap U$.

Then there exists a uniformly continuous function $f: X \to [0,1]$. with $f \leqq \varphi$ on K and $f \geqq \psi$ on L.

Putting $K = X$ and $\psi = \varphi$ on L we obtain the following extension theorem:

8. Corollary. *Let $(X, \mathscr{U})$ be a uniform space, L a subset of X and f_0: $L \to [0,1]$ a uniformly continuous function. If φ: $X \to \bar{\mathsf{R}}_+$ satisfies*

(8.1) $\quad \forall \varepsilon > 0 \, \exists \, U \in \mathscr{U}$ *with* $\varphi(x) \geqq f_0(y) - \varepsilon, \, \forall (x,y) \in U \cap (X \times L)$

then f_0 admits a uniformly continuous extension, $f: X \to [0,1]$, such that $f \leqq \varphi$ on X.

REMARKS (a): If L is compact, φ is lower semicontinuous and $\varphi \geqq f_0$ on L, then (8.1) holds. To see this choose $U_y \in \mathscr{U}$ for $y \in L$ and $U_0 \in \mathscr{U}$ so that

$$(U_y \circ U_y)(y) \subseteqq \{x \mid \varphi(x) > f_0(y) - \tfrac{1}{2}\varepsilon\}$$
$$(U_0 \circ U_0) \cap (L \times L) \subseteqq \{(x,y) \in L \times L \mid |f_0(x) - f_0(y)| < \tfrac{1}{2}\varepsilon\},$$

and consider the covering $\{U_y(y) \mid y \in L\}$ of L. Let $\{U_y(y) \mid y \in \pi\}$ be a finite subcovering, then

$$U = U_0 \cap \bigcap_{y \in \pi} U_y \in \mathscr{U}$$

works in (8.1).

(b): Note that $f_0(y) - \varepsilon \leqq 1 - \varepsilon$, so (8.1) holds whenever we have

(8.2) $\qquad \forall \varepsilon > 0 \; \exists U \in \mathscr{U}$ with $\varphi(x) \geqq 1 - \varepsilon \quad \forall x \in U(L)$.

(c): In particular we find that a *bounded* uniformly continuous function on an arbitrary subset admits a uniformly continuous bounded extension. This is not true for unbounded functions, e.g. if $X = \mathbf{R}$ and $L = \mathbf{N}$ and $f_0(x) = x^2$ for $x \in \mathbf{N}$, then f_0 has no uniformly continuous extension to all of $\mathbf{R}$.

9. EXAMPLE. ($\mathscr{A}$- and $\mathscr{C}\mathscr{A}$-functions). Let $\mathscr{B}$ be a $(\varnothing, \cup c, \cap f)$-*stable paving* on X (i.e. $\mathscr{B}$ is closed under countable unions, finite intersections, and $\varnothing \in \mathscr{B}$). Let $\mathscr{C}$ be the set of functions $f: X \to [0,1]$ of the form

$$f = \sum_n \lambda_n 1_{B_n}$$

where $\lambda_n \geqq 0$, $\sum_n \lambda_n = 1$ and $\{B_n\} \subseteqq \mathscr{B}$. Then clearly $\mathscr{C}$ satisfies (1.1) and since $\mathscr{B}$ is $(\cup c)$-stable we have

(9.1) $\qquad K_1$ and K_2 are (0,1)-separated by $\mathscr{C}$ if and only if there exists $B \in \mathscr{B}$ with $B \cap K_1 = \varnothing$ and $K_2 \subseteqq B$.

If $\mathscr{A}_1$ and $\mathscr{A}_2$ are pavings on X we say that $\mathscr{B}$ *separates* $\mathscr{A}_1$ and $\mathscr{A}_2$ if (9.1) holds for all disjoint sets $(K_1, K_2) \in \mathscr{A}_1 \times \mathscr{A}_2$.

Let $\mathscr{A}$ be a $(\varnothing, \cap c, \cup f)$-stable paving on X. If $K \subseteqq X$ and $f: K \to \mathbf{R}$ is a map, we say that f is an *$\mathscr{A}$-function* on K if

$$K \cap \{f \geqq a\} \in \mathscr{A}, \qquad \forall a \in \bar{\mathbf{R}};$$

and f is a *$\mathscr{C}\mathscr{A}$-function* on K if $(-f)$ is an *$\mathscr{A}$-function on* K, i.e. if

$$K \cap \{f \leqq a\} \in \mathscr{A}, \qquad \forall a \in \bar{\mathbf{R}}.$$

Note that in both cases we have $K \in \mathscr{A}$. Note also that the $\mathscr{A}$-functions are $(\wedge\, c, \vee\, f)$-stable and the $\mathscr{C}\mathscr{A}$-functions are $(\wedge\, f, \vee\, c)$-stable.

Let $\Phi : \mathsf{R}^n \to \mathsf{R}$ be an *increasing function* (i.e. $\Phi(x_1, \ldots, x_n) \leqq \Phi(y_1, \ldots, y_n)$ if $x_j \leqq y_j$, $\forall\, 1 \leqq j \leqq n$). If Φ is upper semicontinuous we have

$$\{x \mid \Phi(x) \geqq a\} = \bigcap_q \bigcup_{j=1}^n \{x \mid x_j \geqq q_j\}$$

where the intersection is taken over all rational vectors $q = (q_1, \ldots, q_n)$ satisfying: $\Phi(q) < a$. Similarly if Φ is lower semicontinuous we then have

$$\{x \mid \Phi(x) \leqq a\} = \bigcap_q \bigcup_{j=1}^n \{x \mid x_j \leqq q_j\}\,,$$

where the intersection is taken over all rational vectors q with $\Phi(q) > a$. Thus the $(\bigcap c, \bigcup f)$-stability of $\mathscr{A}$ implies

(9.2) If Φ is increasing and upper semicontinuous (lower semicontinuous) on R^n and $f_1, \ldots, f_n$ are $\mathscr{A}$-functions ($\mathscr{C}\mathscr{A}$-functions), then $\Phi(f_1, \ldots, f_n)$ is an $\mathscr{A}$-function ($\mathscr{C}\mathscr{A}$-function).

In particular the $\mathscr{A}$-functions and the $\mathscr{C}\mathscr{A}$-functions form a convex cone (put $\Phi(x_1, x_2) = \lambda x_1 + \mu x_2$, $\lambda, \mu \geqq 0$).

Let $f = \sum \lambda_n 1_{B_n}$ be a member of $\mathscr{C}$ and put

$$\Gamma(a) = \{\pi \mid \pi \text{ finite} \subseteqq \mathsf{N},\ \sum_{j \in \pi} \lambda_j > a\}\,.$$

We then have

$$\{f > a\} = \bigcup_{\pi \in \Gamma(a)} \bigcap_{j \in \pi} B_j$$

$$\{f \leqq a\} = \bigcap_{\pi \in \Gamma(a)} (X \setminus \bigcap_{j \in \pi} B_j)$$

$$\{f \geqq a\} = \bigcap_{n=1}^{\infty} \{f > a - 1/n\}\,.$$

Thus by $(\bigcup c, \bigcap f)$-stability of $\mathscr{B}$ and $(\bigcap c)$-stability of $\mathscr{A}$ we find

(9.3) If $K \cap B \in \mathscr{A}$ for all $B \in \mathscr{B}$, then f is an $\mathscr{A}$-function on K for all $f \in \mathscr{C}$.

(9.4) If $K \setminus B \in \mathscr{A}$ for all $B \in \mathscr{B}$, then f is a $\mathscr{C}\mathscr{A}$-function on K for all $f \in \mathscr{C}$.

We can now deduce the following corollary to Theorem 1.

10. COROLLARY. *Let* $\mathscr{B}$ *be a* $(\varnothing, \cup c, \cap f)$*-stable paving and* $\mathscr{A}_1$ *and* $\mathscr{A}_2$ *two* $(\varnothing, \cup f, \cap c)$*-stable pavings such that* $\mathscr{B}$ *separates* $\mathscr{A}_1$ *and* $\mathscr{A}_2$. *Now let* K *and* L *be subsets of* X *and* φ *and* ψ *functions*: $X \to \bar{\mathrm{R}}$ *satisfying*

(10.1) $\quad 0 \leqq \varphi$ *on* K, $\psi \leqq 1$ *on* L *and* $\psi \leqq \varphi$ *on* $K \cap L$.

(10.2) $\quad \varphi$ *is a* $\mathscr{C}\mathscr{A}_1$*-function on* K, *and* ψ *is an* $\mathscr{A}_2$*-function on* L.

(10.3) $\quad K \cap B \in \mathscr{A}_1, \ \forall B \in \mathscr{B}$.

(10.4) $\quad L \setminus B \in \mathscr{A}_2, \ \forall B \in \mathscr{B}$.

Then there exist $\{B_n\} \in \mathscr{B}$ *and* $\lambda_n \geqq 0$ *with* $\sum_n \lambda_n = 1$, *such that* $f = \sum_n \lambda_n 1_{B_n}$ *satisfies*

(10.5) $\quad f(x) \leqq \varphi(x), \quad \forall x \in K$

(10.6) $\quad \psi(x) \leqq f(x), \quad \forall x \in L$.

PROOF. Let $\mathscr{C}$ be defined as above, then $\mathscr{C}$ satisfies (1.1) and (1.2) holds by (10.1). So we only have to verify that (1.3) holds.

Let $u, v \in \mathscr{C}_0$ with $v - u \equiv \lambda$, where $\lambda \in \mathrm{R}_+$. Then $u = a + (1-a)g$ and $v = \lambda + a + (1-a)g$ for some $g \in \mathscr{C}$ and some $a \in [0,1]$. From (9.2) and (9.3) we deduce that $\varphi - u$ is a $\mathscr{C}\mathscr{A}_1$-function, and (9.2) and (9.4) show that $\psi - v$ is an $\mathscr{A}_2$-function. Thus

$$K \cap \{\varphi < u\} \subseteqq K \cap \{\varphi - u \leqq 0\} = A_1 \in \mathscr{A}_1$$

$$L \cap \{\psi > v\} \subseteqq L \cap \{\psi - v \geqq 0\} = A_2 \in \mathscr{A}_2$$

and since $A_1 \cap A_2 = \varnothing$ by (10.1) we see that (1.3) follows from (9.1) since $\mathscr{B}$ separates $\mathscr{A}_1$ and $\mathscr{A}_2$.

REMARKS. (a): If $\mathscr{B}$ is a σ-algebra, then the set $\mathscr{C}$ consists of all measurable functions: $X \to [0,1]$. This follows from Theorem 1, but may be seen directly. Actually, if $\{\lambda_n\}$ is any fixed sequence of non-negative numbers satisfying

$$\sum_{j=1}^{\infty} \lambda_j = 1, \quad \lambda_n \leqq \sum_{j=n+1}^{\infty} \lambda_j, \ \forall n \geqq 1, \tag{10.7}$$

then any measurable function $f: X \to [0,1]$ is the form

$$f = \sum_{n=1}^{\infty} \lambda_n 1_{B_n}$$

for some sequence $\{B_n\} \subseteqq \mathscr{B}$.

(b): If $\mathscr{B}$ is a σ-algebra separating $\mathscr{A}_1$ and $\mathscr{A}_2$, then $\mathscr{B}$ separates $\mathscr{A}_2$ and $\mathscr{A}_1$. Moreover, if $\mathscr{B} \subseteqq \mathscr{A}_1 \cap \mathscr{A}_2$, then (10.3) and (10.4) hold whenever $K \in \mathscr{A}_1$ and $L \in \mathscr{A}_2$.

(c): Let $\mathscr{B}_0$ be an algebra on X, and let $\mathscr{A} = \mathscr{A}_1 = \mathscr{A}_2$ be the paving of all $\mathscr{B}_0$-Souslin sets (see [2]). If

$$\mathscr{B} = \{B \mid B \in \mathscr{A} \text{ and } X \setminus B \in \mathscr{A}\},$$

then $\mathscr{B}$ is a σ-algebra satisfying (10.3) and (10.4) for all $K, L \in \mathscr{A}$, and by the second separation theorem (see [3]) we have that $\mathscr{B}$ separates $\mathscr{A}_1$ and $\mathscr{A}_2$.

(d): Let X be a Hausdorff space and $\mathscr{B}$ a $(\emptyset, \cap c, \cup c)$-stable paving on X, so that $\mathscr{B} \subseteq \mathscr{B}(X)$ (the Borel σ-algebra) and

(10.8) $\quad \forall x_1 \neq x_2 \; \exists B_1, B_2 \in \mathscr{B}$ with $x_j \in \operatorname{int}(B_j)$ for $j = 1,2$ and $B_1 \cap B_2 = \emptyset$.

If $\mathscr{A}_1 = \mathscr{A}_2 = \mathscr{A}$ is the paving of all K-analytic subsets of X (see [3]), then (10.3) and (10.4) hold if $K, L \in \mathscr{A}$ and $\mathscr{B}$ separates $\mathscr{A}_1$ and $\mathscr{A}_2$ (see [3]).

(e): Let X be a Hausdorff space and $\mathscr{B}$ a $(\emptyset, \cap c, \cup c)$-stable paving, so that $\mathscr{B} \subseteq \mathscr{B}(X) \cap S(F(X))$ ($S(F(X))$ is the Souslin sets over the closed sets). Suppose that $\mathscr{B}$ satisfies

(10.9) $\quad \forall x \in X \; \forall U$ a neighbourhood of $x \; \exists B \in \mathscr{B}$, such that $x \in \operatorname{int} B \subseteq U$.

If $\mathscr{A}_1 = S(F(X))$ and $\mathscr{A}_2$ is the paving of all K-analytic sets, then $\mathscr{B}$ separates $\mathscr{A}_1$ and $\mathscr{A}_2$ (but not $\mathscr{A}_2$ and $\mathscr{A}_1$ in general) and (10.3) and (10.4) hold provided that $K \in \mathscr{A}_1$ and $L \in \mathscr{A}_2$ (see [3]).

(f): If $\mathscr{B}$ is a σ-algebra on X, $\mathscr{A}_1$ is the paving of all Blackwell subsets of $(X, \mathscr{B})$ and $\mathscr{A}_2$ is the paving of all $\mathscr{B}$-Souslin sets, then $\mathscr{B}$ separates $\mathscr{A}_1$ and $\mathscr{A}_2$ and (10.3) and (10.4) hold provided that $K \in \mathscr{A}_1$ and $L \in \mathscr{A}_2$.

In [2] Preiss and Vilimovsky proves an "in between theorem" for uniformly continuous $\bar{\mathbb{R}}$-valued functions. By choosing the uniformity appropriately one easily obtains the corollaries in Example 3 and Example 6 from the results in [2]. The "in between theorem" (Theorem 1) presented here and the "in between theorem" presented in [2] are however not comparable. The main difference is that in [2] the functions may be unbounded (and even infinite) whereas in our setting the upper function φ is supposed to bounded from the below by α and the lower function is supposed to be bounded from above by β, however in [2] the "in between function" is chosen from a much larger class, viz. the set of uniformly continuous functions, than in the case studied here, viz. an abstract set of functions which is only supposed to be σ-convex.

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