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A NOTE ON FUNCTORS THAT VANISH ON INDECOMPOSABLE MODULES

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Introduction.

For each module A whose endomorphism ring is local (Le-module) we define a functor $H_A\colon {}_R\mathscr{C} \to {}_{\Delta_A}\mathscr{C}$ that vanishes on indecomposable modules and

$$H_A\Big(\coprod_I M_i\Big) \cong \prod_I H_A(M_i)$$

for any family of left R-modules $\{M_i\}_I$.

Let $\mathscr{C}$ denote the category of left R-modules and $\mathscr{I}$ its radical [5]. We use the functors H_A to show that $\bar{A}$ is an injective and projective object in a full subcategory of $\bar{\mathscr{C}} = \mathscr{C}/\mathscr{I}$ whenever A is an Le-module. We also show that $\bar{E}$ is an injective and projective object in $\bar{\mathscr{C}}$ if E is an injective indecomposable module.

We then apply the functors H_A to study the property that a set of Le-modules complements maximal summands. In [4] we introduced an independence structure on the set $\mathscr{L}(M)$ of Le-modules that are direct summands of the module M. We show in Theorem 3.4 that any basis for $\mathscr{L}(M)$ complements Le-maximal summands, and this generalizes an important part of Azymaya's theorem. We also discuss when a module has an Le-decomposition that complements direct summands (Theorem 3.10).

The main purpose of the last section is to show that the functors H_A may differ from the zero functors. If R is a commutative local ring whose maximal ideal is principal, we obtain a necessary condition on R so that the direct product $\prod_1^\infty R$ is a direct sum of indecomposable modules.

Terminology.

The terminology follows [3] and [4]. We consider left modules over an associative ring R with an identity. We let ${}_R\mathscr{C}$ ($\mathscr{C}_R$) denote the category of left (right) R-modules.

A module A is called an Le-*module* if $A \neq (0)$ and End A is a local ring. We also assume that indecomposable modules are $\neq (0)$.

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A module M has an Le-*decomposition* if M is a direct sum of Le-modules.

If M is an R-module we let J_M denote the Jacobson radical of $\operatorname{End} M$.

The symbol $\oplus$ is used for *internal sum.* The expression $M = N_1 \oplus N_2$ expresses that N_1 and N_2 are submodules of M, $N_1 \cap N_2 = (0)$ and $N_1 + N_2 = M$.

A decomposition $M = \oplus_I M_i$ *complements direct summands,* if for any direct summand K of M, $K \oplus (\oplus_J M_j) = M$ for some subset $J \subset I$.

1.

Let A be an Le-module and M an arbitrary R-module. Let $J(A, M)$ be the set of R-homomorphisms from A to M which are not split monomorphisms. Then $J(A, M)$ is a submodule of the right $\operatorname{End} A$-module $\operatorname{Hom}_R(A, M)$ and $\operatorname{Hom}_R(A, M) J_A \subset J(A, M)$. Hence $F_A(M) = \operatorname{Hom}_R(A, M)/J(A, M)$ is a right Δ_A-module where Δ_A denotes the division ring $\operatorname{End} A/J_A$. The covariant additive functor $F_A \colon {}_R\mathscr{C} \to \mathscr{C}_{\Delta_A}$ commutes with direct sums by Lemma 1.3 [4] and Theorem 25 [3].

Let $J(M, A)$ be the R-homomorphisms from M to A which are not split epimorphisms. Then $J(M, A)$ is a submodule of the left $\operatorname{End} A$-module $\operatorname{Hom}_R(M, A)$ and $J_A \operatorname{Hom}_R(M, A) \subset J(M, A)$. Hence $G_A(M) = \operatorname{Hom}_R(M, A)/J(M, A)$ is a left Δ_A-module. We observe that $G_A \colon {}_R\mathscr{C} \to {}_{\Delta_A}\mathscr{C}$ is an additive contravariant functor.

1.1. LEMMA. Let A be an Le-module and $\{M_i\}_I$ a family of R-modules. Let $\{\varepsilon_i \colon M_i \to \coprod M_i\}_I$ be the canonical injections. Then $G_A(\coprod_I M_i)$ together with the maps $\{G_A(\varepsilon_i)\}_I$ is a direct product in ${}_{\Delta_A}\mathscr{C}$ of the vector spaces $G_A(M_i)$.

PROOF. Let $\theta \colon G(\coprod_I M_i) \to \prod_I G(M_i)$ be the map induced by the family $\{G(\varepsilon_i)\}_I$, that is

$$\theta(f + J(\coprod M_i, A)) = (f\varepsilon_i + J(M_i, A))_I .$$

It is trivially seen that θ is surjective.

Assume that $f \colon \coprod_I M_i \to A$ is a split epimorphism. Let $g \colon A \to \coprod_I M_i$ have the property that $fg = 1_A$. Let $\pi_i \colon \coprod_I M_i \to M_i$ be the canonical projections. By Lemma 1.1 in [4], $(f\varepsilon_{i_0})(\pi_{i_0} g) \notin J_A$ for some $i_0 \in I$. Hence $(f\varepsilon_{i_0})(\pi_{i_0} g)$ is an isomorphism and therfore $f\varepsilon_{i_0}$ is a split epimorphism. This proves that θ is injective.

Let V be any right Δ_A-module. Then $\operatorname{Hom}_{\Delta_A}(V, \Delta_A)$ has a canonical structure as a left Δ_A-module. Hence the functor $\operatorname{Hom}_{\Delta_A}(-, \Delta_A)$ takes right Δ_A-modules to left Δ_A-modules. We let F_A^* denote the contravariant functor $\operatorname{Hom}_{\Delta_A}(-, \Delta_A) \circ F_A$. Hence

$$F_A^*\colon {}_R\mathscr{C} \to {}_{\Delta_A}\mathscr{C} \quad \text{and} \quad F_A^*(M) = \operatorname{Hom}_{\Delta_A}(F_A(M), \Delta_A)\,.$$

Since F_A commutes with direct sums, F_A^* takes direct sums to direct products.

We shall define a natural transformation $\eta^A\colon G_A \to F_A^*$. For any R-module M let $\eta_M^A\colon G_A(M) \to F_A^*(M)$ be defined as follows: Given $f\colon M \to A$ and $g\colon A \to M$ we let $\eta_M^A(\bar{f})(\bar{g}) = \overline{fg}$. The bars on f, g, and fg denotes the images modulo $J(M,A)$, $J(A,M)$, and $J(A,A) = J_A$. It is easily seen that η_M^A is a Δ_A-linear and that η^A is a natural transformation. Moreover, we note that η_M^A is a monomorphism for all R-modules M. For if $\bar{f} \neq 0$, then $f\colon M \to A$ is a split epimorphism and there exists some $g\colon A \to M$ such that $fg = 1_A$. Hence $\eta_M^A(\bar{f})(\bar{g}) \neq 0$ so $\eta_M^A(\bar{f}) \neq 0$.

1.2. Definition. Let A be an Le-module. Let the functor $H_A\colon {}_R\mathscr{C} \to {}_{\Delta_A}\mathscr{C}$ be the cokernel of the natural transformation $\eta_A\colon G_A \to F_A^*$.

1.3. Remark. H_A is a contravariant functor. For any R-homomorphism $f\colon N \to M$, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 \to & G_A(M) & \xrightarrow{\eta_M^A} & F_{A^*}(M) & \xrightarrow{\text{can.}} & H_A(M) & \to 0 \\ & \downarrow{\scriptstyle G_A(f)} & & \downarrow{\scriptstyle F_{A^*}(f)} & & \downarrow{\scriptstyle H_A(f)} & \\ 0 \to & G_A(N) & \xrightarrow{\eta_N^A} & F_{A^*}(N) & \xrightarrow{\text{can.}} & H_A(N) & \to 0\,. \end{array}$$

1.4. Theorem. Let A be an Le-module. The functor $H_A\colon {}_R\mathscr{C} \to {}_{\Delta_A}\mathscr{C}$ vanishes on indecomposable modules and for any family of R-modules $\{M_i\}_I$,

$$H_A\left(\coprod_I M_i\right) \cong \prod_I H_A(M_i)\,.$$

Proof. Let M be an indecomposable module. If $M \cong A$, $G_A(M) \cong F_A^*(M) \cong \Delta_A$ and it is easily seen that η_M^A is an isomorphism. Hence $H_A(M) = 0$. If M is indecomposable and not isomorphic to A, then $G_A(M) = F_A^*(M) = 0$, and again $H_A(M) = 0$.

Let $\{M_i\}_I$ be a family of R-modules. We have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \to & G_A(\coprod M_i) & \to & F_A^*(\coprod M_i) & \to & H_A(\coprod M_i) & \to 0 \\ & \downarrow{\scriptstyle \prod G_A(\varepsilon_i)} & & \downarrow{\scriptstyle \prod F_{A^*}(\varepsilon_i)} & & \downarrow{\scriptstyle \prod H_A(\varepsilon_i)} & \\ 0 \to & \prod G_A(M_i) & \to & \prod F_A^*(M_i) & \to & \prod H_A(M_i) & \to 0\,. \end{array}$$

By Lemma 1.1, $\prod G_A(\varepsilon_i)$ is an isomorphism and by Theorem 25 in [3], $\prod F_{A^*}(\varepsilon_i)$ is an isomorphism. Hence $\prod H_A(\varepsilon_i)$ is an isomorphism. This completes the proof of Theorem 1.4.

The following two corollaries show that the functors H_A vanish on a large class of modules.

1.5. COROLLARY. *Let M be an R-module which is a direct summand of a module with an indecomposable decomposition. Then $H_A(M)=0$ for all* Le-*modules A.*

1.6. COROLLARY. *If the R-module M has an* Le-*decomposition, then $H_A(M)=0$ for all* Le-*modules A.*

1.7. REMARK. We shall in section 4 show that the functors H_A need not be the zero-functors.

2.

Let $\mathscr{C} = {}_R\mathscr{C}$ be the category of left R-modules, $\mathscr{J}$ its radical and $\bar{\mathscr{C}} = \mathscr{C}/\mathscr{J}$ the quotient category. For any Le-module A we shall show that A is a projective and injective object in a full subcategory of $\bar{\mathscr{C}}$.

2.1. DEFINITION. For R-modules M, N let $J(M,N)$ be the set of $f \in \operatorname{Hom}_R(M,N)$ for which $gf \in J_M$ for all $g \in \operatorname{Hom}_R(N,M)$.

2.2. REMARK. If M and N are Le-modules, Definition 2.1 coincides with the earlier definitions of $J(A,N)$ and $J(M,A)$, see [3].

Let $\bar{\mathscr{C}}$ be the category described by $\operatorname{ob}\bar{\mathscr{C}} = \operatorname{ob}_R\mathscr{C}$ and

$$\bar{\mathscr{C}}[M,N] = \operatorname{Hom}_{\bar{\mathscr{C}}}(\bar{M},\bar{N}) = \operatorname{Hom}_R(M,N)/J(M,N).$$

We use the bar to emphasize that $\bar{M}$ and $\bar{N}$ are considered as objects in $\bar{\mathscr{C}}$. The category $\bar{\mathscr{C}}$ is additive and there exists a canonical functor $T\colon {}_R\mathscr{C} \to \bar{\mathscr{C}}$ (see [3] and [5]).

Let $\bar{\mathscr{D}}$ be the full subcategory of $\bar{\mathscr{C}}$ consisting of the objects $\bar{M}$ for which $H_A(M)=0$ for all Le-modules A. Corollaries 1.5 and 1.6 give information about the "size" of $\bar{\mathscr{D}}$. We do not know when $\bar{\mathscr{D}} = \bar{\mathscr{C}}$.

2.3. THEOREM. *For any* Le-*module A the object $\bar{A}$ is projective and injective in $\bar{\mathscr{D}}$.*

PROOF, By Corollary 1.6, $\bar{A} \in \bar{\mathscr{D}}$. Let $\bar{f}\colon \bar{X} \to \bar{Y}$ be a morphism in $\bar{\mathscr{D}}$ and assume first that $\bar{f}$ is an epimorphism. Let A be an Le-module. We want to show that

$$\bar{f}^* = \operatorname{Hom}(\bar{A}, \bar{f})\colon \ \bar{\mathscr{C}}[\bar{A}, \bar{X}] \to \bar{\mathscr{C}}[\bar{A}, \bar{Y}]$$

is surjective. Let $\alpha = \operatorname{Hom}(\bar{f}^*, \Delta_A)$. By definition $\bar{\mathscr{C}}[M, N] = \operatorname{Hom}_R(M, N)/J(M, N)$. We have a commutative diagram

$$\begin{array}{ccc} \bar{\mathscr{C}}[\bar{Y}, \bar{A}] & \xrightarrow{\operatorname{Hom}(\bar{f}, \bar{A})} & \bar{\mathscr{C}}[\bar{X}, \bar{A}] \\ \downarrow \eta_Y^A & & \downarrow \eta_X^A \\ \operatorname{Hom}_{\Delta_A}(\bar{\mathscr{C}}[\bar{A}, \bar{Y}], \Delta_A) & \xrightarrow{\alpha} & \operatorname{Hom}_{\Delta_A}(\bar{\mathscr{C}}[\bar{A}, \bar{X}], \Delta_A). \end{array}$$

The map $\operatorname{Hom}(\bar{f}, \bar{A})$ is a Δ_A-monomorphism, since $\bar{f}$ is an epimorphism. Since $\bar{X}, \bar{Y} \in \bar{\mathscr{D}}$, it follows that η_X^A and η_Y^A are isomorphisms. We get that $\alpha = \operatorname{Hom}(\bar{f}^*, \Delta_A)$ is a Δ_A-monomorphism and therefore the map

$$\bar{f}^*\colon \ \bar{\mathscr{C}}[\bar{A}, \bar{X}] \to \bar{\mathscr{C}}[\bar{A}, \bar{Y}]$$

is an epimorphism in $\mathscr{C}_{\Delta_A}$. Hence $\bar{A}$ is a projective object in $\bar{\mathscr{D}}$.

Assume now that $\bar{f}\colon \bar{X} \to \bar{Y}$ is a monomorphism in $\bar{\mathscr{D}}$. Then

$$\bar{f}^* = \operatorname{Hom}(\bar{A}, \bar{f})\colon \ \bar{\mathscr{C}}[\bar{A}, \bar{X}] \to \bar{\mathscr{C}}[\bar{A}, \bar{Y}]$$

is a monomorphism of vectorspaces. Therefore $\alpha = \operatorname{Hom}(\bar{f}^*, \Delta_A)$ is an epimorphism in ${}_{\Delta_A}\mathscr{C}$. Since the diagram is commutative, we see that $\operatorname{Hom}(\bar{f}, \bar{A})$ is an epimorphism. This proves that $\bar{A}$ is an injective object and completes the proof of Theorem 2.3.

2.4. REMARK. We observe that the proof of Theorem 2.3 only used that η_M^A is an isomorphism for all R-modules M. Hence $\bar{A}$ is a projective and injective object in $\bar{\mathscr{C}}$ provided $H_A = 0$.

2.5. COROLLARY. *Let R be a ring for which every left R-module is a direct sum of indecomposable modules. Then $\bar{A}$ is a projective and injective object in $\bar{\mathscr{C}}$ for any* Le-*module A.*

PROOF. By Corollary 1.5, $H_A = 0$ for any Le-module A. The corollary follows now from Remark 2.4.

2.6. DEFINITION. (i) Let $\mathscr{L}(M)$ be the family of direct summands of M which are Le-modules.

(ii) Let A be an Le-module. Then $\mathscr{L}_A(M)$ denotes the subset of $\mathscr{L}(M)$ consisting of the Le-modules isomorphic to A.

In [4] we defined an independence structure on $\mathscr{L}(M)$ and $\mathscr{L}_A(M)$. A subset $E \subset \mathscr{L}(M)$ is *independent* if for any finite subset $\{A_1, \ldots, A_k\} \subset E$ the sum $A_1 + \ldots + A_k$ is direct and a direct summand of M. *Shortly we write* $A_1 \oplus \ldots \oplus A_k \lhd M$.

Any independent set in an independence structure is contained in a maximal independent set which is called a basis for the independence structure.

The next lemma is useful when discussing the vanishing of the functors H_A.

2.7. LEMMA. *Let A be an* Le-*module and M any R-module. The following properties are equivalent:*

(i) $H_A(M)=0$.

(ii) *For any independent set $\{A_i\}_I \subset \mathscr{L}_A(M)$ and any map $g\colon \oplus_I A_i \to A$ there exists a map $\hat{g}\colon M \to A$ such that for all $i \in I$, $\hat{g}\,|\,A_i - g\,|\,A_i\colon A_i \to A$ is not an isomorphism.*

(iii) *There exists a basis $\{A_i\}_I \subset \mathscr{L}_A(M)$ such that for any map $g\colon \oplus_I A_i \to A$ there exists a map $\hat{g}\colon M \to A$ such that for all $i \in I$, $\hat{g}\,|\,A_i - g\,|\,A_i\colon A_i \to A$ is not an isomorphism.*

PROOF. (i) $\Rightarrow$ (ii). By definition $H_A(M)=0$, iff η_M^A is an isomorphism. Let $\{A_i\}_I$ be an independent set in $\mathscr{L}_A(M)$. Let $f_i\colon A \to M$ be a family of split monomorphisms from A to M such that $\operatorname{im} f_i = A_i$ for all $i \in I$. By Theorem 2.2 in [4], $\{f_i + J(A,M)\}$ is an independent set in the vector-space $F_A(M)$. Let $g\colon \oplus_I A_i \to A$ and let $g_i = g\,|\,A_i$. Furthermore let $f_i'\colon A \to A_i$ be the isomorphism induced by f_i. Since $\{f_i + J(A,M)\}_I$ are linearly independent over Δ_A, there exists a Δ_A-linear map $\theta\colon F_A(M) \to \Delta_A$ such that

$$\theta(f_i + J(A,M)) = g_i f_i' + J_A .$$

Since η_M^A is surjective there exists a map $\hat{g}\colon M \to A$ such that $\eta_M^A(\hat{g} + J(M,A)) = \theta$. This implies that $\hat{g} f_i - g_i f_i' \in J_A$ for all $i \in I$. It is then easily seen that $\hat{g}\,|\,A_i - g\,|\,A_i$ is not an isomorphism.

(ii) $\Rightarrow$ (iii) Trivial.

(iii) $\Rightarrow$ (i) Let $\{A_i\}_I$ be a basis for $\mathscr{L}_A(M)$ which satisfies condition (iii). Let $f_i\colon A \to M$ be a family of split monomorphisms such that $\operatorname{im} f_i = A_i$ for all $i \in I$. By theorem 2.2 in [4], the family $\{f_i + J(A,M)\}_I$ is a basis for the vectorspace $F_A(M)$. Let $\theta\colon F_A(M) \to \Delta_A$ be a Δ_A-linear map and let $\theta(f_i + J(A,M)) = \delta_i + J_A$ where $\delta_i \in \operatorname{End} A$. Define an R-homomorphism

$$g : \oplus A_i \to A \quad \text{by} \quad g(f_i(a)) = \delta_i(a) \quad \text{for all } a \in A \text{ and } i \in I .$$

By (iii) there exists a $\hat{g}\colon M \to A$ such that

$$\hat{g}\,|\,A_i - g\,|\,A_i : A_i \to A$$

is not an isomorphism. One then easily verifies that $\eta_M^A(g+J(M,A))$ and θ agree on the basis $\{f_i+J(A,M)\}_I$. Hence $\eta_M^A=\theta$. This shows that η_M^A is surjective and therefore $H_A(M)=0$.

2.8. COROLLARY. *Let E be an injective indecomposable module. Then η_M^E is an isomorphism for all R-modules M.*

PROOF. If E is an injective Le-module, the equivalence between (i) and (ii) in Lemma 2.7 implies that $H_E=0$. Hence η_M^E is an isomorphism for all R-modules M.

2.9. THEOREM. *If E is an injective indecomposable module, then $\bar{E}$ is an injective and projective object in $\bar{\mathscr{C}}$.*

PROOF. Let E be an injective indecomposable module. By Corollary 2.8, η_M^E is an isomorphism for all modules M. Hence $H_E=0$ and by Remark 2.4 this implies the theorem.

If $J(M,N)=0$ for any pair of R-Modules M,N then $\mathscr{C}=\bar{\mathscr{C}}$. Hence Theorem 2.9 implies that any injective indecomposable module is projective. We shall show that R actually is a semisimple artinian ring.

2.10 THEOREM. *The following statements are equivalent for a ring R:*
(i) *R is a semisimple artinian ring.*
(ii) *For any pair of left R-modules M and N, $J(M,N)=0$.*
(iii) *For any left R-module M, $J_M=0$.*

PROOF. (i) $\Rightarrow$ (ii) Suppose that R is semisimple. Let f: $M \to N$ and suppose that $f \neq 0$. Let $\ker f \oplus M' = M$. Then $f(M') \oplus N' = N$ for some $N' \subset N$ and we can define a map g: $N \to M$ such that $gf(m')=m'$ for all $m' \in M'$. Then $1_M - gf$ is not an isomorphism and hence $f \notin J(M,N)$.

(ii) $\Rightarrow$ (iii) Trivial.

(iii) $\Rightarrow$ (ii) By Kelly [5] or [3],

$$J(M\oplus N, M\oplus N) \cong J(M,M)\oplus J(N,N)\oplus J(M,N)\oplus J(N,M)\,.$$

Hence (iii) implies (ii).

(ii) $\Rightarrow$ (i) We assume (ii) and shall show that every left R-module M is injective. Let $M \subset E$ where E is injective and the extension $M \subset E$ is essential. We claim the canonical map f: $E \to E/M$ is in $J(E,E/M)$. Let g: $E/M \to E$ be an arbitrary map. Then $\ker gf \supset M$ which is essential in E. Hence by Proposition 18.20 in [1], $gf \in J_E$. By Definition 2.1, this implies that $f \in J(E,E/M)=0$. Hence $M=E$.

3.

In this section we apply the functors H_A to study the property that a set of Le-modules complements maximal summands.

Let K be a direct summand of a module M. If M/K is indecomposable we say that K is a *maximal summand* in M, [2]. If M/K is an Le-module we say that K is an Le-*maximal summand.* If A is an Le-module we say that K is an A-*maximal summand* if $M/K \cong A$.

3.1. Definition. Let $\{A_i\}_I \subset \mathscr{L}(M)$. The family $\{A_i\}_I$ complements Le-maximal summands if for any Le-maximal summand K, $K \oplus A_i = M$ for some $i \in I$.

Let A be an Le-module. The family $\{A_i\}_I \subset \mathscr{L}(M)$ complements A-maximal summands if for any A-maximal summand K, $K \oplus A_i = M$ for some $i \in I$.

3.2. Remark. If a module M has an Le-decomposition then any indecomposable summand of M is an Le-module (Azumaya). Hence a family complements maximal summands if it complements Le-maximal summands.

3.3. Theorem. *Let A be an* Le-*module and let f_i: $A \to M$ $(i \in I)$ be a family of split monomorphisms from A to M. Let V be the subspace of $F_A(M)$ generated by the elements $\{f_i + J(A, M)\}_I$. The following statements are equivalent*:

(i) *The family $\{f_i(A)\}_I$ complements A-maximal summands.*

(ii) *For any split epimorphism g: $M \to A$, $\eta_M^A(g + J(M, A))$ is not zero on all of V.*

Proof. (i) $\Rightarrow$ (ii) Assume that the family $\{f_i(A)\}_I$ complements A-maximal summands. Let f: $M \to A$ be a split epimorphism. Then $\ker g$ is an A-maximal summand. Hence $\ker g \oplus f_i(A) = M$ for some $i \in I$. Hence gf_i: $A \to A$ is an isomorphism, so

$$\eta_M^A(g + J(M, A))(f_i + J(A, M)) \neq 0 .$$

(ii) $\Rightarrow$ (i) Let K be an A-maximal summand and assume (ii). Let g: $M \to A$ be a split epimorphism such that $\ker g = K$. By (ii)

$$\eta_M^A(g + J(M, A))(f_{i_0} + J(A, M)) \neq 0 \quad \text{for some } i_0 \in I .$$

Hence $gf_{i_0} \notin J_A$ and therefore gf_{i_0} is an isomorphism. This implies that $\operatorname{im} f_{i_0}$ is a complement of K. Hence the family $\{f_i(A)\}_I$ complements A-maximal summands.

3.4. Theorem. *Let M be an R-module. Let $\{A_i\}_I$ be a basis for $\mathscr{L}(M)$. Then the family $\{A_i\}_I$ complements* Le-*maximal summands. If A is an* Le-*module and $\{A_i\}_I$ is a basis for $\mathscr{L}_A(M)$, then $\{A_i\}_I$ complements A-maximal summands.*

PROOF. Let A be an Le-module and assume that $\{A_i\}_I$ is a basis for $\mathscr{L}_A(M)$. Let f_i: $A \to M$ be split monomorphisms such that $f_i(A)=A_i$. By Theorem 2.2 in [4], the set $\{f_i+J(A,M)\}_I$ is a basis for $F_A(M)$. Hence the vectorspace V in Theorem 3.3 becomes $F_A(M)$ and since η_M^A is injective, property (ii) is satisfied. Hence $\{A_i\}_I=\{f_i(A)\}_I$ complements A-maximal summands.

By Proposition 2.1 in [4], a set $E\subset\mathscr{L}(M)$ is a basis for $\mathscr{L}(M)$, iff $E\cap\mathscr{L}_A(M)$ is a basis for $\mathscr{L}_A(M)$ for all Le-modules A. This completes the proof of Theorem 3.4.

3.5. REMARK. If $M=\bigoplus_I A_i$ is an Le-decomposition, then $\{A_i\}_I$ is a basis for $\mathscr{L}(M)$. Hence Theorem 3.4 implies that the family $\{A_i\}_I$ complements Le-maximal summands.

3.6. THEOREM. *Let $\{A_i\}_I$ be a family of* Le-*modules that complements* Le-*maximal summands. Then the family $\{A_i\}_I$ contains an independent set which complements* Le-*maximal summands.*

PROOF. Let $\{A_i\}_I$ be a family of Le-modules isomorphic to A that complements A-maximal summands, and let $\{f_i\}_I$ be a family of split monomorphisms from A to M such that $f_i(A)=A_i$ for all $i\in I$. Let V be the vectorspace generated by the elements $\{f_i+J(A,M)\}_{i\in I}$. Let I' be a subset of I such that $\{f_i+J(A,M)\}_{i\in I'}$ is a basis for V. Since the elements $\{f_i+J(A,M)\}_{i\in I}$ and the elements $\{f_i+J(A,M)\}_{i\in I'}$ generate the same vectorspace, it follows from Theorem 3.3 that the family $\{A_i\}_{I'}$ complements A-maximal summands.

The passage from this "local" case to the global situation in Theorem 3.6 is by Proposition 2.1 in [4], which says that a set $E\subset\mathscr{L}(M)$ is independent, iff $E\cap\mathscr{L}_A(M)$ is independent for all Le-modules A. We leave the simple details to the reader.

3.7. REMARK. Let $E\subset\mathscr{L}(M)$ and suppose that E contains a basis for $\mathscr{L}(M)$. By Theorem 3.4, the set E complements Le-maximal summands. Let M be an R-module such that $H_A(M)=0$ for all Le-modules A. Condition (ii) in Theorem 3.3 then implies that $V=F_A(M)$, and hence a set $E\subset\mathscr{L}(M)$ complements Le-maximal summands, iff E contains a basis for $\mathscr{L}(M)$. A special case is the next theorem.

3.8. THEOREM. *Let M be an R-module which is a direct summand of a module with an* Le-*decomposition. A family $E\subset\mathscr{L}(M)$ complements maximal summands iff E contains a basis for $\mathscr{L}(M)$.*

PROOF. Since any indecomposable summand of a module with an Le-decomposition is an Le-module, a set $E \subset \mathscr{L}(M)$ complements maximal summands, iff the set complements Le-maximal summands. Theorem 3.8 follows then from Remark 3.7.

3.9. REMARK. Let V be a vectorspace over a division ring D and let $\{A_i\}_I$ be a family of one dimensional subspaces. If the family $\{A_i\}_I$ complements maximal summands then one easily proves that $\sum_I A_i = V$.

3.10. THEOREM. *A module $M \neq (0)$ has an* Le-*decomposition that complements direct summands, iff for any family $\{A_i\}_I$ of* Le-*modules that complements* Le-*maximal summands we have that $\sum_I A_i = M$.*

PROOF. Assume that M has an Le-decomposition that complements direct summands. Let $\{A_i\}_I$ complement Le-maximal summands. Since any maximal summand is an Le-maximal summand it follows from Theorem 3.8 that $\{A_i\}_I$ contains a basis $\{A_j\}_J$ for $\mathscr{L}(M)$. By Theorem 3.10 in [4], $\oplus_J A_j = M$ and hence $\sum_I A_i = M$ whenever $\{A_i\}_I$ complements Le-maximal summands. Let $\{A_i\}_I$ be any basis for $\mathscr{L}(M)$. By Theorem 3.4, $\{A_i\}_I$ complements Le-maximal summands. Hence $\oplus_I A_i = M$ for any basis $\{A_i\}_I$ for $\mathscr{L}(M)$. By Theorem 3.10 in [4] it follows that M has an Le-decomposition that complements direct summands.

4.

The purpose of this section is to show that the functors H_A introduced in the first section need not be identically zero. The results we prove can be made considerably stronger, but since we only are interested in providing an example of a functor $H_A \neq 0$ we try to do this as economically as possible.

4.1. LEMMA. *Let R be a commutative local ring where the maximal ideal P is a principal ideal. Let $M = \prod_{i=1}^{\infty} R_i$ where $R_i = R$ for $i = 1, 2, \ldots$. If there exists an R-homomorphism $f\colon M \to R$ such that $f \mid R_i\colon R_i \to R$ is an isomorphism for all $i = 1, 2, \ldots$, then R is complete in the P-adic topology. (Our definition of complete does not require the topology to be Hausdorff.)*

PROOF. Let $P = (\pi)$ and assume that $f\colon \prod_{i=1}^{\infty} R_i \to R$ induces an isomorphism on each R_i. Since an automorphism on R is multiplication with a unit, we may assume that

$$f(\{r_n\}_{n=1}^{\infty}) = r_1 + \ldots + r_n \quad \text{if } r_k = 0 \text{ for } k > n\,.$$

Let $\{r_i\}_1^\infty$ be a Cauchy sequence. We claim that the series $\sum_1^\infty r_i$ converge to $f(\{r_n\})$. For any $n \geqq 1$ there exists an integer N such that $r_k \in P^n = (\pi^n)$ for all $k > N$. Hence

$$f(\{r_n\}) - (r_1 + \ldots + r_N) = f(0, \ldots, 0, r_{N+1}, r_{N+2}, \ldots)$$

and since $r_k = \pi^n r_k'$ for $k > N$ we see that

$$f(\{r_n\}) - (r_1 + \ldots + r_N) \in P^n .$$

This shows that $\sum_1^\infty r_i = f(\{r_n\})$.

4.2. THEOREM. *Let R be a commutative local ring whose maximal ideal P is principal. Let $M = \prod_1^\infty R_i$ ($R_i = R$). If $H_R(M) = 0$ then R is complete in the P-adic topology.*

PROOF. Assume that $H_R(M) = 0$. The family $\{R_i\}_1^\infty$ is an independent set in $\mathscr{L}_R(M)$ since any finite sum is direct and a direct summand of M. Let $g \colon \oplus_1^\infty R_i \to R$ be an isomorphism on each R_i. By Lemma 2.4 there exists a map $\hat{g} \colon \prod R_i \to R$ such that $\hat{g} \mid R_i - g \mid R_i \colon R_i \to R$ is not an isomorphism. Since R is a local ring this implies that $\hat{g} \mid R_i$ is an isomorphism for all $i \in I$. The theorem follows now from Lemma 4.1.

Since the functors H_A vanish on any module which is a direct summand of a module with an indecomposable decomposition we can conclude

4.3. THEOREM. *Let R be a commutative local ring whose maximal ideal P is a principal ideal. If the module $\prod_1^\infty R$ is a direct summand of a module with an indecomposable decomposition, then R is complete in the P-adic topology.*

4.4. REMARK. If R is Noetherian and satisfies the conditions of Theorem 4.3, then R is necessarily artinian. There are, however, non-Noetherian local rings whose maximal ideal is principal, for instance valuation rings of rank greater than one with discrete value group.

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