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COHEN-MACAULAY HOMOLOGICAL DIMENSIONS

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Abstract

We introduce new homological dimensions, namely the Cohen-Macaulay projective, injective and flat dimensions for homologically bounded complexes. Among other things we show that (a) these invariants characterize the Cohen-Macaulay property for local rings, (b) Cohen-Macaulay flat dimension fits between the Gorenstein flat dimension and the large restricted flat dimension, and (c) Cohen-Macaulay injective dimension fits between the Gorenstein injective dimension and the Chouinard invariant.

1. Introduction

A commutative Noetherian local ring R is regular if the residue field k has finite projective dimension and only if all R-modules have finite projective dimension [2], [24]. This theorem of Auslander, Buchsbaum and Serre is a main motivation of studying homological dimensions. The injective and flat dimensions have similar behavior.

Auslander and Bridger [1], introduced a homological dimension for finitely generated modules designed to single out modules with properties similar to those of modules over Gorenstein rings. They called it G-dimension and it is a refinement of the projective dimension and showed that a local Noetherian ring $(R, \mathfrak{m}, k)$ is Gorenstein if the residue field k has finite G-dimension and only if all finitely generated R-modules have finite G-dimension.

To extend the G-dimension beyond the realm of finitely generated modules over Noetherian rings, Enochs and Jenda [12] introduced the notion of Gorenstein projective module. Then the notion of Gorenstein projective dimension was studied in [7].

The notion of Gorenstein injective module is dual to that of Gorenstein projective module and were introduced in the same paper by Enochs and Jenda [12]. Then the notion of Gorenstein injective dimension was studied in [7].

Another extension of the G-dimension is based on Gorenstein flat modules, a notion due to Enochs, Jenda, and Torrecillas [13]. Then the notion of Gorenstein flat dimension was studied in [7].

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More recently, the complete intersection dimension has been introduced for finitely generated R-modules, using quasi-deformations and projective dimension, to characterize the complete intersection property of local rings [3]. Parallel to Gorenstein projective, injective and flat dimensions, the complete intersection projective, injective and flat dimensions have been introduced and studied in [20], [21], [23] and [22].

The *Cohen-Macaulay dimension* of a finitely generated R-module M, as defined by Gerko [16] is

$$\begin{aligned}&\text{CM-dim}_R(M)\\ &\quad := \inf\left\{\text{G-dim}_Q(M \otimes_R R') - \text{G-dim}_Q(R') \;\middle|\; \begin{array}{c} R \to R' \leftarrow Q \text{ is a} \\ \text{CM-quasi-deformation}\end{array}\right\}\end{aligned}$$

(see Section 2 for the definition of CM-quasi-deformation).

The purpose of this paper is to develop a similar theory of projective, injective and flat analogue for Cohen-Macaulay case. Thus we introduce Cohen-Macaulay projective dimension (CM_*-pd), Cohen-Macaulay injective dimension (CM_*-id) and Cohen-Macaulay flat dimension (CM_*-fd) for homologically bounded complexes over commutative Noetherian local rings $(R, \mathfrak{m}, k)$ with identity (see Definition 3.1). In particular $\text{CM-dim}_R(M) = \text{CM}_*\text{-pd}_R(M)$ $= \text{CM}_*\text{-fd}_R(M)$, for a finitely generated R-module M. Among other things, we show that these invariants characterize the Cohen-Macaulay property for local rings. We also show that if M is a homologically bounded R-complex, then we have the inequalities

$$\text{Rfd}_R(M) \leq \text{CM}_*\text{-fd}_R(M) \leq \text{Gfd}_R(M),$$

with equality to the left of any finite value. In particular if $\text{Gfd}_R(M) < \infty$, then $\text{CM}_*\text{-fd}_R(M) = \text{Gfd}_R(M)$, and if $\text{CM}_*\text{-fd}_R(M) < \infty$, then

$$\text{CM}_*\text{-fd}_R(M) = \sup\{\text{depth } R_\mathfrak{p} - \text{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \mid \mathfrak{p} \in \text{Spec}(R)\},$$

where $\text{Rfd}_R(M)$ is the large restricted flat dimension. Also, we show that there are inequalities

$$\sup\{\text{depth } R_\mathfrak{p} - \text{width}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \text{Spec}(R)\} \leq \text{CM}_*\text{-id}_R(M) \leq \text{Gid}_R(M),$$

such that if $\text{Gid}_R(M) < \infty$, then $\text{CM}_*\text{-id}_R(M) = \text{Gid}_R(M)$, and if $\text{CM}_*\text{-id}_R(M) < \infty$ for a homologically finite R-complex M, then

$$\begin{aligned}\text{CM}_*\text{-id}_R(M) &= \sup\{\text{depth } R_\mathfrak{p} - \text{width}_{R_\mathfrak{p}}(M_\mathfrak{p}) \mid \mathfrak{p} \in \text{Spec}(R)\}\\ &= \text{depth } R - \inf(M).\end{aligned}$$

Finally we compare our Cohen-Macaulay homological dimensions with the homological dimensions of Holm and Jørgensen [17].

2. Definitions and Notations

Let $(R, \mathfrak{m}, k)$ and $(S, \mathfrak{n}, \ell)$ be commutative local Noetherian rings.

We work in the derived category $\mathscr{D}(R)$ of complexes of R-modules, indexed homologically. A complex M is *homologically bounded* if $\mathrm{H}_i(M) = 0$ for all $|i| \gg 0$; and it is *homologically finite* if $\bigoplus_i \mathrm{H}_i(M)$ is finitely generated.

Fix R-complexes M and N. Let $M \otimes^{\mathbf{L}}_R N$ and $\mathbf{R}\mathrm{Hom}_R(M, N)$ denote the left-derived tensor product and right-derived homomorphism complexes, respectively. Let $\inf(M)$ and $\sup(M)$ denote the infimum and supremum, respectively, of the set $\{n \in \mathbb{Z} \mid \mathrm{H}_n(M) \neq 0\}$.

DEFINITION/NOTATION 2.1. A homologically finite R-complex M is *reflexive* if the complex $\mathbf{R}\mathrm{Hom}_R(M, R)$ is homologically bounded and the biduality morphism $\delta_M \colon M \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M, R), R)$ is an isomorphism in $\mathscr{D}(R)$. Set

$$\text{G-dim}_R(M) := -\inf(\mathbf{R}\mathrm{Hom}_R(M, R)),$$

if M is reflexive, and $\text{G-dim}_R(M) := \infty$ otherwise. Set also $\text{G-dim}_R(0) = -\infty$. This is the G-dimension of Auslander and Bridger [1] and Yassemi [26].

DEFINITION/NOTATION 2.2. An R-module G is *G-projective* if there exists an exact sequence of R-modules

$$X = \cdots \xrightarrow{\partial_2^X} P_1 \xrightarrow{\partial_1^X} P_0 \xrightarrow{\partial_0^X} P_{-1} \xrightarrow{\partial_{-1}^X} P_{-2} \xrightarrow{\partial_{-2}^X} \cdots$$

such that $G \cong \mathrm{Coker}(\partial_1^X)$, each P_i is projective, and $\mathrm{Hom}_R(X, Q)$ is exact for each projective R-module Q.

An R-module G is *G-flat* if there exists an exact sequence of R-modules

$$Y = \cdots \xrightarrow{\partial_2^Y} F_1 \xrightarrow{\partial_1^Y} F_0 \xrightarrow{\partial_0^Y} F_{-1} \xrightarrow{\partial_{-1}^Y} F_{-2} \xrightarrow{\partial_{-2}^Y} \cdots$$

such that $G \cong \mathrm{Coker}(\partial_1^Y)$, each F_i is flat, and $I \otimes_R Y$ is exact for each injective R-module I.

An R-module G is *G-injective* if there exists an exact sequence of R-modules

$$Z = \cdots \xrightarrow{\partial_2^Z} I_1 \xrightarrow{\partial_1^Z} I_0 \xrightarrow{\partial_0^Z} I_{-1} \xrightarrow{\partial_{-1}^Z} I_{-2} \xrightarrow{\partial_{-2}^Z} \cdots$$

such that $G \cong \mathrm{Coker}(\partial_1^Z)$, each I_i is injective, and $\mathrm{Hom}_R(I, Z)$ is exact for each injective R-module I.

Let M be a homologically bounded R-complex. A G-projective resolution of M is an isomorphism $H \simeq M$ in $\mathscr{D}(R)$ where H is a complex of G-projective R-modules such that $H_i = 0$ for all $i \ll 0$. The G-projective dimension of M is

$$\operatorname{Gpd}_R(M) := \inf\big\{\sup\{n \mid H_n \neq 0\} \mid H \simeq M \text{ is a } G\text{-projective resolution}\big\}.$$

The G-flat dimension of M is defined similarly and denoted $\operatorname{Gfd}_R(M)$, while the G-injective dimension $\operatorname{Gid}_R(M)$ is dual [7]. These are the G-projective, G-flat, and G-injective dimensions of Enochs, Jenda and Torrecillas (which they consider only in the case of modules) [12] and [13].

REMARK 2.3. (1) It is known that, for a homologically bounded R-complex M, $\operatorname{Gpd}_R(M)$ and $\operatorname{Gfd}_R(M)$ are simultaneously finite [23, Proposition 4.3].

(2) Let $R \to S$ be a flat local homomorphism and M a finitely generated R-module. Then it is well-known that, $\text{G-dim}_R(M) = \text{G-dim}_S(M \otimes_R S)$ and $\text{G-dim}_R(M) = \operatorname{Gfd}_R(M) = \operatorname{Gpd}_R(M)$ [9].

(3) The finiteness of G-projective, G-flat, and G-injective dimensions characterize the Gorenstein property of local rings [7].

DEFINITION/NOTATION 2.4. A finitely generated R-module M is called *G-perfect* if $\text{G-dim}_R M = \operatorname{grade}_R M := \inf\{i \mid \operatorname{Ext}^i_R(M, R) \neq 0\}$. Let Q be a local ring and J an ideal of Q. By abuse of language we say that J is *G-perfect* if the Q-module Q/J has the corresponding property.

A CM-*deformation* of R is a surjective local homomorphism $Q \to R$ such that $J = \ker(Q \to R)$ is a G-perfect ideal in Q. A CM-*quasi-deformation* of R is a diagram of local homomorphisms $R \to R' \leftarrow Q$, with $R \to R'$ a flat extension and $R' \leftarrow Q$ a CM-deformation.

The *Cohen-Macaulay dimension* of a nonzero finitely generated R-module M, as defined by Gerko [16] is

$$\text{CM-dim}_R(M) := \inf\left\{\text{G-dim}_Q(M \otimes_R R') - \text{G-dim}_Q(R') \;\middle|\; \begin{array}{c} R \to R' \leftarrow Q \text{ is a} \\ \text{CM-quasi-deformation} \end{array}\right\},$$

and set $\text{CM-dim}_R(0) = -\infty$.

REMARK 2.5. By [16, Theorems 3.8 and 3.9, and Proposition 3.10] we have

(1) R is Cohen-Macaulay if and only if $\text{CM-dim}_R(k) < \infty$.

(2) If M is a finitely generated R-module such that $\text{CM-dim}_R(M) < \infty$, then

$$\text{CM-dim}_R(M) = \operatorname{depth} R - \operatorname{depth}_R(M).$$

(3) For each prime ideal $\mathfrak{p}$ of R, $\text{CM-dim}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leq \text{CM-dim}_R(M)$.

DEFINITION/NOTATION 2.6. A finitely generated R-module C is *semidualizing* if the homothety morphism $\chi_C^R\colon R \to \mathbf{R}\mathrm{Hom}_R(C, C)$ is an isomorphism in $\mathscr{D}(R)$. A finitely generated R-module D is *canonical* if it is semidualizing and $\mathrm{id}_R(D)$ is finite.

Let $\varphi\colon R \to S$ be a local ring homomorphism. We denote $\widehat{R}$ the completion of R at its maximal ideal and let $\varepsilon_R\colon R \to \widehat{R}$ denote the natural map. The *completion of* φ is the unique local ring homomorphism $\widehat{\varphi}\colon \widehat{R} \to \widehat{S}$ such that $\widehat{\varphi} \circ \varepsilon_R = \varepsilon_S \circ \varphi$. The *semi-completion of* φ is the composition $\varepsilon_S \circ \varphi\colon R \to \widehat{S}$.

3. Cohen-Macaulay projective, flat and injective dimensions

In this section we introduce a Cohen-Macaulay projective dimension, Cohen-Macaulay flat dimension, and Cohen-Macaulay injective dimension for homologically bounded R-complexes and derive their basic properties. When M is a module, Definition 3.1 is from [20], which is in turn modeled on [3] and [16].

DEFINITION 3.1. Let $(R, \mathfrak{m})$ be a local ring. For each homologically bounded R-complex M, define the Cohen-Macaulay projective dimension, Cohen-Macaulay flat dimension and Cohen-Macaulay injective dimension of M as,

$$\mathrm{CM}_*\text{-}\mathrm{pd}_R(M) := \inf\left\{\mathrm{Gpd}_Q(M \otimes_R R') - \mathrm{Gfd}_Q(R') \;\middle|\; \begin{array}{c} R \to R' \leftarrow Q \text{ is a} \\ \text{CM-quasi-deformation} \end{array}\right\}$$

$$\mathrm{CM}_*\text{-}\mathrm{fd}_R(M) := \inf\left\{\mathrm{Gfd}_Q(M \otimes_R R') - \mathrm{Gfd}_Q(R') \;\middle|\; \begin{array}{c} R \to R' \leftarrow Q \text{ is a} \\ \text{CM-quasi-deformation} \end{array}\right\}$$

$$\mathrm{CM}_*\text{-}\mathrm{id}_R(M) := \inf\left\{\mathrm{Gid}_Q(M \otimes_R R') - \mathrm{Gfd}_Q(R') \;\middle|\; \begin{array}{c} R \to R' \leftarrow Q \text{ is a} \\ \text{CM-quasi-deformation} \end{array}\right\}$$

respectively.

REMARK 3.2. (1) It is known that $\mathrm{Gpd}_R(M)$ and $\mathrm{Gfd}_R(M)$ are simultaneously finite by Remark 2.3(1). Hence $\mathrm{CM}_*\text{-}\mathrm{pd}_R(M)$ and $\mathrm{CM}_*\text{-}\mathrm{fd}_R(M)$ are simultaneously finite.

(2) By taking the trivial CM-quasi-deformation $R \to R \leftarrow R$, one has

$$\mathrm{CM}_*\text{-}\mathrm{pd}_R(M) \le \mathrm{Gpd}_R(M),$$
$$\mathrm{CM}_*\text{-}\mathrm{fd}_R(M) \le \mathrm{Gfd}_R(M),$$
$$\mathrm{CM}_*\text{-}\mathrm{id}_R(M) \le \mathrm{Gid}_R(M).$$

(3) By Remark 2.3(2) it can be seen that if M is a finitely generated R-module then, $\mathrm{CM}_*\text{-pd}_R(M) = \mathrm{CM}_*\text{-fd}_R(M) = \text{CM-dim}_R(M)$.

The following two theorems show that the finiteness of these dimensions characterize the Cohen-Macaulay rings.

THEOREM 3.3. *The following conditions are equivalent:*

(1) *The ring R is Cohen-Macaulay.*

(2) $\mathrm{CM}_*\text{-pd}_R(M) < \infty$ *for every homologically bounded R-complex M.*

(3) $\mathrm{CM}_*\text{-pd}_R(k) < \infty$.

(4) $\mathrm{CM}_*\text{-fd}_R(M) < \infty$ *for every homologically bounded R-complex M.*

(5) $\mathrm{CM}_*\text{-fd}_R\, k < \infty$.

PROOF. (1) $\Rightarrow$ (2) Let $\widehat{R}$ be the $\mathfrak{m}$-adic completion of R. Since R is Cohen-Macaulay, so is $\widehat{R}$. Therefore by Cohen's structure theorem, $\widehat{R}$ is isomorphic to Q/J, where Q is a regular local ring. By Cohen-Macaulay-ness of $\widehat{R}$ and regularity of Q, the ideal J is G-perfect. Thus $R \to \widehat{R} \leftarrow Q$ is a CM-quasi-deformation. Since Q is regular $\mathrm{Gpd}_Q(M \otimes_R \widehat{R}) < \infty$ for every homologically bounded R-complex M. Thus $\mathrm{CM}_*\text{-pd}_R(M)$ is finite.

(2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are trivial.

(2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (5) are trivial since $\mathrm{CM}_*\text{-fd}_R(M) \leq \mathrm{CM}_*\text{-pd}_R(M)$.

(5) $\Rightarrow$ (1) It follows from Remark 3.2(3) that $\text{CM-dim}_R(k) = \mathrm{CM}_*\text{-fd}_R(k) < \infty$. Now Remark 2.5(1), completes the proof.

THEOREM 3.4. *The following conditions are equivalent.*

(1) *The ring R is Cohen-Macaulay.*

(2) $\mathrm{CM}_*\text{-id}_R(M) < \infty$ *for every homologically bounded R-complex M.*

(3) $\mathrm{CM}_*\text{-id}_R(k) < \infty$.

PROOF. (1) $\Rightarrow$ (2) is the same as proof of part (1) $\Rightarrow$ (2) of Theorem 3.3.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Suppose $\mathrm{CM}_*\text{-id}_R(k) < \infty$. So that there is a CM-quasi-deformation $R \to R' \leftarrow Q$, such that $\mathrm{Gid}_Q(k \otimes_R R')$ is finite. It is clear that $k \otimes_R R'$ is a cyclic Q-module. Consequently Q is a Gorenstein ring by [15, Theorem 4.5]. We plan to show that R' is a Cohen-Macaulay ring. Let $I = \ker(Q \to R')$ which is G-perfect by definition. We have

$$\begin{aligned}
\operatorname{ht} I &= \operatorname{grade}(I, Q) = \text{G-dim}_Q R' \\
&= \operatorname{depth} Q - \operatorname{depth}_Q R' = \operatorname{depth} Q - \operatorname{depth} R' \\
&= \dim Q - \operatorname{depth} R' = \operatorname{ht} I + \dim R' - \operatorname{depth} R',
\end{aligned}$$

in which the equalities follow from Cohen-Macaulay-ness of Q; G-perfectness of I; Auslander-Buchsbaum formula; [5, Exercise 1.2.26]; Cohen-Macaulay-ness of Q; and [5, Corollary 2.1.4] respectively. Therefore we obtain that $\dim R' - \operatorname{depth} R' = 0$, that is R' is Cohen-Macaulay. Now [5, Theorem 2.1.7] gives us the desired result.

The proof of the above theorem says some thing more, viz., a local ring R is Cohen-Macaulay if and only if there exists a cyclic R-module of finite Cohen-Macaulay injective dimension.

COROLLARY 3.5. *Assume that* $C \neq 0$ *is a cyclic* R*-module. Then* R *is a Cohen-Macaulay ring if and only if* $\text{CM}_*\text{-id}_R\, C < \infty$.

REMARK 3.6. Let M be a homologically finite R-complex such that $\operatorname{Gid}_R(M) < \infty$. Then by [15, Theorem 3.6], we obtain that $\operatorname{Gid}_{\widehat{R}}(M \otimes_R \widehat{R}) < \infty$. Hence using [11, Corollary 2.3], we have

$$\operatorname{Gid}_{\widehat{R}}(M \otimes_R \widehat{R}) = \operatorname{depth} \widehat{R} - \inf(M \otimes_R \widehat{R}) = \operatorname{depth} R - \inf(M) = \operatorname{Gid}_R(M).$$

PROPOSITION 3.7. *Let* M *be a homologically finite* R*-complex. Then*

$$\text{CM}_*\text{-id}_R(M) = \inf\left\{ \operatorname{Gid}_Q(M \otimes_R R') - \operatorname{Gfd}_Q(R') \;\middle|\; \begin{array}{c} R \to R' \leftarrow Q \textit{ is a} \\ \text{CM-}\textit{quasi-deformation} \\ \textit{such that } Q \textit{ is complete} \end{array} \right\}.$$

PROOF. It is clear that the left hand side is less than or equal to the right hand side. Now let $R \to R' \leftarrow Q$ be a CM-quasi-deformation. Then note that $R \to \widehat{R'} \leftarrow \widehat{Q}$ is also a CM-quasi-deformation such that

$$\operatorname{Gid}_Q(M \otimes_R R') = \operatorname{Gid}_{\widehat{Q}}(M \otimes_R R' \otimes_Q \widehat{Q}) = \operatorname{Gid}_{\widehat{Q}}(M \otimes_R \widehat{R'}),$$

and $\operatorname{Gfd}_Q(R') = \operatorname{Gfd}_{\widehat{Q}}(\widehat{R'})$, where the first equality holds by Remark 3.6. So we can assume in the CM-quasi-deformation $R \to R' \leftarrow Q$ that, Q is a complete local ring. This shows the equality.

PROPOSITION 3.8. *Let* M *be a homologically bounded* R*-complex. Then*

$$\text{CM}_*\text{-fd}_R(M) = \inf\left\{ \operatorname{Gfd}_Q(M \otimes_R R') - \operatorname{Gfd}_Q(R') \;\middle|\; \begin{array}{c} R \to R' \leftarrow Q \textit{ is a} \\ \text{CM-}\textit{quasi-deformation} \\ \textit{such that } Q \textit{ is complete} \end{array} \right\}.$$

PROOF. The proof is the same as proof of Proposition 3.7, but here use [19, Corollary 8.9] instead of Remark 3.6.

Let M be homologically bounded R-complex. Then Foxby showed that

$$\mathrm{Gpd}_{\widehat{R}}(M \otimes_R \widehat{R}) \leq \mathrm{Gpd}_R(M)$$

(see [10, Ascent table II(b)]).

PROPOSITION 3.9. *Let M be a homologically bounded R-complex. Then*

$$\mathrm{CM}_*\text{-pd}_R(M) = \inf \left\{ \mathrm{Gpd}_Q(M \otimes_R R') - \mathrm{Gfd}_Q(R') \;\middle|\; \begin{array}{c} R \to R' \leftarrow Q \text{ is a} \\ \text{CM-quasi-deformation} \\ \text{such that } Q \text{ is complete} \end{array} \right\}.$$

PROOF. The proof is the same as proof of Proposition 3.7, but here use the comment just before the proposition instead of Remark 3.6.

A homological dimension should not grow under localization. Let $\mathfrak{p}$ be a prime ideal of R and M a homologically bounded R-complex. It is well known that

$$\mathrm{Gfd}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leq \mathrm{Gfd}_R(M),$$

and Foxby showed that (when R has finite Krull dimension)

$$\mathrm{Gpd}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leq \mathrm{Gpd}_R(M)$$

(see [9, Page 262]). On the other hand if R has a dualizing complex then,

$$\mathrm{Gid}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leq \mathrm{Gid}_R(M)$$

by [9, Proposition 5.5].

THEOREM 3.10. *Let M be a homologically finite R-complex. For each prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$ there is an inequality*

$$\mathrm{CM}_*\text{-id}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leq \mathrm{CM}_*\text{-id}_R(M).$$

PROOF. Assume that $\mathrm{CM}_*\text{-id}_R(M) < \infty$. Let $R \to R' \leftarrow Q$ be a CM-quasi-deformation with Q a complete local ring, such that $\mathrm{Gid}_Q(M \otimes R') < \infty$ and $\mathrm{CM}_*\text{-id}_R(M) = \mathrm{Gid}_Q(M \otimes R') - \mathrm{Gfd}_Q(R')$ by Proposition 3.7. Hence Q admits a dualizing complex.

Let $\mathfrak{p}$ be a prime ideal of R. Since $R \to R'$ is a faithfully flat extension of rings, there is a prime ideal $\mathfrak{p}'$ in R' lying over $\mathfrak{p}$. Let $\mathfrak{q}$ be the inverse image of $\mathfrak{p}'$ in Q. The map $R_\mathfrak{p} \to R'_{\mathfrak{p}'}$ is flat, and $R'_{\mathfrak{p}'} \leftarrow Q_\mathfrak{q}$ is a CM-deformation and note that $\mathrm{Gfd}_{Q_{\mathfrak{q}}}(R'_{\mathfrak{p}'}) = \mathrm{Gfd}_Q R'$. Therefore the diagram $R_{\mathfrak{p}} \to R'_{\mathfrak{p}'} \leftarrow Q_{\mathfrak{q}}$ is a CM-quasi-deformation with

$$\mathrm{Gid}_{Q_{\mathfrak{q}}}(M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} R'_{\mathfrak{p}'}) = \mathrm{Gid}_{Q_{\mathfrak{q}}}((M \otimes_R R') \otimes_Q Q_{\mathfrak{q}}) \leq \mathrm{Gid}_Q(M \otimes_R R') < \infty,$$

where the inequality holds by [9, Proposition 5.5]. Hence $\mathrm{CM}_*\text{-id}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) < \infty$. So we obtain

$$\begin{aligned} \mathrm{CM}_*\text{-id}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) &\leq \mathrm{Gid}_{Q_{\mathfrak{q}}}(M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} R'_{\mathfrak{p}'}) - \mathrm{Gfd}_{Q_{\mathfrak{q}}}(R'_{\mathfrak{p}'}) \\ &\leq \mathrm{Gid}_Q(M \otimes_R R') - \mathrm{Gfd}_Q(R') \\ &= \mathrm{CM}_*\text{-id}_R(M). \end{aligned}$$

Thus the desired inequality follows.

We do not know when the inequality $\mathrm{CM}_*\text{-id}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq \mathrm{CM}_*\text{-id}_R(M)$ holds in general. However for $\mathrm{CM}_*\text{-pd}_R(M)$ and $\mathrm{CM}_*\text{-fd}_R(M)$ we have

THEOREM 3.11. *Let M be a homologically bounded R-complex. For each prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$ there is an inequality*

(1) $\mathrm{CM}_*\text{-pd}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq \mathrm{CM}_*\text{-pd}_R(M)$.

(2) $\mathrm{CM}_*\text{-fd}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq \mathrm{CM}_*\text{-fd}_R(M)$.

PROOF. The proof is the same as proof of Theorem 3.10, but here we do not need Q is a complete local ring.

PROPOSITION 3.12. *Let M be a homologically finite R-complex. Then there is an equality*

$$\mathrm{CM}_*\text{-id}_R(M) = \inf\left\{ \mathrm{Gid}_Q(M \otimes_R R') - \mathrm{Gfd}_Q(R') \;\middle|\; \begin{array}{c} R \to R' \leftarrow Q \text{ is a} \\ \text{CM-quasi-deformation} \\ \text{such that the closed fibre} \\ \text{of } R \to R' \text{ is Artinian} \end{array} \right\}.$$

PROOF. It is clear that the left hand side is less than or equal to the right hand side. Let $R \to R' \leftarrow Q$ be a CM-quasi-deformation with Q a complete local ring, such that $\mathrm{CM}_*\text{-id}_R(M) = \mathrm{Gid}_Q(M \otimes R') - \mathrm{Gfd}_Q(R')$ by Proposition 3.7. Hence Q admits a dualizing complex. Now choose $\mathfrak{p}' \in \mathrm{Spec}(R')$ such that it is a minimal prime ideal containing $\mathfrak{m}R'$; thus $\mathfrak{m} = \mathfrak{p}' \cap R$ and $\mathfrak{p}' = \mathfrak{q}/J$ for some $\mathfrak{q} \in \mathrm{Spec}(Q)$, where $J = \ker(Q \to R')$. Now the diagram $R \to R'_{\mathfrak{p}'} \leftarrow Q_{\mathfrak{q}}$ is a CM-quasi-deformation such that the closed fiber of $R \to R'_{\mathfrak{p}'}$ is Artinian.

It is clear that $\mathrm{Gfd}_Q\, R' = \mathrm{Gfd}_{Q_\mathfrak{q}}\, R'_{\mathfrak{p}'}$. Also we have

$$\begin{aligned}\mathrm{Gid}_{Q_\mathfrak{q}}(M \otimes_R R'_{\mathfrak{p}'}) &= \mathrm{Gid}_{Q_\mathfrak{q}}(M \otimes_R (R' \otimes_Q Q_\mathfrak{q}))\\ &= \mathrm{Gid}_{Q_\mathfrak{q}}((M \otimes_R R') \otimes_Q Q_\mathfrak{q})\\ &\leq \mathrm{Gid}_Q(M \otimes_R R'),\end{aligned}$$

where the inequality holds by [9, Proposition 5.5]. Hence $\mathrm{Gid}_{Q_\mathfrak{q}}(M \otimes_R R'_\mathfrak{p}) - \mathrm{Gfd}_{Q_\mathfrak{q}}(R'_\mathfrak{p}) \leq \mathrm{CM}_*\text{-}\mathrm{id}_R(M)$. So the proof is complete.

PROPOSITION 3.13. *Let M be a homologically bounded R-complex. Then there are equalities*

$$\mathrm{CM}_*\text{-}\mathrm{pd}_R(M) = \inf\left\{ \mathrm{Gpd}_Q(M \otimes_R R') - \mathrm{Gfd}_Q(R') \;\middle|\; \begin{array}{c} R \to R' \leftarrow Q \textit{ is a} \\ \mathrm{CM}\textit{-quasi-deformation} \\ \textit{such that the closed fibre} \\ \textit{of } R \to R' \textit{ is Artinian} \end{array} \right\},$$

$$\mathrm{CM}_*\text{-}\mathrm{fd}_R(M) = \inf\left\{ \mathrm{Gfd}_Q(M \otimes_R R') - \mathrm{Gfd}_Q(R') \;\middle|\; \begin{array}{c} R \to R' \leftarrow Q \textit{ is a} \\ \mathrm{CM}\textit{-quasi-deformation} \\ \textit{such that the closed fibre} \\ \textit{of } R \to R' \textit{ is Artinian} \end{array} \right\}.$$

PROOF. The proof is the same as proof of Proposition 3.12, but here we do not need Q is a complete local ring.

REMARK 3.14. (1) Let M be a homologically finite R-complex. Then, one can combine the proofs of Propositions 3.7 and 3.12, to obtain an equality

$$\mathrm{CM}_*\text{-}\mathrm{id}_R(M) = \inf\left\{ \mathrm{Gid}_Q(M \otimes_R R') - \mathrm{Gfd}_Q(R') \;\middle|\; \begin{array}{c} R \to R' \leftarrow Q \text{ is a} \\ \text{CM-quasi-deformation} \\ \text{such that } Q \text{ is complete} \\ \text{and the closed fibre of} \\ R \to R' \text{ is Artinian} \end{array} \right\}.$$

(2) Likewise for a homologically bounded R-complex M, one can combine the proofs of Propositions 3.8, 3.9 and 3.12, to obtain the equalities

$$\mathrm{CM}_*\text{-}\mathrm{fd}_R(M) = \inf\left\{ \mathrm{Gfd}_Q(M \otimes_R R') - \mathrm{Gfd}_Q(R') \;\middle|\; \begin{array}{c} R \to R' \leftarrow Q \text{ is a} \\ \text{CM-quasi-deformation} \\ \text{such that } Q \text{ is complete} \\ \text{and the closed fibre of} \\ R \to R' \text{ is Artinian} \end{array} \right\},$$

$$\operatorname{CM}_*\text{-pd}_R(M) = \inf\left\{ \operatorname{Gpd}_Q(M \otimes_R R') - \operatorname{Gfd}_Q(R') \;\middle|\; \begin{array}{c} R \to R' \leftarrow Q \text{ is a} \\ \text{CM-quasi-deformation} \\ \text{such that } Q \text{ is complete} \\ \text{and the closed fibre of} \\ R \to R' \text{ is Artinian} \end{array} \right\}.$$

4. Large restricted flat dimension and Chouinard's invariant

Recall from [8], that the *large restricted flat dimension* is defined by

$$\operatorname{Rfd}_R(M) := \sup\left\{\sup(F \otimes_R^{\mathbf{L}} M) \mid F \text{ an } R\text{-module with } \operatorname{fd}_R(F) < \infty\right\}.$$

This number is finite, as long as $\mathrm{H}(M)$ is nonzero and the Krull dimension of R is finite; see [8, Proposition 2.2]. It is useful to keep in mind an alternative formula [8, Theorem 2.4] for computing this invariant:

$$\operatorname{Rfd}_R(M) = \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \operatorname{Spec}(R)\}.$$

Recall here that the *depth* of a homologically bounded R-complex M is defined by

$$\operatorname{depth}_R(M) = -\sup(\mathbf{R}\operatorname{Hom}_R(k, M)),$$

and it is shown that $\operatorname{depth}_R(M) \geq -\sup(M)$.

It is proved in [19, Theorem 8.8] that for an R-complex M, $\operatorname{Rfd}_R(M)$ is a refinement of $\operatorname{Gfd}_R(M)$, that is

$$\operatorname{Rfd}_R(M) \leq \operatorname{Gfd}_R(M),$$

with equality if $\operatorname{Gfd}_R(M)$ is finite.

First, we plan to show that, when the Cohen-Macaulay flat dimension of a homologically bounded R-complex M is finite, then it is equal to the large restricted flat dimension of M. The following proposition is the main tool.

Proposition 4.1. *Let $R \to S \leftarrow Q$ be a* CM*-quasi-deformation, and let M be a homologically bounded R-complex. Then*

$$\operatorname{Rfd}_R(M) = \operatorname{Rfd}_Q(M \otimes_R S) - \operatorname{Rfd}_Q(S).$$

Proof. First we prove the equality

$$\operatorname{Rfd}_S(Y) = \operatorname{Rfd}_Q(Y) - \text{G-dim}_Q(S),$$

for a homologically bounded S-complex Y. To this end, choose by [8, Theorem 2.4(b)] a prime ideal $\mathfrak{p}$ of S such that the first equality below holds. Let

$\mathfrak{q}$ be the inverse image of $\mathfrak{p}$ in Q. Therefore there is an isomorphism $Y_{\mathfrak{p}} \cong Y_{\mathfrak{q}}$ of $Q_{\mathfrak{q}}$-modules and a CM-deformation $Q_{\mathfrak{q}} \to S_{\mathfrak{p}}$. Hence

$$\begin{aligned}\mathrm{Rfd}_S(Y) &= \operatorname{depth} S_{\mathfrak{p}} - \operatorname{depth}_{S_{\mathfrak{p}}} Y_{\mathfrak{p}} \\ &= \operatorname{depth}_{Q_{\mathfrak{q}}} S_{\mathfrak{p}} - \operatorname{depth}_{Q_{\mathfrak{q}}} Y_{\mathfrak{p}} \\ &= \operatorname{depth} Q_{\mathfrak{q}} - \text{G-dim}_{Q_{\mathfrak{q}}} S_{\mathfrak{p}} - \operatorname{depth}_{Q_{\mathfrak{q}}} Y_{\mathfrak{p}} \\ &\leq \mathrm{Rfd}_Q(Y) - \text{G-dim}_{Q_{\mathfrak{q}}}(S_{\mathfrak{p}}) \\ &= \mathrm{Rfd}_Q(Y) - \text{G-dim}_Q(S).\end{aligned}$$

The second equality holds since $Q_{\mathfrak{q}} \to S_{\mathfrak{p}}$ is surjective and [18, Proposition 5.2(1)]; the third equality holds by Auslander-Bridger formula [1]; the fourth equality is due to the G-perfectness assumption of S over Q; while the inequality follows from [8, Theorem 2.4(b)]. Now by [25, Proposition 3.5] we have

$$\begin{aligned}\mathrm{Rfd}_Q(Y) &\leq \mathrm{Rfd}_S(Y) + \mathrm{Rfd}_Q(S) \\ &\leq \mathrm{Rfd}_Q(Y) - \text{G-dim}_Q(S) + \mathrm{Rfd}_Q(S) = \mathrm{Rfd}_Q(Y),\end{aligned}$$

which is the desired equality.

Now we have

$$\begin{aligned}\mathrm{Rfd}_Q(M \otimes_R S) &\leq \mathrm{Rfd}_S(M \otimes_R S) + \mathrm{Rfd}_Q(S) \\ &= \mathrm{Rfd}_S(M \otimes_R S) + \text{G-dim}_Q(S) \\ &= \mathrm{Rfd}_Q(M \otimes_R S),\end{aligned}$$

where the inequality is in [25, Proposition 3.5], the first equality follows from the hypotheses, and the second equality follows from the above observation. Hence

$$\mathrm{Rfd}_Q(M \otimes_R S) - \mathrm{Rfd}_Q(S) = \mathrm{Rfd}_S(M \otimes_R S) = \mathrm{Rfd}_R(M)$$

where the second equality holds by [19, Lemma 8.5(1)].

COROLLARY 4.2. *Let M be a homologically bounded R-complex. Then we have the inequalities*

$$\mathrm{Rfd}_R(M) \leq \text{CM}_*\text{-fd}_R(M) \leq \mathrm{Gfd}_R(M),$$

with equality to the left of any finite value. In particular if $\text{CM}_*\text{-fd}_R(M) < \infty$, *then*

$$\begin{aligned}\text{CM}_*\text{-fd}_R(M) &= \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \operatorname{Spec}(R)\} \\ &\leq \dim R + \sup(M).\end{aligned}$$

Now using Corollary 4.2, we investigate the effect of change of ring on Cohen-Macaulay flat dimension.

PROPOSITION 4.3. *Let M be a homologically bounded R-complex. Let $R \to R'$ be a local flat extension, and $M' = M \otimes_R R'$. Then*

$$\text{CM}_*\text{-fd}_R(M) \le \text{CM}_*\text{-fd}_{R'}(M')$$

with equality when $\text{CM}_*\text{-fd}_{R'}(M')$ *is finite.*

PROOF. Suppose that $\text{CM}_*\text{-fd}_{R'}(M') < \infty$, and let $R' \to R'' \leftarrow Q$ be a CM-quasi-deformation with $\text{Gfd}_Q(M' \otimes_{R'} R'') < \infty$. Since $R \to R'$ and $R' \to R''$ are flat extensions, the local homomorphism $R \to R''$ is also flat. Hence $R \to R'' \leftarrow Q$ is a CM-quasi-deformation with $\text{Gfd}_Q(M \otimes_R R'') < \infty$. It follows that $\text{CM}_*\text{-fd}_R(M)$ is finite. Now by Corollary 4.2 and [19, Lemma 8.5(1)], we have

$$\text{CM}_*\text{-fd}_R(M) = \text{Rfd}_R(M) = \text{Rfd}_{R'}(M') = \text{CM}_*\text{-fd}_{R'}(M'),$$

to complete the proof.

PROPOSITION 4.4. *For every homologically bounded R-complex M*

$$\text{CM}_*\text{-fd}_R(M) = \text{CM}_*\text{-fd}_{\widehat{R}}(M \otimes_R \widehat{R}).$$

PROOF. If $\text{CM}_*\text{-fd}_R(M) = \infty$, then we obtain that $\text{CM}_*\text{-fd}_{\widehat{R}}(M \otimes_R \widehat{R}) = \infty$ by Proposition 4.3. Now assume that $\text{CM}_*\text{-fd}_R(M) < \infty$. Using Proposition 4.3, it is sufficient to prove that $\text{CM}_*\text{-fd}_{\widehat{R}}(M \otimes_R \widehat{R})$ is finite. To this end, choose a CM-quasi-deformation $R \to R' \leftarrow Q$ of R such that $\text{Gfd}_Q(M \otimes_R R') < \infty$. So we have $\widehat{R} \to \widehat{R'} \leftarrow \widehat{Q}$ is a CM-quasi-deformation of $\widehat{R}$ with respect to their maximal ideal-adic completions. Now using [19, Corollary 8.9] we obtain

$$\text{Gfd}_{\widehat{Q}}((M \otimes_R \widehat{R}) \otimes_{\widehat{R}} \widehat{R'}) = \text{Gfd}_Q(M \otimes_R R') < \infty.$$

Hence $\text{Gfd}_{\widehat{Q}}((M \otimes_R \widehat{R}) \otimes_{\widehat{R}} \widehat{R'})$ is finite which in turn implies that $\text{CM}_*\text{-fd}_{\widehat{R}}(M \otimes_R \widehat{R})$ is finite.

Next, recall that the *width* of an R-complex M is defined by

$$\text{width}_R(M) = \inf(M \otimes^{\mathbf{L}}_R k),$$

and that $\text{width}_R(M) \ge \inf(M)$. Also, if M is homologically finite, then

$$\text{width}_R(M) = \inf(M).$$

It is the dual notion for $\operatorname{depth}_R(M)$. In particular by [8, Proposition 4.8], we have

$$\operatorname{width}_R(M) = \operatorname{depth}_R(\mathbf{R}\operatorname{Hom}_R(M, \mathrm{E}_R(k))),$$

where $\mathrm{E}_R(k)$ denotes the injective envelope of k over R.

The *Chouinard invariant* [6, Corollary 3.1] is denoted by $\operatorname{Ch}_R(M)$ and

$$\operatorname{Ch}_R(M) := \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \operatorname{Spec}(R)\}.$$

It is proved in [11, Theorem 2.2] that for an R-complex M, $\operatorname{Ch}_R(M)$ is a refinement of $\operatorname{Gid}_R(M)$, that is

$$\operatorname{Ch}_R(M) \leq \operatorname{Gid}_R(M),$$

with equality if $\operatorname{Gid}_R(M)$ is finite. Now we want to show that the Cohen-Macaulay injective dimension is bounded below by the Chouinard's invariant.

LEMMA 4.5. *Suppose that $Q \to S$ is a surjective local homomorphism and Y is an S-complex. Then we have*

$$\operatorname{width}_S(Y) = \operatorname{width}_Q(Y).$$

PROOF. We have the following equalities:

$$\begin{aligned}
\operatorname{width}_S(Y) &= \operatorname{depth}_S \mathbf{R}\operatorname{Hom}_S(Y, \mathrm{E}_S(k)) \\
&= \operatorname{depth}_S \mathbf{R}\operatorname{Hom}_S(Y, \operatorname{Hom}_Q(S, \mathrm{E}_Q(k))) \\
&= \operatorname{depth}_S \mathbf{R}\operatorname{Hom}_Q(Y, \mathrm{E}_Q(k)) \\
&= \operatorname{depth}_Q \mathbf{R}\operatorname{Hom}_Q(Y, \mathrm{E}_Q(k)) \\
&= \operatorname{width}_Q(Y),
\end{aligned}$$

where the first one is by [8, Proposition 4.8]; the second one is by [4, Lemma 10.1.15]; the third one is by adjointness of Hom and tensor; the fourth one is true since $Q \to S$ is surjective and [18, Proposition 5.2(1)]; while the last one is again by [8, Proposition 4.8]. Here we used k for the residue fields of Q and S, and $\mathrm{E}_Q(k)$ and $\mathrm{E}_S(k)$ for the injective envelopes of k over respectively Q and S.

LEMMA 4.6. *Suppose that $R \to S$ is a flat local ring homomorphism, and M is a homologically bounded R-complex. Then we have*

$$\operatorname{width}_S(M \otimes_R S) = \operatorname{width}_R(M).$$

PROOF. A standard application of the Künneth formula yields the equality.

PROPOSITION 4.7. *Let $R \to S$ be a flat local homomorphism and let M be a homologically bounded R-complex. Then*

$$\mathrm{Ch}_R(M) \le \mathrm{Ch}_S(M \otimes_R S).$$

PROOF. Let $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $\mathrm{Ch}_R(M) = \operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$. Let $\mathfrak{q} \in \mathrm{Spec}(S)$ contain $\mathfrak{p}S$ minimally. Since $R \to S$ is a flat local homomorphism we have $\mathfrak{p} = \mathfrak{q} \cap R$. Hence:

$$\begin{aligned}\mathrm{Ch}_R(M) &= \operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})\\ &= \operatorname{depth} S_{\mathfrak{q}} - \operatorname{width}_{S_{\mathfrak{q}}}(M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{q}})\\ &= \operatorname{depth} S_{\mathfrak{q}} - \operatorname{width}_{S_{\mathfrak{q}}}(M \otimes_R S)_{\mathfrak{q}}\\ &\le \mathrm{Ch}_S(M \otimes_R S),\end{aligned}$$

in which the second equality holds by Lemma 4.6 and the fact that $R_{\mathfrak{p}} \to S_{\mathfrak{q}}$ has Artinian closed fibre.

PROPOSITION 4.8. *Let $Q \to S$ be a* CM*-deformation, and Y be a homologically bounded S-complex. Then*

$$\mathrm{Ch}_S(Y) \le \mathrm{Ch}_Q(Y) - \mathrm{Gfd}_Q(S).$$

PROOF. Choose a prime ideal $\mathfrak{p}$ of S such that the first equality below holds. Let $\mathfrak{q}$ be the inverse image of $\mathfrak{p}$ in Q. Therefore there is an isomorphism $Y_{\mathfrak{p}} \cong Y_{\mathfrak{q}}$ of $Q_{\mathfrak{q}}$-complexes and a CM-deformation $Q_{\mathfrak{q}} \to S_{\mathfrak{p}}$. Hence

$$\begin{aligned}\mathrm{Ch}_S(Y) &= \operatorname{depth} S_{\mathfrak{p}} - \operatorname{width}_{S_{\mathfrak{p}}}(Y_{\mathfrak{p}})\\ &= \operatorname{depth}_{Q_{\mathfrak{q}}} S_{\mathfrak{p}} - \operatorname{width}_{Q_{\mathfrak{q}}}(Y_{\mathfrak{p}})\\ &= \operatorname{depth} Q_{\mathfrak{q}} - \mathrm{Gfd}_{Q_{\mathfrak{q}}} S_{\mathfrak{p}} - \operatorname{width}_{Q_{\mathfrak{q}}}(Y_{\mathfrak{p}})\\ &\le \mathrm{Ch}_Q(Y) - \mathrm{Gfd}_{Q_{\mathfrak{q}}}(S_{\mathfrak{p}})\\ &= \mathrm{Ch}_Q(Y) - \mathrm{Gfd}_Q(S).\end{aligned}$$

The second equality holds since $Q_{\mathfrak{q}} \to S_{\mathfrak{p}}$ is surjective; the third equality holds by Auslander-Bridger formula [1]; the fourth equality is due to the G-perfectness assumption of S over Q.

THEOREM 4.9. *Let M be a homologically bounded R-complex. Then there is the inequality*

$$\mathrm{Ch}_R(M) \le \mathrm{CM}_*\text{-}\mathrm{id}_R(M).$$

PROOF. We can assume that $\mathrm{CM}_*\text{-id}_R(M) < \infty$. Choose a CM-quasi-deformation $R \to R' \leftarrow Q$, such that $\mathrm{CM}_*\text{-id}_R(M) = \mathrm{Gid}_Q(M \otimes_R R') - \mathrm{Gfd}_Q(R')$. Hence we have

$$\begin{aligned}\mathrm{CM}_*\text{-id}_R(M) &= \mathrm{Gid}_Q(M \otimes_R R') - \mathrm{Gfd}_Q(R') \\ &= \mathrm{Ch}_Q(M \otimes_R R') - \mathrm{Gfd}_Q(R') \\ &\geq \mathrm{Ch}_{R'}(M \otimes_R R') \geq \mathrm{Ch}_R(M),\end{aligned}$$

in which the second equality comes by [11, Theorem 2.2], and inequalities follow Propositions 4.8 and 4.7 respectively.

COROLLARY 4.10. *Let M be a homologically bounded R-complex. Then there are inequalities*

$$\mathrm{Ch}_R(M) \leq \mathrm{CM}_*\text{-id}_R(M) \leq \mathrm{Gid}_R(M),$$

such that if $\mathrm{Gid}_R(M) < \infty$*, then* $\mathrm{Gid}_R(M) = \mathrm{CM}_*\text{-id}_R(M)$.

PROOF. The inequalities hold by Theorem 4.9 and Remark 3.2(2). And if $\mathrm{Gid}_R(M) < \infty$, then the equality holds by [11, Theorem 2.2].

COROLLARY 4.11. *Let M be a homologically finite R-complex such that* $\mathrm{CM}_*\text{-id}_R(M)$ *is finite. Then*

$$\begin{aligned}\mathrm{CM}_*\text{-id}_R(M) &= \mathrm{Ch}_R(M) = \operatorname{depth} R - \inf(M) \\ &\leq \dim R - \inf(M).\end{aligned}$$

PROOF. By Proposition 3.12 there is a CM-quasi-deformation $R \to R' \leftarrow Q$ such that the closed fibre of $R \to R'$ is Artinian and the first equality below holds. So that

$$\begin{aligned}\mathrm{CM}_*\text{-id}_R(M) &= \mathrm{Gid}_Q(M \otimes_R R') - \mathrm{Gfd}_Q(R') \\ &= \operatorname{depth} Q - \inf(M \otimes_R R') - \operatorname{depth} Q + \operatorname{depth} R' \\ &= \operatorname{depth} R' - \inf(M \otimes_R R') \\ &= \operatorname{depth} R - \inf(M).\end{aligned}$$

The second equality holds by [11, Corollary 2.3] and the Auslander-Bridger formula [1], while the lase equality holds, because the closed fiber of $R \to R'$ is Artinian and [5, Proposition 1.2.16].

Now by Theorem 4.9, $\operatorname{depth} R - \inf(M) \leq \mathrm{Ch}_R(M) \leq \mathrm{CM}_*\text{-id}_R(M) = \operatorname{depth} R - \inf(M)$. Therefore $\mathrm{CM}_*\text{-id}_R(M) = \mathrm{Ch}_R(M) = \operatorname{depth} R - \inf(M)$.

In concluding, recall that there are notions of Cohen-Macaulay projective dimension, Cohen-Macaulay flat dimension and Cohen-Macaulay injective dimension of Holm and Jørgensen, which are different with our Definition 3.1.

DEFINITION 4.12. (cf., [17, Definition 2.3]) Let $(R, \mathfrak{m})$ be a local ring. For each homologically bounded R-complex M, the Cohen-Macaulay projective, flat and injective dimension, of M is defined as, respectively,

$$\operatorname{CM}\operatorname{pd}_R(M) := \inf\{\operatorname{Gpd}_{R\ltimes C}(M) \mid C \text{ is a semidualizing module}\}$$

$$\operatorname{CM}\operatorname{fd}_R(M) := \inf\{\operatorname{Gfd}_{R\ltimes C}(M) \mid C \text{ is a semidualizing module}\}$$

$$\operatorname{CM}\operatorname{id}_R(M) := \inf\{\operatorname{Gid}_{R\ltimes C}(M) \mid C \text{ is a semidualizing module}\}.$$

Here $R \ltimes C$ denotes the trivial extension ring of R by C; it is the R-module $R \oplus C$ equipped with the multiplication $(r, c)(r', c') = (rr', rc' + r'c)$.

REMARK 4.13. (1) For each homologically bounded R-complex M, we have

$$\operatorname{CM}_*\text{-}\operatorname{pd}_R(M) \leq \operatorname{CM}\operatorname{pd}_R(M)$$

$$\operatorname{CM}_*\text{-}\operatorname{fd}_R(M) \leq \operatorname{CM}\operatorname{fd}_R(M)$$

$$\operatorname{CM}_*\text{-}\operatorname{id}_R(M) \leq \operatorname{CM}\operatorname{id}_R(M).$$

More precisely, assume that $\operatorname{CM}\operatorname{pd}_R(M) < \infty$ and choose a semidualizing R-module C such that $\operatorname{CM}\operatorname{pd}_R(M) = \operatorname{Gpd}_{R\ltimes C}(M)$. Then by [16, Lemma 3.6], we have the CM-quasi-deformation $R \to R \overset{\tau}{\leftarrow} Q$ where $Q := R \ltimes C$ and $\tau(r, c) = r$, such that $\text{G-dim}_Q(R) = 0$. Thus we obtain

$$\operatorname{Gpd}_Q(M \otimes_R R) - \operatorname{Gfd}_Q(R) = \operatorname{Gpd}_{R\ltimes C}(M).$$

This shows the first inequality. The proof of the other two inequalities are the same as the first one.

(2) The finiteness of the Cohen-Macaulay homological dimensions in Definition 4.12, characterize Cohen-Macaulay rings admitting a canonical module [17, Theorem 5.1].

(3) Assume that $(R, \mathfrak{m}, k)$ is a Cohen-Macaulay ring, not admitting a canonical module (e.g., see [14] for such an example). Then $\operatorname{CM}_*\text{-}\operatorname{pd}_R(k) < \infty$ (and, $\operatorname{CM}_*\text{-}\operatorname{fd}_R(k) < \infty$, $\operatorname{CM}_*\text{-}\operatorname{id}_R(k) < \infty$) but $\operatorname{CM}\operatorname{pd}_R(k) = \infty$ (and, $\operatorname{CM}\operatorname{fd}_R(k) = \infty$, $\operatorname{CM}\operatorname{id}_R(k) = \infty$).

LEMMA 4.14. *Assume that C is a semidualizing R-module and let M be a homologically bounded R-complex. Consider M as a $R \ltimes C$-complex via the natural surjection $\tau\colon R \ltimes C \to R$.*

(1) *If* $\operatorname{Gfd}_{R\ltimes C}(M) < \infty$, *then* $\operatorname{Gfd}_{R\ltimes C}(M) = \operatorname{Rfd}_R(M)$.

(2) *If* $\operatorname{Gid}_{R\ltimes C}(M) < \infty$, *then* $\operatorname{Gid}_{R\ltimes C}(M) = \operatorname{Ch}_R(M)$.

PROOF. Note that $\operatorname{Spec}(R \ltimes C) = \{\mathfrak{p} \ltimes C \mid \mathfrak{p} \in \operatorname{Spec}(R)\}$ and $(R \ltimes C)_{\mathfrak{p}\ltimes C} \cong R_{\mathfrak{p}} \ltimes C_{\mathfrak{p}}$ by [4, Exercise 6.2.12]. Let L be an R-module which is an $R \ltimes C$-module via the surjection $\tau\colon R \ltimes C \to R$, and let $\mathfrak{p}$ be a prime ideal of R. Then $\varphi\colon L_{\mathfrak{p}\ltimes C} \to L_{\mathfrak{p}}$ sending $\ell/(r, c)$ to ℓ/r is an $R_{\mathfrak{p}}$-isomorphism. By [19, Theorem 8.8] we have the first equality below.

$$\begin{aligned}
&\operatorname{Gfd}_{R\ltimes C}(M)\\
&\quad= \sup\bigl\{\operatorname{depth}(R \ltimes C)_{\mathfrak{p}\ltimes C} - \operatorname{depth}_{(R\ltimes C)_{\mathfrak{p}\ltimes C}}(M_{\mathfrak{p}\ltimes C}) \mid \mathfrak{p} \in \operatorname{Spec}(R)\bigr\}\\
&\quad= \sup\bigl\{\operatorname{depth}_{R_{\mathfrak{p}}\ltimes C_{\mathfrak{p}}}(R_{\mathfrak{p}} \ltimes C_{\mathfrak{p}}) - \operatorname{depth}_{R_{\mathfrak{p}}\ltimes C_{\mathfrak{p}}}(M_{\mathfrak{p}\ltimes C}) \mid \mathfrak{p} \in \operatorname{Spec}(R)\bigr\}\\
&\quad= \sup\bigl\{\operatorname{depth}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}} \oplus C_{\mathfrak{p}}) - \operatorname{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \operatorname{Spec}(R)\bigr\}\\
&\quad= \sup\bigl\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \operatorname{Spec}(R)\bigr\}\\
&\quad= \operatorname{Rfd}_R(M).
\end{aligned}$$

The third equality holds since there is a surjection $R_{\mathfrak{p}} \ltimes C_{\mathfrak{p}} \to R_{\mathfrak{p}}$ and [18, Proposition 5.2(1)]. The fourth equality uses

$$\operatorname{depth}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}} \oplus C_{\mathfrak{p}}) = \min\{\operatorname{depth} R_{\mathfrak{p}}, \operatorname{depth}_{R_{\mathfrak{p}}}(C_{\mathfrak{p}})\} = \operatorname{depth} R_{\mathfrak{p}}.$$

The proof of (2) is the same as (1) using [11, Theorem 2.2] instead of [19, Theorem 8.8], and Lemma 4.5, instead of [18, Proposition 5.2(1)].

COROLLARY 4.15. *Let M be a homologically bounded R-complex.*

(1) *If* $\operatorname{CM\,fd}_R(M) < \infty$, *then* $\operatorname{CM\,fd}_R(M) = \operatorname{CM}_*\text{-}\operatorname{fd}_R(M)$.

(2) *If* $\operatorname{CM\,id}_R(M) < \infty$, *then* $\operatorname{CM\,id}_R(M) = \operatorname{CM}_*\text{-}\operatorname{id}_R(M)$.

PROOF. Note that there are the inequalities

$$\operatorname{Rfd}_R(M) \leq \operatorname{CM}_*\text{-}\operatorname{fd}_R(M) \leq \operatorname{CM\,fd}_R(M) = \operatorname{Rfd}_R(M)$$

(resp., $\operatorname{Ch}_R(M) \leq \operatorname{CM}_*\text{-}\operatorname{id}_R(M) \leq \operatorname{CM\,id}_R(M) = \operatorname{Ch}_R(M)$) by Corollary 4.2 (resp., Theorem 4.9), and Lemma 4.14.

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