MATH. SCAND. 49 (1981), 275–282

ON STRONG MARKOV DUALS

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Introduction.

A strong Markov process with an excessive reference measure always has a moderate Markov dual. Necessary and sufficient conditions for the existence of a strong Markov dual are known. However these conditions seem hard to check. In this note we show that if the potential kernel of a Hunt Process is "nice", then there is a strong Markov dual. Specifically let X_t be a transient Hunt process with excessive reference measure dx. Let u be the potential kernel of the process. If for each x, $y \to u(x,y)$ is lower semi continuous and there is a positive function f such that $\int u(x,y)f(x)\,dx$ is continuous and positive, then the process admits a strong Markov dual. This dual may have branch points, but as shown in [3], these form a polar and copolar set iff the given process satisfies Hypothesis B of Hunt. We are unable to give reasonable conditions more general than that of [4] under which this happens. This paper is related to [4]. However the conditions, methods, and results are diferent.

Notations and terminology will generally be that of [1]. For a real function f: non-negative means $f(x) \geqq 0\ \forall x$, whereas strictly positive means $f(x) > 0\ \forall x$. X_t will denote a transient Hunt process with a locally compact second countable state space E. We assume that there is a Radon measure dx and a non-negative function $u(\cdot,\cdot)$ on $E \times E$ such that

1) $y \to u(x,y)$ is lower semi continuous for each $x \in E$. If $y \neq z$, the functions $u(\cdot,y)$ and $u(\cdot,z)$ differ on a set of strictly positive dx measure.
2) There is a strictly positive function φ such that $\int u(x,y)\varphi(x)\,dx$ is continuous and non-negative.
3) For each non-negative Borel function f we have

$$E^x\left[\int_0^\infty f(X_t)\,dt\right] = \int u(x,y)f(y)\,dy\,.$$

The path we take is the following: We show that a time change of X_t has a strong Markov dual. This implies that the right continuous version of the time changed moderate Markov dual is in fact strong Markov. Changing time we

Received June 17, 1980; in revised form November 6, 1980.

conclude that the right continuous version of the moderate Markov dual is itself strong Markov.

1.

Let for $f \geqq 0$ measurable

$$Uf(x) = \int u(x,y) f(y)\, dy$$

$$\hat{U}f(y) = \int u(x,y) f(x)\, dx\,.$$

PROPOSITION 1. *$\hat{U}$ satisfies the complete maximum principle: For $f, g \geqq 0$ and constant $b > 0$,*

$$b + \hat{U}f \geqq \hat{U}g \quad \text{on} \quad (g > 0) \tag{1}$$

implies the inequality everywhere.

PROOF. By increasing b slightly we may assume that in (1) strict inequality obtains. We may also assume that g has compact support; the general case then follows by monotone convergence. $\hat{U}(n\varphi)$ is continuous for any n, where φ is the function in condition (2) of the introduction. But since $\hat{U}(n\varphi) = \hat{U}(n\varphi \wedge g) - \hat{U}(g - (n\varphi \wedge g))$ and both summands are lower semi continuous, it follows that $\hat{U}(n\varphi \wedge g)$ is also continuous. This sequence increases monotonically to $\hat{U}g$. Therefore if we can prove the statement for g of the form $n\varphi \wedge g$ it also holds for general g. Hence we may assume that in (1), $\hat{U}g$ is continuous and g has compact support. Since we have assumed that there is strict inequality in (1), the same limit procedure can be applied to f and we may assume that $\hat{U}f$ is continuous also.

Now it is well known that there is a dual potential kernel V, for example the potential kernel of a moderate Markov dual. Obviously $Vh = \hat{U}h$ almost everywhere for each Borel h. And V satisfies the complete maximum principle. Hence (1) holds almost everywhere and by continuity everywhere. The proof is complete.

Using standard techniques we can find a strictly positive function a such that

$$\int a\, dx = 1,\ Ua, \hat{U}a \text{ are non-negative and bounded and } \hat{U}a \text{ is continuous}\,. \tag{2}$$

The kernels U_a, $\hat{U}_a$ defined by

$$U_a f = U(af),\ \hat{U}_a f = \hat{U}(af)$$

also satisfy the complete maximum principle. U_a corresponds to the Hunt process obtained by time change of X_t using the additive functional

$$A_t = \int_0^t a(X_s)\,ds\ .$$

On the other hand, since $\hat{U}_a$ is a uniform kernel, by a theorem of Hunt, there is a submarkov resolvent $\hat{U}_a^\alpha$ such that $\hat{U}_a^0 = \hat{U}_a$.

It is seen that U_a and $\hat{U}_a$ are in duality relative to the probability measure $a(x)dx$. That the corresponding resolvents are also in duality can be seen, for example by series expansions.

Finally we remark that for each bounded measurable f, $\hat{U}_a f$ is bounded and continuous.

In the next article we show using the Ray–Knight approach that $\hat{U}_a^\alpha$, corresponds to a strong Markov process on E.

2.

In this article we let R, S denote two potential kernels which are in duality relative to a finite measure m on E. R^α and S^α denote the corresponding resolvents. We also assume that

(1)

$S^\alpha f$ is continuous if f is bounded measurable.
The set $\{S^\alpha f\}$ separates points in E.
R^α corresponds to a Hunt process Y_t.

Using a known technique [see for example 2] we can form a min-stable, separable, convex cone $\mathscr{H}$ of bounded non-negative continuous functions on E such that

$\mathscr{H}$ contains constant and $\bigcup_\alpha S^\alpha \mathscr{H}$ separates points of E and is contained in $\mathscr{H}$.

Let $\bar{E}$ denote the compactification of E using $\mathscr{H}$. $\bar{E}$ is a compact metric space. $\bar{E}$ contains a continuous (because elements of $\mathscr{H}$ are continuous) and 1 -1 (because $\mathscr{H}$ separates points) image of E. Since topology on compact sets is not disturbed by $1-1$ continuous maps, the original topology on E agrees with the topology on E inherited from $\bar{E}$, in particular E can be regarded as a σ-compact borel set of $\bar{E}$.

The definition of $\bar{E}$ implies that elements of $\mathscr{H}$ can be extended to be continuous on $\bar{E}$. Denoting the extension by a bar on top, the Stone–Weierstrass theorem implies that $\bar{\mathscr{H}} - \bar{\mathscr{H}}$ is dense in $C(\bar{E})$.

The operators S^α have a natural definition denoted $\bar{S}^\alpha$ on $C(\bar{E})$ into itself and thus defined they satisfy the resolvent equation. Also the range of $\bar{S}^1$ will separate points of $\bar{E}$. In other words $(\bar{S}^\alpha,\ \alpha>0)$ is a Ray resolvent on $\bar{E}$.

According to a theorem of Ray this resolvent gives rise to a Ray process $(\bar{Y}_t, \bar{P}^x)$ on $\bar{E}$.

We will have use for the following two notes.

NOTE 1. For all $x \in E$ and $\alpha > 0$,

$$\bar{S}^\alpha(x, E^c) = 0 .$$

To see this, let $x \in E$ and $\alpha \geqq 0$ be given. Given compacts $K \subset E$ and $L \subset E^c$ let $f \in C(\bar{E})$ be such that $0 \leqq f \leqq 1$, f equal to 0 on K and 1 on L. Then

$$\begin{aligned}\bar{S}^\alpha(x, L) &\leqq \bar{S}^\alpha f(x) = \overline{S^\alpha(f|_E)(x)} \\ &\leqq S^\alpha(1_{E \setminus K})(x)\end{aligned}$$

and the last quantity is small if K is large.

NOTE 2. $m\bar{P}_0 = m$. Here we are regarding m as a measure on $\bar{E}$ and $\bar{P}_0 f = \lim_{\alpha \to \infty} \alpha \bar{S}^\alpha f$ for all $f \in C(\bar{E})$.

REMARK. From the construction it follows that

$$\alpha \bar{S}^\alpha f(x) = \alpha S^\alpha f|_E(x)$$

for $\alpha > 0$, $x \in E$ and $f \in C(\bar{E})$; therefore $\bar{P}_0 f(x) = f(x)$ for $x \in E$ and $f \in C(\bar{E})$. Thus all points in E are non-branch points.

Note 2 follows from

$$\begin{aligned} m\bar{P}_0 f &= \int \bar{P}_0 f \, dm = \lim_{\alpha \to \infty} \alpha \int \bar{S}^\alpha f \, dm \\ &= \lim_{\alpha \to \infty} \alpha \int S^\alpha f \, dm = \int f \, dm \end{aligned}$$

because $\lim \alpha S^\alpha f = f$ m-almost everywhere, and everything is bounded.

By regarding points not in E as traps for example, Y_t can be considered a right-continuous with left limits strong Markov process on $\bar{E}$. Y_t and $\bar{Y}_t$ will then be two strong Markov processes on $\bar{E}$ which are in weak duality relative to m. We also have

$$mP_0 = m \quad \text{and} \quad m\bar{P}_0 = m$$

as measures on $\bar{E}$. In other words we are in a set-up studied in Walsh [6] in which paper the arguments about duality used in the proof of the following theorem can be found.

THEOREM. *$(\bar{Y}_t, \bar{P}^x, x \in E)$ is a right continuous strong Markov process on E. Its left limits also belong to E. It is in weak duality with the given process Y_t.*

PROOF. For compact $K \subset E$, duality implies

$$\bar{P}^m\{\exists\, t \leqq n,\ \bar{Y}_t \in \bar{E} \setminus K,\ \bar{\zeta} \geqq n\}$$

$$= P^m\{\exists\, t \leqq n,\ Y_t \in \bar{E} \setminus K,\ \zeta \geqq n\} = \mathscr{E}_n(K)$$

say. $\mathscr{E}_n(K)$ tends to zero as K increases to E.

The Borel–Cantelli lemma implies the existence of a sequence K_n of compacts in E satisfying

for $\bar{P}^m$-almost all ω $\quad \exists\, n$ such that
$\bar{Y}_t(\omega) \in K_n$ for all $t \leqq n$ if $\bar{\zeta}(\omega) \geqq n$.

In other words $\bar{P}^m$-almost surely both $\bar{Y}_t$ and $\bar{Y}_{t-}$ belong to E. The function s defined by

$$s(x) = \bar{P}^x[\exists\, t > 0, \bar{Y}_t \in \bar{E} \setminus E \text{ or } \bar{Y}_{t-} \in \bar{E} \setminus E]$$

is seen to be excessive. $s = 0$ m-almost everywhere, as shown above. Since $\alpha \bar{S}^\alpha s$ increases to s, we see that $s \equiv 0$ on E. Hence for all $x \in E$, $\bar{P}^x$-almost purely $\bar{Y}_t$ and $\bar{Y}_{t-}$ both belong to E. This is the assertion in the theorem.

3.

We continue with section 1. As we remarked in section 1, U_a is the potential kernel of the time change of X_t using the additive functional with density $a(X_t)$. Let us call this process X_t^a. The results of article 2 applied to this situation give a strong Markov process $\bar{X}_t^a$, which is in weak duality with X_t^a, relative to the measure $a(x)\,dx$. The assumption concerning point separation of the resolvent is fulfilled by the last part of assumption 1) above. It is now obvious that time changing $\bar{X}_t^a$ using the additive functional with density $(a(\bar{X}_t^a))^{-1}$ would give us a strong Markov dual to X_t. However this seems to need some justification which we proceed to give.

Let us denote by $(\hat{X}_t, \hat{P}^x)$ a moderate Markov dual of X_t. The process $\hat{X}_t^a$ obtained by time change of $\hat{X}_t$ using the additive functional

$$\int_0^t a(\hat{X}_s)\,ds$$

is moderate Markov. See [1, Example 2.11 pp. 212]. Here we are using that a is strictly positive. This implies that the inverse, say $\{\tau(t)\}$ of $t \to \int_0^t a(\hat{X}_s)\,ds$ is continuous and strictly increasing and in particular that each $\tau(t)$ is a predictable stopping time. It is easily verified that it is in weak duality with X_t^a relative to the measure $a(x)dx$.

On the other hand, the left continuous version of $\bar{X}_t^a$, namely $(\bar{X}_{t-}^a, \bar{P}^x)$, is also moderate Markov. To see this, recall that the resolvent $\hat{U}_a^\lambda$ corresponding to the uniform kernel $\hat{U}_a$ maps bounded Borel functions into continuous functions. Now very simple modifications in the proof of [1, pp. 42] are needed.

This left continuous version is also in weak duality with X_t^a. Hence, except for a polar set of x, the processes $(\hat{X}_t^a, \hat{P}^x)$ and $(\bar{X}_{t-}^a, \bar{P}^x)$ are indistinguishable [3].

Now it is intuitively clear that a time change of $\hat{X}_t^a$ using

$$\int_0^t \frac{1}{a}(\hat{X}_s^a)\, ds \tag{1}$$

leads us to $\hat{X}_t$. This is validated once we show that for sufficiently small t, the integral in (1) is finite. But this is an easy consequence of Lemma (2.2) [1, pp. 206]. However we only need the fact that (1) is finite for small enough t. But then except perhaps for x in a polar set (1) is finite, almost surely $\bar{P}^x$ if $\hat{X}_s^a$ is replaced by $\bar{X}_s^a$ provided of course that t is small enough. In other words we can time change $\bar{X}^a$ using $\int_0^t (1/a)(\bar{X}_s^a)\, ds$. The time changed process will then be strong Markov and will clearly be dual to the original process. Thus we have proved:

THEOREM. *A strong Markov process, satisfying the conditions spelled out in the introduction admits a strong Markov dual.*

REMARK. We have proved above that the right continuous version of any moderate Markov dual is in fact strong Markov, provided the conditions of the Introduction hold.

4. An example.

We shall here give a simple example of a potential kernel which satisfies all our conditions, and since this kernel is not necessarily symmetric, the existence of a strong markov dual is not obvious. Let G denote the 1-potential kernel of the d-dimensional Brownian motion with $d \geqq 3$. Let $P(x, dy)$ be a submarkov kernel on $\mathbf{R}^d$ satisfying the following two conditions:

$$Ps \leqq s \quad \text{for all excessive functions } s\,. \tag{1}$$

$$\int P(x, dz) G(z, y) \quad \text{is continuous in } y\,. \tag{2}$$

Define inductively kernels $V_n(\cdot,\cdot)$ as follows:

$$V_0(x,y) = G(x,y)$$

and

$$V_{n+1}(x,y) = G(x,y)+GPV_n(x,y), \quad n \geqq 0 . \tag{3}$$

By induction it is seen that V_n increase with n. We put

$$V(x,y) = \lim_n V_n(x,y) . \tag{4}$$

If G_0 denotes the Newtonian kernel, then using (1), it is seen by induction that $V_n \leqq G_0$, and hence

$$V(x,y) \leqq G_0(x,y) . \tag{5}$$

Since G is infinite on the diagonal, and G_0 is finite off the diagonal, we see that V has these two properties.

We now show that V is continuous off the diagonal. By condition (2) $PG(x,y)$ is continuous in y. Since GG_0 is continuous and bounded in both variables, it follows using (1) that GPG is continuous in both variables and bounded. So if $L=GP$, then $L^nG(x,y)$ is continuous and bounded in both variables. Now for any N we can write

$$V = \sum_0^N L^nG+L^NV$$

but using (1) and (5) we get

$$L^NV \leqq G^NG_0$$

from which it follows that V is continuous off the diagonal. Thus V is lower semicontinuous in both variables, infinite on the diagonal and continuous and finite off the diagonal. Of course V is strictly positive.

If for a continuous function f with compact support we let

$$Vf(x) = \int V(x,y)(f(y))\,dy$$

we see

$$Vf = G[f+PVf] . \tag{6}$$

Further $Vf \leqq G_0f$, so that Vf is continuous and bounded. For any $0<\alpha<1$, using (6) and resolvent equation we see

$$Vf = G_\alpha[f+PVf-(1-\alpha)Vf] , \tag{7}$$

where G_α is the α-potential kernel of Brownian motion. Using (7) it is not difficult to see that V satisfies the complete maximum principle. A theorem of Hunt asserts that V is the potential kernel of a Hunt process.

If f is bounded with compact support $\hat{V}f(y) = \int V(x,y)f(x)\,dx$ is continuous in y. This is so because V is extended continuous in both variables, $V \leqq G_0$ and G_0 has this property. For any fixed bounded non-negative g with compact support the measure $\hat{V}g(y)\,dy$ is excessive for the Hunt process determined by V and with respect to this excessive measure the potential kernel is $(\hat{V}g(y))^{-1}V(x,y)$. With this change all the conditions in the Introduction will be satisfied.

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