CHARACTERIZATIONS OF POISSON INTEGRALS ON SYMMETRIC SPACES

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Introduction.

It is well known that for the hyperbolic unit disk \( U = \{ |z| < 1 \} \) the eigenfunctions \( u \) of the Laplacian are given by generalized Poisson integrals of hyperfunctions \( T \) on the boundary ([4, Ch. IV]). More generally, such an integral representation holds for the joint eigenfunctions of the invariant differential operators on a symmetric space \( X \), see [6]. If \( X \) has rank 1, the kernels used here are powers of the ordinary Poisson kernel. In that case, \( T \) is a distribution if and only if \( u \) grows at most exponentially with the distance, see Lewis [J. Funct. Anal. 29 (1978), 287–307]. For \( U \), the functional \( T \) is given by an \( L^p \) function on the boundary, \( 1 < p < \infty \), if and only if the \( L^p \) norm of \( u/\phi \) on \( |z| = r \) is bounded as \( r \to 1 \), as follows by standard arguments, cf. Remark 2 in Section 4. Here \( \phi \) is the circular mean value of \( u \) (assuming \( u(0) \neq 0 \)), i.e., the generalized Poisson integral of a constant function on the boundary. The present paper deals with another way of characterizing those \( u \) for which \( T \) is an \( L^p \) function. These characterizations use weak \( L^p \) spaces and work for symmetric spaces of arbitrary rank. They extend the author’s work in [12] on ordinary Poisson integrals in \( \mathbb{R}^n \).

In \( U \), the results read as follows. Let \( dm_0 \) be the measure \( (1 - |z|^2)^{-1 - s} dx dy \), so that \( s = 1 \) gives the invariant measure and \( s = -1 \) Lebesgue measure. Then \( T \) is an \( L^p \) function if and only if \( (1 - |z|^2)^{p/s} u/\phi \) is in weak \( L^p (m_0) \). When \( p > 1 \), this holds for all \( s \neq 0 \), when \( p = 1 \) only for \( s \notin [0, 1] \).

For rank \( X = r > 1 \), it turns out that \( r - 1 \) logarithmic factors must be introduced in the weak \( L^p \) condition, which can be done in several ways. The case (called \( \lambda = 0 \)) corresponding to the square root of the ordinary Poisson kernel must be treated differently, although the results are essentially the same.

Lohoué and Rychener [9] have proved a special case of our results and applied it to convolution operators on \( L^p \) in a Lie group. Most of our techniques are suitable generalizations of those of [12]. See also [13], where more general kernels are considered.

The preparatory Section 2 contains, among other things, some known facts

Received June 9, 1980.
about the behaviour of joint eigenfunctions, and there the measures and weak $L^p$ spaces we use are defined. In Section 3, we prove two auxiliary technical results in the setting of a "half-space" over a nilpotent Lie group. The idea of the proof of Theorem 3.1 is taken from that of Theorem 1 in [12].

The main results are given in Section 4 ($\lambda \neq 0$) and Section 5 ($\lambda = 0$). For $\lambda = 0$, we also give in Section 5 a method of recovering $f$ from its Poisson integral, which replaces the ordinary convergence result at the boundary. Finally, those results which hold only for $p > 1$ are studied in the last section, which also contains two counterexamples.

2. Preliminaries.

Let $X = G/K$ be a Riemannian symmetric space of noncompact type. Here $G$ is a connected semisimple Lie group with finite center, and $K$ a maximal compact subgroup of $G$. In fact, all our results are valid even when $X$ is reducible, but for simplicity we treat only the irreducible case. Denoting by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of $G$ and $K$, we choose a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Then $r = \dim \mathfrak{a}$ is the rank of $X$. This gives a root space decomposition of $\mathfrak{g}$. Choosing as usual a positive Weyl chamber $\mathfrak{a}_+$ and the associated ordering of the roots, we call $\mathfrak{n}$ (or $\mathfrak{n}$) the sum of the root spaces corresponding to positive (negative) roots. Letting $A$, $N$, and $\bar{N}$ be the subgroups of $G$ having Lie algebras $\mathfrak{a}$, $\mathfrak{n}$, and $\bar{\mathfrak{n}}$, respectively, we have Iwasawa decompositions $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ and $G = KAN$. Thus, any $g \in G$ can be written uniquely as $g = k(g) \exp H(g)n(g)$ with $k(g) \in K$, $H(g) \in \mathfrak{a}$, and $n(g) \in N$. We let $e$ be the unit element of $G$, $K$, or $N$, and set $0 = eK \in X$.

Denoting $\bar{A}_+ = \exp \bar{\mathfrak{a}}_+$, where $\bar{\mathfrak{a}}_+$ is the closure of $\mathfrak{a}_+$ in $\mathfrak{a}$, we have the Cartan decomposition $G = K\bar{A}_+ K$. This means that for any $g \in G$ or $x \in X$ there is a unique element $H$ in $\mathfrak{a}_+$ such that $g$ or $x$ belongs to $K(\exp H)K$. We then set $H = H'(g)$ or $H = H'(x)$, respectively. A function $F = F(H)$ defined on $\bar{\mathfrak{a}}_+$ will be considered also as a function on $G$ and on $X$, by means of $F(g) = F(H'(g))$ and $F(x) = F(H'(x))$.

Let $\mathfrak{a}^* (\bar{\mathfrak{a}}^*)$ be the real (complex) dual of $\mathfrak{a}$. Then

$$\mathfrak{a}_+^* = \{ \lambda \in \mathfrak{a}^* : \lambda(H) > 0 \quad \text{for} \quad H \in \mathfrak{a}_+ \}$$

is the open cone generated by the positive roots, by the bipolar theorem. The Killing form makes $\mathfrak{a}$ into an inner product space, so that there is a canonical map $\mathfrak{a} \rightarrow \mathfrak{a}^*$, and we denote by $\mathfrak{a}_+^*$ the image of $\mathfrak{a}_+$ under this map. Then $\mathfrak{a}_+^* \subset \mathfrak{a}^*$, as proved in Harish-Chandra [1, Lemma 35, p. 279]. Let $2q$ be the sum of the positive roots, so that $q \in \mathfrak{a}_+^*$ (see [1, p. 281]). We denote by $S$ the "slice" $\{ H_0 \in \mathfrak{a}_+ : 2q(H_0) = 1 \}$, and often write any $H \in \mathfrak{a}_+$ as $H = tH_0$, with $t > 0$ and $H_0 \in S$, and put $t = |H|$. Thus, $|\cdot| = |H'(x)|$ is also a function on $X$. A restricted
domain in $X$ is defined to be one of type $\{ x \in X : H'(x) \in \mathbb{R}_+S' \}$, where $S'$ is a nonempty, open, and relatively compact subset of $S$ and $\mathbb{R}_+S'$ the open cone it generates in $a$. In such a domain, we see that $H'(x)$ stays far from the boundary of the positive Weyl chamber, except for $x$ near $o$. We call $\mathbb{R}_+S'$ a restricted cone.

The (maximal) boundary of $X$ is by definition the quotient $K/M$, where $M$ is the centralizer of $A$ in $K$. This boundary has a unique normalized $K$-invariant measure $dkM$. If $\lambda \in a_\mathbb{K}$, the associated Poisson kernel is

$$P_\lambda(g, k) = e^{-(i\lambda + \varphi)(Hg^{-1}k)}, \quad \text{for } g \in G, \ k \in K.$$ 

Since this expression is right $K$-invariant in $g$ and right $M$-invariant in $k$, we may also consider $P_\lambda$ as a function on $X \times K/M$ and write $P_\lambda(x, kM)$ for $x \in X$. The $\lambda$-Poisson integral of a Borel measure $\mu$ in $K/M$ is defined by

$$P_\lambda \mu(x) = \int P_\lambda(x, kM) d\mu(kM).$$

For integrable functions $f$ on the boundary, $P_\lambda f$ means $P_\lambda(fdkM)$. Then the (spherical) function $\varphi_\lambda = P_\lambda 1$ is a left $K$-invariant function on $X$. As is well known, any $P_\lambda \mu$ is an eigenfunction for all $K$-invariant differential operators on $X$, and the eigenvalues depend only on $\lambda$. Let $s_\lambda$ be the space of all eigenfunctions for these operators with the same eigenvalues as $\varphi_\lambda$. Whenever convenient, we consider the functions in $s_\lambda$ as defined on $G$ rather than on $X$. These functions are smooth since some invariant operators are elliptic.

In this paper, $C$ will denote many different constants, and we will generally not indicate precisely which parameters $C$ depends on at each occurrence. The relation $f \sim g$ means $C^{-1} \leq f/g \leq C$. The following lemma describes the asymptotic behavior of $\varphi_\lambda$.

**Lemma 2.1.** One has $\varphi_\lambda(\exp H) \sim e^{i\lambda - \varphi(H)}$ if $i\lambda \in a_\mathbb{R}$, uniformly for $H \in a_+$. This can be written simply $\varphi_\lambda \sim e^{i\lambda - \varphi}$, with our conventions.

**Proof.** With $h = \exp H$, we set $\bar{n}^h = h\bar{n}h^{-1}$ and $\bar{n} = h^{-1}\bar{n}h$. Denoting by $d\bar{n}$ a suitable Haar measure on $\bar{N}$, we may transform the Poisson integral to an integral over $\bar{N}$ by means of $\bar{n} \rightarrow k(\bar{n})M$, getting

$$\varphi_\lambda(h) = \int_{\bar{N}} e^{-(i\lambda + \varphi)(H(k^{-1}\bar{n}))-2\varphi(H(k))} d\bar{n}$$

$$= e^{(i\lambda - \varphi)(H)} \int_{\bar{N}} e^{-(i\lambda + \varphi)(H(\bar{n})) + 2\varphi(H) + (i\lambda - \varphi)(H(\bar{n})))} d\bar{n}$$

$$= e^{(i\lambda - \varphi)(H)} \int_{\bar{N}} e^{-(i\lambda + \varphi)(H(\bar{n})) + (i\lambda - \varphi)(H(\bar{n})))} d\bar{n}$$
(see e.g. Helgason [4, pp. 129–130]; distinguish between $H(.)$ and $H$). The last step here was the transformation $\tilde{n} \rightarrow \tilde{n}^h$ which has Jacobian $e^{-2\phi(H)}$. If a canonical coordinate system is used in $\tilde{N}$, the conjugation $\tilde{n} \rightarrow \tilde{n}^h$ has the effect of decreasing all coordinates when $h \in \tilde{A}_+$, so $\tilde{n}^h$ stays in a compact set as $\tilde{n}$ varies in a compact set $L \subset \tilde{N}$ and $h \in \tilde{A}_+$. Therefore, the integrand in the last integral is $\sim 1$ in $L$. From this we get $\varphi_\lambda(\exp H) \geq e^{(i\lambda - \omega(H))}/C$.

For the converse inequality, we estimate the same integrand from above by $e^{-(\omega + \delta i\lambda)H(\tilde{n})}$, where $\delta > 0$ is small, as in Helgason [4, p. 130] or Michelson [10, p. 262], and this expression is integrable and independent of $H$. The lemma is proved.

A measure $\mu$ in $\tilde{N}$ may also be considered as a measure in $K/M$ by means of the transformation $\tilde{n} \rightarrow k(\tilde{n})M$. From the proof just given, we then see that

$$P_\lambda \mu(\tilde{m}h) = e^{(i\lambda - \omega)(H)} \int e^{-(i\lambda + \omega(H^h(\tilde{m}^{-1}\tilde{n}) + 2\phi(H) + (i\lambda + \omega)(H(\tilde{n})))} d\mu(\tilde{n}),$$

for $\tilde{m} \in N$, $h \in A$, which will be used later.

For $\lambda = 0$, we put

$$\varphi_0(\exp H) = e^{-\omega(H)}p(H), \quad H \in \tilde{A}_+.$$ Harish–Chandra [1, p. 279] has proved that there is a natural number $q$ such that $\psi(H)/(1 + |H|)^q$ is bounded on $\tilde{A}_+$ and $\psi(H_0) \sim t^q$ as $t \to \infty$ for each $H_0 \in S$, but this last relation is not uniform in $H_0$ when $r > 1$. If $r = 1$, then $q = 1$.

**Lemma 2.2.** Given $i\lambda \in \tilde{a}_+$ and a compact set $L \subset \mathbb{R}$, there is a constant $C = C(L, \lambda)$ such that any nonnegative $u \in \mathcal{D}_\lambda$ satisfies

$$u(gx) \leq Cu(gy)$$

for all $x, y \in L$ and any $g \in G$.

This lemma is a form of Harnack’s inequality. Except for some cases, it is a consequence of Lemma 2.1 in Michelson [10], but we indicate another proof: For $g = e$, the lemma follows by well-known elliptic operator techniques (cf. Serrin [11], or the fact that $\mathcal{D}_\lambda$ defines a sheaf satisfying Brelot’s axiomatic potential theory). The general case is then immediate from the translation invariance of $\mathcal{D}_\lambda$.

We shall work with several positive measures on $X$. The invariant measure $m$ is given by

$$\int_X \varphi dm = \int_{K/M \times A_+} \varphi(k \exp H) \prod_\mathfrak{a} \sinh (\alpha(H)) dk M dH,$$
where the product is taken over the positive roots, counted according to multiplicity. We write $\varphi(k \exp H)$ rather than $\varphi(k(\exp H)K)$, and $dH$ is a Euclidean measure in $a$.

Writing $H = tH_0$, we have $dH = t^{r-1} dt dH_0$ in $a_+$ for a fixed measure $dH_0$ in $S$ which is an $(r-1)$-dimensional Euclidean measure on $S$. As $H \to \infty$, which means that $\alpha(H) \to \infty$ for all positive roots, $dm = e^{2\sigma} dk M dH$. Setting $dm_\sigma = e^{\sigma-2\sigma} dm$ for $\sigma \in a^*$, we obtain a family of measures, and $dm_\sigma \sim e^{\sigma} dk M dH$ as $H \to \infty$. Notice that Lebesgue measure in the unit disk is included here, up to the $\sim$ relation, and that $m_\sigma$ is finite if and only if $\sigma \in -a^*$.

Next, we define weak $L^p$ spaces. If $\mu$ is any positive measure on $X$ and $f$ a $\mu$-measurable real- or complex-valued function, the distribution function of $f$ is

$$
\lambda_f(\alpha) = \mu\{x \in X : |f(x)| > \alpha\}, \quad \alpha > 0.
$$

The decreasing rearrangement of $f$ is

$$f^*(t) = \inf\{\alpha : \lambda_f(\alpha) \leq t\}, \quad 0 < t < \mu(X) .
$$

Notice the simple inequality

$$
\int_E |f| \, d\mu \leq \int_0^{\mu(E)} f^*(t) \, dt
$$

valid for any $\mu$-measurable set $E \subset X$. Weak $L^p$, denoted $A_p$, consists of those $f$ for which

$$f^*(t) \leq Ct^{-1/p}, \quad 0 < t < \mu(X),
$$

and the smallest possible $C$ here is the quasi-norm of $f$ in weak $L^p$. Setting

$$\log^b t = (1 + \|\log t\|^b), \quad \text{all } t > 0 \text{ and } b \geq 0,
$$

we define $A^s_p$ by the inequality $f^*(t) \leq C t^{-1/p} \log^b_{*} t$, and call $C$ the quasinorm as before. Notice that $A^0_p = A_p$, that $A^s_p$ may also be defined by

$$
\lambda_f(\alpha) \leq C \alpha^{-p} \log^b_{*} \alpha
$$

with another $C$, and that

$$
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$$

(2.3)

$$f \in A^s_p \iff |f|^p \in A^s_1.
$$

When $\mu = m_\sigma$, we denote by $A_{p,\sigma}$ and $A_{s,p,\sigma}$ the spaces obtained. Finally, $A^s_{p,\sigma}$ is weak $L^p$ with respect to the measure $(1 + |H|)^{1-r} dm_\sigma$; observe that this measure is slightly smaller than $m_\sigma$ and behaves like $e^{s} dk M dtdH_0$ as $H = tH_0 \to \infty$.

3. Auxiliary theorems.

In this section, $H_0$ will be in $S$ and we set $h = \exp tH_0$, $t \in R$. For $\tilde{n} \in \tilde{N}$, we write $\tilde{n}_t = h_t \tilde{n} h_{-t}$. In the space $\tilde{N} \times R$, let $dm_\sigma^t$ denote the measure $e^{t_s(H_0)} \tilde{n}_t d\tilde{n} dt$. 

$$
\tilde{N} \times R
$$

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(\tilde{n} \in \tilde{N}, t \in \mathbb{R}). \text{ Here } \sigma \in a^* \text{ as before. Because of (2.1), the integral formed in the following theorem is closely related to } P_{\lambda \mu}(mh_i).

**Theorem 3.1.** Let \( i \lambda \in a_+^* \), and \( \sigma \in (2\Omega + a_+^*) \cup (-a_+^*) \). With \( H_0 \in S \) and \( \mu \) a probability measure in \( \tilde{N} \), define a function \( v \) in \( \tilde{N} \times \mathbb{R} \) by

\[
v(\tilde{m}, t) = e^{-i\sigma(H_0)} \int e^{-i(\lambda + \phi)H((\tilde{m}^{-1}\tilde{n})_t)} \, d\mu(\tilde{n}).
\]

Then \( v \) is in weak \( L^1 \) with respect to \( m_\sigma' \) in \( \tilde{N} \times \mathbb{R} \), and the corresponding quasinorm is bounded uniformly for \( H_0 \in S \).

**Proof.** The product \( \tilde{N} \times \mathbb{R} \) should be seen as a "half-space" \( \{(\tilde{m}, t') : \tilde{m} \in \tilde{N}, t' > 0\} \) over \( \tilde{N} \), but we use \( t = -\log t' \) instead of \( t' \) as a coordinate. Call \( \tilde{N} \times [j, j + 1] \subset \tilde{N} \times \mathbb{R} \) the \( j \)-layer, for any integer \( j \).

Let \( B \) be a compact neighborhood of \( e \in \tilde{N} \) which is symmetric (\( B^{-1} = B \)), of Haar measure 1, and such that \( hBh^{-1} \subset B \) for all \( h \in \tilde{A}_+ \). Of course, \( B_t \) means \( h_1 B_{h^{-1}_1} \), and the Haar measure of \( B_t \) equals the Jacobian of the map \( \tilde{n} \rightarrow \tilde{n}_t \), which is \( e^{-2\sigma(tH_0)} = e^{-t} \). To obtain the claimed uniformity in \( H_0 \), we fix an element \( H_1 \in S \), putting \( h_1' = \exp yH_1 \) and \( B_{t,y} = h_1'B_{t,y}h_1' \). Notice that \( B_{t,y} \) is decreasing in \( t \) and increasing in \( y \). Take \( \beta > 0 \) so that \( BB \subset B_{0,\beta} \).

The sets \( B_t \) will serve as building-blocks to discretize the problem. For each integer \( j \), we choose a maximal set \( \{\tilde{n}_jB_j\}_j \) of pairwise disjoint translates of \( B_j \) in \( \tilde{N} \), and each \( \tilde{n}_jB_j \) is called a \( j \)-base. Thus, for any \( \tilde{m} \in \tilde{N} \), the translate \( \tilde{m}B_j \) must intersect some \( j \)-base. The sets \( \tilde{n}_jB_j \times [j, j + 1] \) are called \( j \)-pieces, and they are disjoint and contained in the \( j \)-layer.

In the "half-space", the \( j \)-layer should be thought of as situated at height \( \sim e^{-j} \) and having width \( \sim e^{-j} \). And the \( j \)-pieces essentially correspond to a subdivision of the \( j \)-layer into cubes of side \( e^{-j} \). (The \( j \)-pieces do not cover the \( j \)-layer, but this is unimportant.)

We shall need three observations. First, Lemma 2.2 gives a property of \( v \). For if \( \tilde{p} \in B_t \) and \( |\tau - t| \leq 1 \), then \( \tilde{m}\tilde{p}h_t = \tilde{m}h_{\tau} \tilde{p}h_t \), and the last two factors here belong to a compact set. Applying Lemma 2.2 to the Poisson kernel, we conclude that for any \( k \in K \) and any \( \tilde{m} \in \tilde{N} \),

\[
e^{-i(\lambda + \phi)(H((\tilde{m}h_t)^{-1}k))}
\]

does not change more than by a factor \( C \) if \( \tilde{m} \) is replaced by \( \tilde{m}\tilde{p} \) and \( t \) by \( \tau \). Since \( (\tilde{m}^{-1}\tilde{n})_t = (\tilde{n}^{-1}\tilde{m}h_t)^{-1}h_t \), this implies

\[
(3.1) \quad v(\tilde{m}\tilde{p}, \tau) \sim v(\tilde{m}, t) \quad \text{for } \tilde{p} \in B_n, |t - \tau| \leq 1.
\]

Next, we see that
(3.2) \[ \int e^{-(i\hat{\alpha} + \varphi)(H(\hat{m}, t)) + t} \, d\hat{m} \] is independent of \( t \),

by transforming \( \hat{m} \to \hat{m}_r \). This integral is known to be finite ([1, Lemma 45, p. 289]). Finally

(3.3) \[ \int_{\mathcal{N} \setminus B_{0, y}} e^{-(i\hat{\alpha} + \varphi)(H(\hat{m}))} \, d\hat{m} = O(e^{-a y}), \quad y \to \infty, \]

for some \( a > 0 \). This follows from the facts that

\[ \int e^{-(1-\eta)(i\hat{\alpha} + \varphi)(H(\hat{m}))} \, d\hat{m} < \infty \]

and \( e^{-(i\hat{\alpha} + \varphi)H(\hat{m})} = O(|\hat{m}|^{-a'}) \) for some \( \eta, a' > 0 \) and some norm on \( \hat{N} \) (Knapp–Williamson [7, Proposition 5.5]).

Now let \( \sigma = \sigma(H_0) \) so that \( \sigma \notin [0, 1] \) and \( s \) is bounded away from \([0, 1]\), uniformly as \( H_0 \in S \). Consider first the case \( s > 1 \). We have \( v \leq e^{-(s-1)\eta} \) everywhere since \( H(\hat{m}) \in \tilde{a} \) for any \( \tilde{m} \in \tilde{N} \) (see [1, Lemma 43, p. 287]). Take \( \alpha > 0 \). Since \( v \to 0 \) as \( t \to \infty \), we may let \( j_0 \) be the largest integer for which the set \( \{ (\hat{m}, t) : v(\hat{m}, t) > \alpha \} \) intersects the \( j_0 \)-layer.

By induction in decreasing \( j \), we shall now construct for \( j = j_0, j_0 - 1, \ldots \) measures \( v_j \) in \( \hat{N} \times \mathbb{R} \), and \( \text{supp} v_j \) will be a set of \( j \)-pieces. Each time we decide to place \( v_k \)-mass in a certain \( k \)-piece, we simultaneously forbid placing mass near this \( k \)-piece in the sequel, i.e. for \( j < k \). This is done by introducing “above” the \( k \)-piece a forbidden region which becomes wider as we move upwards (\( j \) decreases). At each step in the construction, mass is placed in a piece if and only if this piece intersects \( \{ v > \alpha \} \) and is not already in a forbidden region.

When we now carry out this in detail, \( F_p, j = j_0, j_0 - 1, \ldots \), will be sets of (forbidden) pieces. For some \( k \leq j_0 \), assume \( v_j \) and \( F_j \) defined for \( j_0 \geq j > k \). Then we let \( v_k \) be the restriction of the measure \( m'_\sigma \) to the union of all those \( k \)-pieces which intersect \( \{ (\hat{m}, t) : v(\hat{m}, t) > \alpha \} \) and do not belong to \( \bigcup_{j \geq j} F_j \), i.e. which are not already forbidden. (In case \( k = j_0 \), this union is empty.) Set \( P_k = \pi(\text{supp} v_k) \), where \( \pi: \hat{N} \times \mathbb{R} \to \hat{N} \) is the projection. Then \( P_k \) is a union of \( k \)-bases. Now \( F_k \) is defined as the set of those \( j \)-pieces, all \( j < k \), whose projections intersect the set \( P_k B_{j, \alpha(k-j) + \rho} \), where \( \alpha \) is a fixed number satisfying \( 0 < \alpha < s - 1 \). This defines \( v_j, j \leq j_0 \).

We claim that \( v = \sum_{j = -\infty}^{j_0} v_j \) satisfies

(i) \( m'_\sigma \{ v > C\alpha \} \leq C\| v \| \)

(ii) \( v > \alpha/C \) in \( \text{supp} v \)

(iii) \( U^r \leq C \) in \( \hat{N} \),

where
\[ U^v(\tilde{n}) = \int e^{-st - (i\xi + \varphi)(H((\tilde{m}^{-1}\tilde{n})_\infty)) + t} \, dv(\tilde{m}, t). \]

These three inequalities imply Theorem 3.1, since

\[ m'_\sigma\{v > C\alpha\} \leq C\|v\| \leq C\alpha^{-1} \int v \, dv = C\alpha^{-1} \int U^v \, d\mu \leq C\alpha^{-1}, \]

by Fubini’s theorem; cf. [12, p. 183].

To prove (i), we first observe that the \( m'_\sigma \) measure of a \( k \)-piece is \( Ce^{(s-1)k} \). Take a point \((\tilde{m}, t)\) with \( v(\tilde{m}, t) > C\alpha \), and let \( k = [t] \). Then \( \tilde{m}B_k \times [k, k + 1] \) intersects some \( k \)-piece \( \tilde{n}_iB_k \times [k, k + 1] \), and because of (3.1), one has \( v > \alpha \) in the intersection, if \( C \) is suitably chosen. It also follows that \( \tilde{m} \in \tilde{n}_iB_k \tilde{B}_k^{-1} = \tilde{n}_iB_k \tilde{B}_k \subseteq \tilde{n}_iB_{k, \beta} \). But the Haar measure of \( \tilde{n}_iB_{k, \beta} \) is at most \( C \) times that of \( \tilde{n}_iB_k \), and it follows that \( m'_\sigma\{v > C\alpha\} \) is bounded by \( C \) times the total \( m'_\sigma \) measure of those pieces which intersect \( \{v > \alpha\} \). And such a piece is either in \( \supp v \) or in \( \bigcup F_k \) by construction, and \( m'_\sigma(\supp v) = \|v\| \). Thus, it remains to estimate the total measure of the pieces in \( \bigcup F_k \) by \( C\|v\| \). To this end, notice that a \( j \)-piece in \( F_k, j < k \), must intersect \( \pi(Q)B_j, \pi(k-j)+\beta \times [j, j + 1] \), for some \( k \)-piece \( Q \) in \( \supp v_k \). Therefore, this \( j \)-piece is contained in \( \pi(Q)B_j, \pi(k-j)+\beta \times [j, j + 1] \) for some \( C \). The Haar measure of \( \pi(Q)B_j, \pi(k-j)+\beta \times [j, j + 1] \) is \( O(e^{-j + k - j}) \). Hence, the total \( m'_\sigma \) measure of all pieces in \( F_k \) associated with \( Q \) in this way is at most

\[ C \sum_{j < k} e^{s\alpha_j - j + k - j} \leq C e^{(s-1)k} \sum_{j < k} e^{-j + k - j} \leq C e^{(s-1)k} = Cm'_\sigma(Q). \]

Summing over all the pieces \( Q \) in \( \supp v_k \) and then over \( k \), we see that the total measure of the pieces in \( \bigcup F_k \) is at most

\[ Cm'_\sigma(\supp v) = C\|v\|. \]

Thus, (i) is proved.

Inequality (ii) is an immediate consequence of (3.1).

To prove (iii), we need two lemmas. The first one expresses that if the projection \( P_j \) of \( \supp v_j \) is far from \( \tilde{n} \), then \( U^{v_j} \) is small at \( \tilde{n} \).

**Lemma 3.2.** Let \( b > 0 \). If \( \tilde{n} \in \tilde{N} \) and \( P_j \cap \tilde{n}B_{j, \sigma b} = \emptyset \), then \( U^{v_j}(\tilde{n}) \leq C_0 e^{-\varepsilon b} \), where \( \varepsilon > 0 \) and \( C_0 \) are constants.

**Proof.** In view of the reasoning leading to (3.1),

\[ U^{v_j}(\tilde{n}) \leq C \int_{P_j} e^{-(i\xi + \varphi)(H((\tilde{m}^{-1}\tilde{n})_\infty)) + j} \, d\tilde{m}. \]
By assumption, \( P_j^{-1} \bar{n} \subset \bar{N} \setminus B_{j, x_b} \), so a transformation \( \bar{n} \to \bar{n} \bar{m}_j^{-1} \) takes us to the integral in (3.3). The lemma follows.

By making \( C_0 \) larger if necessary, we may assume

\[
(3.5) \quad U^{\nu_l}(\bar{n}) \leq C_0
\]

for all \( j \) and all \( \bar{n} \), because of (3.2). Inequality (iii) is a consequence of the following lemma, where \( \varepsilon \) and \( C_0 \) are as just described.

**Lemma 3.3.** For any \( \bar{n} \in \bar{N} \) and any \( j \leq j_0 \), it is possible to rearrange the sum \( \sum_{k=0}^{j_0} U^{\nu_l}(\bar{n}) \) so that it becomes dominated term by term by \( \sum_{k=0}^{j_0} C_0 e^{-\varepsilon k} \).

**Proof.** The case \( j = j_0 \) is clear from (3.5), so assume the lemma holds for \( j + 1 \). Let \( m \) be the nonnegative integer satisfying

\[
(3.6) \quad C_0 e^{-\varepsilon(m+1)} < U^{\nu_l}(\bar{n}) \leq C_0 e^{-\varepsilon m}.
\]

Lemma 3.2 then implies that \( P_j \cap \bar{n} B_j, x(m+1) \neq \emptyset \) so that \( \bar{n} \in P_j B_j, x(m+1) \). If \( k \geq j + m + 1 \) we have

\[
\bar{n} B_{k, x(k-j)} \subset P_j B_{j, x(m+1)} B_{k, x(k-j)} \subset P_j B_{j, x(k-j)} B_{j, x(k-j)}
\]

\[
\subset P_j B_{j, x(k-j)+\beta} \subset \bar{N} \setminus P_k,
\]

the last inclusion by the construction of \( F_k \). So for \( j_0 \geq k \geq j + m + 1 \), Lemma 3.2 implies \( U^{\nu_l}(\bar{n}) \leq C_0 e^{-\varepsilon(k-j)} \). By our induction assumption, the terms \( U^{\nu_l}(\bar{n}) \), \( j + m + 1 > k > j \), are in some order dominated by \( C_0 e^{-\varepsilon k} \), \( 0 \leq k \leq m - 1 \). These two estimates together with the right-hand inequality of (3.6) end the induction step. Lemma 3.3, (iii) and Theorem 3.1 (case \( s > 1 \)) are proved. The claimed uniformity in \( H_0 \) follows since none of the constants used depend on \( H_0 \).

When \( s < 0 \) in Theorem 3.1, we need only modify a few details. Then \( j_0 \) is the smallest integer \( j \) for which the \( j \)-layer intersects \( \{ v > x \} \), and the construction is carried out from smaller to greater \( j \)-values. The set \( F_k \) consists of those \( j \)-pieces, \( j > k \), whose projections intersect \( P_k B_{k, x(j-k)+\beta} \). Here \( 0 < x < -s \). We leave the rest to the reader.

This ends the proof of Theorem 3.1.

For \( \lambda = 0 \) we replace Theorem 3.1 by a weaker local result. Let \( R_+ = \{ t \in R : t > 0 \} \).

**Theorem 3.4.** Fix a compact set \( L \subset \bar{N} \), and let \( \mu \) be a probability measure carried by \( L \). Let \( \sigma, H_0 \), and \( v \) be as in Theorem 3.1 but set \( \lambda = 0 \). Then

\[
m^{\mu}_v(\bar{m}, t) \in L \times R_+ : v(\bar{m}, t) > x \leq C x^{-1} \psi(C (\log \alpha) H_0)
\]

for \( x > 0 \), where \( C \) depends on \( L \) but not on \( H_0 \).
**Proof.** Again let \( s = \sigma(H_0) \) and consider first the case \( s > 1 \). In this proof, we use a measure \( \nu \) as in the proof of Theorem 3.1, but the construction of \( \nu \) is much easier this time. In fact, \( \nu \) is carried by the "lower" boundary of the set \( \{ \nu > \alpha \} \) and has an area density there.

Set

\[
S(\tilde{m}) = \sup \{ t : \nu(\tilde{m}, t) > \alpha \}
\]

when \( \tilde{m} \in L' \) and \( L' \) is the set of \( \tilde{m} \in L \) for which \( S(\tilde{m}) > 0 \). Let \( \nu \) be the measure in \( L \times \mathbb{R}_+ \) defined by

\[
\int_{L'} \phi(\tilde{m}, t) d\nu(\tilde{m}, t) = \int_{L'} e^{S(\tilde{m})} \phi(\tilde{m}, S(\tilde{m})) d\tilde{m}.
\]

Then clearly,

(ii') \quad \nu = \alpha \quad \text{in supp} \nu ,

and moreover,

(i') \quad m'_\alpha(L \times \mathbb{R}_+ \cap \{ \nu > \alpha \}) \leq \int_{L'} d\tilde{m} \int_0^{S(\tilde{m})} e^{st} dt

\[
\leq s^{-1} \int_{L'} e^{S(\tilde{m})} d\tilde{m} = s^{-1} \| \nu \| .
\]

Now define \( U^\nu \) as in the preceding proof (\( \lambda = 0 \)). Theorem 3.4 follows if we show

(iii') \quad U^\nu \leq C\psi(C(\log \alpha)H_0) \quad \text{in} \ L ,

cf. (3.4).

**Lemma 3.5.** For any \( \tilde{n} \in \tilde{N} \), the quantity \( e^{-q(H(\tilde{n}, \cdot)) + t} \) increases with \( t \).

**Proof.** The square of this quantity is

\[
e^{-2q(H(\tilde{n}, \cdot)) + t} = P_{-i\varrho}(\tilde{n}^{-1}h, e)e^{2q(tH_0)}
\]

and \( P_{-i\varrho} = P \) is the ordinary Poisson kernel. From the expansion

\[
P(\tilde{n}^{-1}h, e)^{-1} = \sum s G_s(h_i)D_s(\tilde{n}),
\]

\[
G_s(h_i) = \exp \sum s \pm \alpha(tH_0),
\]

given in Knapp and Williamson [7, Proposition 5.1, p. 71], the lemma easily follows.
Proof of (iii). Since \( v \leq e^{-(s-1)t} \), we need only consider small \( \alpha \), and we have \( S(\tilde{m}) \leq t_0 \) for all \( \tilde{m} \in L' \) if \( t_0 \) is defined as \( C \log_\alpha \alpha \). Any \( \tilde{n} \in L \) then satisfies

\[
U^\gamma(\tilde{n}) = \int_{L'} e^{-\varrho(H((\tilde{m}^{-1}\tilde{n}) - S(\tilde{m}))) + S(\tilde{m})} d\tilde{m} \\
\leq C \int_{L'} e^{-\varrho(H((\tilde{m}^{-1}\tilde{n}) - r_0)) + t_0} d\tilde{m},
\]

where Lemma 3.5 was used. Since \( \tilde{m} \) and \( \tilde{n} \) stay in a compact set, we may subtract \( \varrho(H(\tilde{m}^{-1}\tilde{n})) \) in the exponent in the last integral if we change the value of \( C \). Transforming \( \tilde{m} \rightarrow \tilde{n} \tilde{m}^{-1} \), we obtain

\[
U^\gamma(\tilde{n}) \leq C \int e^{-\varrho(H(\tilde{m}^{-1}\tilde{n})) + t_0 - \varrho(H(\tilde{m}))} d\tilde{m} = \psi(t_0 H_0),
\]

where the last equality is seen from the proof of Lemma 2.1. This proves (iii') and Theorem 3.4 for \( s > 1 \).

When \( s < 0 \), we may assume \( \alpha \) is large since \( m'_\sigma(L \times \mathbb{R}_+ +) \) is finite, and we may neglect the set where \( t > t_0 = C \log_\alpha \alpha \), since the \( m'_\sigma \) measure of this set is \( O(\alpha^{-1}) \). Now \( S(\tilde{m}) \) is defined as \( \inf \{ t : \nu(\tilde{m}, t) > \alpha \} \) for \( \tilde{m} \) in the set \( L' \subset L \) where this \( \inf \) is positive but smaller than \( t_0 \). The rest goes as for \( s > 1 \).

Theorem 3.4 is proved. Notice that the proof given is based on that of Theorem 2 in [13].

4. Results for \( \text{Re } i \lambda \in \alpha_+^* \).

If \( \lambda \in \alpha_+ \), we define \( \lambda' \) by \( i\lambda' = \text{Re } i \lambda \).

Theorem 4.1. Let \( \text{Re } i \lambda \in \alpha_+^* \) and \( \sigma \in (2q + +_+ \alpha^*) \cup (-_+ \alpha^*) \), and take \( p \in [1, \infty[ \). For any \( u \in \mathcal{S}_\lambda \), the following are equivalent:

(a) \( u = P_\lambda f \) for some \( f \in L^p(K/M) \) when \( p > 1 \), or \( u = P_\lambda \mu \) for some Borel measure \( \mu \) on \( K/M \), when \( p = 1 \).

(b1) \( e^{-\sigma/|P_\lambda u|/\varphi_{\lambda'}} \in \Lambda_{p, \sigma}^{-1} \).

(b2) \( (1 + |.|)^{-r-1/2} e^{-\sigma/|P_\lambda u|/\varphi_{\lambda'}} \in \Lambda_{p, \sigma} \).

(b3) \( e^{-\sigma/|P_\lambda u|/\varphi_{\lambda'}} \in \Lambda_{p, \sigma}^* \).

Observe that for \( r = 1 \) the (b1) conditions coincide. Before the proof, we give a lemma.

Lemma 4.2. Let \( L \subset \bar{N} \) be compact. Any nonnegative \( \varphi \in \mathcal{S}_v \), \( v \in \alpha_+^* \), satisfies

\[
\varphi(\tilde{m}h) \sim \varphi(k(\tilde{m})h)
\]

for \( \tilde{m} \in L, h \in A_+ \). The same relation holds when \( \varphi \) is replaced by \( e^\nu \), any \( \nu \in \alpha^* \).
PROOF. Let \( \tilde{m} = \text{kan} \) be the Iwasawa decomposition of \( \tilde{m} \), so that \( \tilde{m} h = k h a^b n \). If \( m \in L \), then also \( a, n, \) and \( h \) stay in compact sets, so (4.1) follows from Lemma 2.2. This is true in particular for \( q = \phi, iv \in a^*_r \). But \( \phi \sim e^{iv \cdot \phi} \) by Lemma 2.1, so considering quotients \( \phi / \phi \cdot v \), we see that any \( e^v \) must satisfy (4.1), and the lemma is proved.

PROOF OF THEOREM 4.1. (a) \( \Rightarrow \) (b). Since \( |P_\lambda f| \leq P_\lambda |f| \), we may assume \( i \lambda \in a^*_+ \). The \( p = 1 \) case then immediately implies the other cases, because of (2.4) and since, by Hölder’s inequality, \( |P_\lambda f|^p \leq P_\lambda |f|^p \cdot \phi_\lambda^{p-1} \).

Now let \( i \lambda \in a^*_+ \) and \( u = P_\lambda u \), where \( \mu \) is a probability measure carried by \( k(LM) \subset K/M \), and \( L \) is as in Lemma 4.2. To begin with, we prove that \( u \) satisfies (b) in \( k(L)A_+ \subset X \), and start with (b1). Let \( w = e^{-\sigma u} / P_\lambda \). Since \( dm_\mu \leq e^\sigma d k M dH \) and because of (2.3), it suffices to prove that for all \( \alpha > 0 \)

\( I \equiv \int_D e^{\sigma(H)} dk M dH \leq C \alpha^{-1} \log_{**}^{-1} \alpha \)

where \( D = \{ (k M, H) \in k(LM) \times a^*_+ : w(k \exp H) > \alpha \} \).

Setting \( k = k(\tilde{m}) \), we know that \( dkM \) corresponds to \( e^{-2\sigma(H(\tilde{m}))} d\tilde{m} \) which is majorized by \( d\tilde{m} \), so

\[ I \leq \int_{D'} e^\sigma H d\tilde{m} dH , \]

with \( D' = \{ (\tilde{m}, H) \in L \times a_+ : w(\tilde{m} \exp H) > \alpha \} \). Because of Lemma 4.2,

\( D' \subset D'' = \{ (\tilde{m}, H) \in L \times a_+ : w(\tilde{m} \exp H) > \alpha / C \} \).

The inverse image of \( \mu \) under \( \bar{n} \rightarrow k(\bar{n})M \) is a measure in \( L \) which is also called \( \mu \). For \( H_0 \in S \), let \( v \) be as in Theorem 3.1. Because of (2.1), we have \( w(\tilde{m} \exp t H_0) \leq C v(\tilde{m}, t) \). When \( r = 1 \), we see that \( I \leq C \) times the \( m_\nu \) measure of that part of \( L \times \mathbb{R}_+ \) where \( v > \alpha / C \). So by Theorem 3.1, \( I \leq C \alpha^{-1} \) which is (4.2).

For \( r > 1 \), we get

\( I \leq \int_{D''} t^{r-1} e^{\sigma(H_0)} d\tilde{m} dt dH_0 = \int dH_0 \int d\tilde{m} dt \ldots \).

As in the proof of Theorem 3.4, we may neglect the subset \( E \) of \( L \times a_+ \) where \( t = |H| > C \log_\alpha \alpha \), either because \( v \) is small in \( E \) or because the measure of \( E \) is small. This means that \( t^{r-1} \) can be estimated by \( C \log_{**}^{-1} \alpha \) in (4.3). Hence, the inner integral in (4.3) is \( O(\alpha^{-1} \log_{**}^{-1} \alpha) \), uniformly in \( H_0 \), and (4.2) follows again.

To obtain (b2) in \( k(L)A_+ \), notice that we may also neglect the set where \( t < (\log_\alpha \alpha) / C \) for similar reasons. But when \( t \sim \log_\alpha \alpha \), the factor \( 1 + |H| \) behaves
like a constant and \( (b_3) \) follows from \( (b_1) \). Finally, \( (b_3) \) is a consequence of Theorem 3.1 in a similar way.

To complete the proof of \((a) \Rightarrow (b_j)\), we must get rid of \( L \). Since \( k(\mathcal{N})M \) is open and dense in \( K/M \), it is easy to find a compact set \( L \) and finitely many points \( k_1, \ldots, k_n \) so that the sets \( k_j k(L)M \) together cover \( K/M \) and their intersection is a neighborhood \( U \) of \( eM \). Decomposing a given measure \( \mu \) in \( K/M \) into parts carried by the \( k_j k(L)M \), we see that \( u = P_x \mu \) satisfies \( (b_j) \) in \( UA_+ \). Hence by translation, \( (b_j) \) holds in all of \( X \), \( j = 1, 2, 3 \).

\( (b_j) \Rightarrow (a) \). Assume \( u \in \mathcal{E}_A \) satisfies some \( (b_j) \) and that the associated quasinorm is at most 1. We start with a crude preliminary estimate.

**Lemma 4.3.** \( |u| \leq Ce^{Cq} \) in \( X \).

**Proof.** Because of the mean value theorem (see Helgason [3, p. 438]), we have for any \( g \in G \) and \( x \in X \)

\[
\int_K u(gkx) \, dk = \lambda_x u(g),
\]

where \( \lambda_x \to 1 \) as \( x \to o \) and \( dk \) is the normalized Haar measure in \( K \). The use of the mean value theorem at this point was suggested by T. Rychener. Let \( B_R \) denote the geodesic ball in \( X \) with center \( o \) and radius \( R \). Now integrate (4.4) with respect to \( dm(x) \) over \( B_R \), when \( R > 0 \) is small. We get

\[
|u(g)| \leq C m(B_R)^{-1} \int_K dk \int_{B_R} |u(gkx)| \, dm(x)
\]

\[
= C m(B_R)^{-1} \int_{B_R} |u(gx)| \, dm(x)
\]

because of the \( K \)-invariance of \( m \) and \( B_R \). Fix \( g \in G \). By Lemmas 2.1 and 2.2, the functions \( e^v, v \in \mathfrak{a}^* \), are approximately constant in \( gB_1 \), so \( dm_\sigma/dm_\beta \equiv e^{\sigma|v| - 2\sigma|u|} \) in \( gB_1 \). For some \( C \), the function \( v = e^{-Cq|u|} \) is in \( \Lambda_p^{-1} \), with a quasinorm \( < C \), when \( (b_1) \) or \( (b_2) \) is satisfied. In the \( (b_3) \) case, we replace \( \sigma \) by a slightly smaller \( \sigma' \), and reason in the same way. Clearly, \( v \sim e^{-Cq|u|} \) in \( gB_1 \). Now let \( v^\ast \) be the decreasing rearrangement of the restriction of \( v \) to \( gB_1 \) with respect to \( m \). Considering distribution functions with respect to \( m_\sigma \) and \( m \), we get

\[
v^\ast(t) \leq C (\beta t)^{-1/p} \log(\sigma^{-1})_p (\beta t).
\]

When \( p > 1 \), the lemma follows at once from (4.5–4.6) and (2.2), so assume
Let $s_j$ be the sup of $v$ in $gB_{1 - 2^{-j}}, j = 1, 2, \ldots$. Set $n = \dim X$, so that $m(B_R) \sim R^n$ for $R < 1$. For $x \in gB_{1 - 2^{-j}}$, (4.5) implies

$$v(x) \leq C2^{nj} \int_{xB_{2^{-j-1}}} v \, dm.$$ 

Now $xB_{2^{-j-1}} \subset gB_{1 - 2^{-j-1}}$, so $v \leq s_{j+1}$ there. Applying (4.6) and (2.2), we therefore have

$$v(x) \leq C2^{nj} \int_0^{C2^{-nj}} \min(s_{j+1}, (\beta t)^{-1} \log \beta^{-1}(\beta t)) \, dt$$

$$\leq C2^{nj} \beta^{-1} + C2^{nj} \int_{1/\beta s_{j+1}}^{C2^{-nj}} (\beta t)^{-1} \log \beta^{-1}(\beta t) \, dt.$$ 

Transforming $t \to t/\beta$ in the last integral, we see that

$$v(x) \leq C2^{nj} \beta^{-1} + C2^{nj} \beta^{-1} (\log \beta s_{j+1} + \log \beta).$$

It is possible to assume that all the $s_j$ are $\beta \pm 2$, so that

$$\log \beta s_{j+1} + \log \beta \sim \log \beta (\beta s_{j+1}),$$

since otherwise the lemma follows at once. Letting $x$ vary, we have proved

$$s_j \leq C2^{nj} \beta^{-1} + C2^{nj} \beta^{-1} \log \beta (\beta s_{j+1}).$$

It is elementary to see from this inequality that if $A > 0$ is large enough, and if the inequality

$$2^{-nj} \beta s_j > A2^j$$

holds for $j = 1$, then it holds for all $j$. But this would mean that $v$ is unbounded in $gB_1$, which is false. Hence, (4.7) cannot hold for $j = 1$, and this gives the desired estimate for $v(g)$ and $u(g)$. The lemma is proved.

Continuing the proof of (b) $\Rightarrow$ (a), we shall show that

$$\liminf_{H \to \infty} I_{\lambda}(H) < \infty,$$

where $I_{\lambda}(H) = \int_{K/M} \frac{|u(k \exp H)|^p}{|\varphi_{\lambda}(\exp H)|} \, dk \, M.$

If (b) is satisfied, take a compact set $S' \subset S$. For $T > 1$, clearly

$$\int_1^T t^{-1} dt \int_{S'} I_{\lambda}(tH_0) \, dH_0 = \int_{D_T} |e^{-\sigma(H)/p}u/\varphi_{\lambda}|^p e^{\sigma(H)} \, dk \, M \, dH$$

when $D_T = \{k \exp H \in X : k \in K, \ H \in R_+ S', \ 1 \leq |H| \leq T\}$. Notice that $e^{\sigma(H)}dkdH \leq Cdm_{\sigma}$ in $D_T$ and that $m_{\sigma}(D_T) \leq Ce^{CT}$. Lemma 4.3 and (b) give two estimates for $|e^{-\sigma(H)/p}u/\varphi_{\lambda}|^p$. From (2.4) and (2.2), applied to $m_{\sigma}$, it follows that both sides of (4.9) are dominated by
\[ C \int_0^{e^{CT}} \min (e^{CT}, t^{-1} \log_*^{-1} t) \, dt \leq CT'. \]

But then necessarily
\[
\lim \inf_{t \to \infty} \int_{S'} I_{\lambda'}(tH_0) \, dH_0 < \infty,
\]
so \( \lim \inf_{t \to \infty} I_{\lambda'}(tH_0) < \infty \) for some \( H_0 \in S' \) by Fatou's lemma. From this (4.8) follows, since \( |\varphi_{\lambda}| \) and \( \varphi_{\lambda'} \) have the same asymptotic behavior on a ray \( \{tH_0\} \), as proved by Harish–Chandra [1, p. 291].

When \( u \) satisfies \( (b_2) \), we write instead
\[
\int dt \int_{S'} I_{\lambda'}(tH_0) \, dH_0 = \int [H]^{-(r-1)/p} e^{-\sigma(H)/p} u/\varphi_{\lambda'} |\rho_e^{\sigma(H)} \, dk \, M \, dH,
\]
where the integrals are taken over the same sets as before. Then this is estimated by \( O(T) \) in the same way. The details, as well as the \( (b_3) \) case, are left to the reader.

Finally, we must show that (4.8) yields the representation of \( u \) as a Poisson integral. For each irreducible representation \( \delta \) of \( K \), let \( \alpha_\delta = d_\delta \bar{\alpha}_\delta \), where \( d \) denotes dimension, \( \chi \) character, and the bar complex conjugate. As in Helgason [4, p. 138], we expand \( u \) in
\[
u = \sum_\delta \alpha_\delta \ast u,
\]
where the convolution is performed in \( K \).

Harish–Chandra [2, Corollary 1, p. 13] has proved that this series converges in \( C^\infty (X) \). Now every \( \alpha_\delta \ast u \) is a \( K \)-finite function in \( \mathcal{S}_\delta \), so by [5, Corollary 7.4, p. 207], \( \alpha_\delta \ast u = P_\lambda f_\delta \) for some \( K \)-finite function \( f_\delta \) in \( K/M \). Because of (4.8), we may take a sequence \( H_j \to \infty \) for which \( u(k \exp H_j)/\varphi_\lambda(\exp H_j) \) converges weakly to a measure \( \mu \) in \( K/M \), and \( \mu \) is an \( L^p \) function if \( p > 1 \). Then
\[
\alpha_\delta \ast u(k \exp H_j)/\varphi_\lambda(\exp H_j) \to \alpha_\delta \ast \mu(kM), \quad j \to \infty,
\]
uniformly for \( k \in K \). Michelson [10, Theorem 1.3] has proved that \( P_\lambda f_\delta /\varphi_\lambda \to f_\delta \) as \( H \to \infty \), so we conclude \( f_\delta = \alpha_\delta \ast \mu \). Thus,
\[
u = \sum_\delta P_\lambda (\alpha_\delta \ast \mu),
\]
and it remains to prove that this last sum equals \( P_\lambda \mu \). And this follows from a direct calculation since \( P_\lambda (x, \cdot) \) is smooth and thus has a convergent \( \alpha_\delta \) expansion. Theorem 4.1 is completely proved.

**Remark 1.** As to the last part of this proof, cf. also the general representation theorem in [6].
Remark 2. In the case when $r = 1$ and $i \lambda \in a_+^*$, we sketch a proof that (4.8) implies the desired representation for $u$ which does not use any general representation theorem. Let the measure $\mu$ on the boundary be a weak* accumulation point of $u(\cdot \exp H)/\varphi_\lambda$ as $H \to \infty$, and regularize by convolving in $K$ by a smooth approximate identity $\psi_\varepsilon$. Then $\psi_\varepsilon \ast u(\cdot \exp H_j)/\varphi_\lambda$ will converge uniformly to $\psi_\varepsilon \ast \mu$ for some sequence $H_j \to \infty$. Now if $v \in \mathcal{E}_\lambda$, then $v/\varphi_\lambda$ must assume its maximum in the domain $\{k \exp H : H \in a_+, H < H_j\}$ on the boundary $K \exp H_j$. This follows from Hopf's maximum principle applied to $v/\varphi_\lambda$ and the operator $w \to \Delta(\varphi_\lambda w) - w \Delta \varphi_\lambda$, where $\Delta$ is the Laplacian of $X$. Applying this with $v = \pm (\psi_\varepsilon \ast u - P_\lambda(\psi_\varepsilon \ast \mu))$ and letting $j \to \infty$ gives $\psi_\varepsilon \ast u = P_\lambda(\psi_\varepsilon \ast \mu)$ and thus $u = P_\lambda \mu$.

5. Results for $\lambda = 0$.

Theorem 5.1. If $r = 1$, Theorem 4.1 holds when $\lambda = 0$. For $r > 1$ and $\lambda = 0$, let $\sigma$ and $p$ be as in Theorem 4.1, and assume $u \in \mathcal{E}_0$. Then $u$ has a representation as in condition (a) of Theorem 4.1 if and only if (b) holds in some (or every) restricted domain. Here $j$ is 1, 2, or 3.

We do not know whether Theorem 4.1 holds for $\lambda = 0$, $r > 1$, although this seems plausible in view of Theorem 3.4. However, conditions like

$$e^{-\sigma/p}u/e^{-q} \in L^{q/p+1}_{p,\sigma}$$

also characterize the Poisson integrals of $L^p$ functions or measures for $\lambda = 0$. The proof of this is left to the reader.

In the preceding section, we already used a convergence result of type $P_\lambda f/\varphi_\lambda \to f$ at the boundary, for $\text{Re} i \lambda \in a_+$. Michelson [10] obtains such results by proving that $P_\lambda(\exp H, kM)/\varphi_\lambda(\exp H)$ is an approximate identity in $K/M$ as $H \to \infty$. Since this expression has integral 1 and bounded $L^1$ norm in $K/M$, it defines an approximate identity if and only if its $L^1$ norm in $K/M \setminus U$ tends to 0 as $H \to \infty$ for any neighborhood $U$ of $eM$ in $K/M$. Whether this is true for $\lambda = 0$ and $r > 1$ seems to be unknown. The following weaker result will be needed in the proof of Theorem 5.1.

Theorem 5.2. Let $H_0 \in S$ and $\varepsilon > 0$, and set $h_t = \exp tH_0$. There exists a Lebesgue measurable set $F \subset R_+$ such that for any $T > 1$ the measure of $F \cap [T, 2T]$ is larger than $(1 - \varepsilon)T$ and such that for any neighborhood $U$ of $eM$ in $K/M$

$$\frac{1}{\varphi_0(h_t)} \int_{K/M \setminus U} P_0(h_t, kM) dkM \to 0 \quad \text{as } t \to \infty, \; t \in F.$$
PROOF. As usual, we transform the integral to $\bar{N}$. Assume $U = k(B)M$, for a compact neighborhood $B$ of $e \in \bar{N}$. Writing $B_t$ as in Section 3, we have

$$ I(B_t) = e^{q(tH_0)} \int_{K/M \setminus U} P_0(h_t, kM) \, dkM $$

$$ = \int_{\bar{N} \setminus B} e^{-q(H(\bar{\eta}))) + t - q(H(\bar{\eta}))} \, d\bar{\eta} . $$

Now

$$ \bar{N} \setminus B = \bigcup_{j=1}^{\infty} (B_{-j} \setminus B_{-j+1}) , $$

and

$$ e^{-q(H(\bar{\eta}))} \leq Ce^{-\delta j} \quad \text{for } \bar{\eta} \notin B_{-j+1} \text{ and some } \delta > 0 , $$

by [7, Proposition 5.5]. Thus,

$$ I(B_t) \leq C \sum_{j=1}^{\infty} e^{-\delta j} \int_{B_{-j} \setminus B_{-j+1}} e^{-q(H(\bar{\eta})) + t} \, d\bar{\eta} . \quad (5.2) $$

Notice that the quantity in (5.1) is $I(B_t)/\psi(tH_0)$ and that $\psi(tH_0) \sim t^q$, $t \to \infty$. We must thus determine $F$ so that $I(B_t) = o(t^q)$, $t \to \infty$, $t \in F$.

In the terms with $j > t$ in (5.2), we transform $\bar{\eta} \to \bar{\eta}_{-j}$, getting

$$ \sum_{j > t} \ldots \leq C \sum_{j > t} e^{-\delta j} \int_{B \setminus B_1} e^{-q(H(\bar{\eta}_{-j})) + j + t} \, d\bar{\eta}_1 . $$

Since $H(\bar{\eta})$ is bounded in $B \setminus B_1$, the integral in the last sum is dominated by

$$ C \int_{\bar{N}} e^{-q(H(\bar{\eta}_{-j})) + j + t - q(H(\bar{\eta}))} \, d\bar{\eta} = C\psi((j + t)H_0) \leq C(j + t)^q . $$

Hence,

$$ \sum_{j > t} \ldots \leq C \sum_{j > t} e^{-\delta j (j + t)^q} \to 0 \quad \text{as } t \to \infty . $$

As to the other terms in (5.2), we have

$$ \sum_{t=1}^{T} \sum_{j=1}^{t} e^{-\delta j} \int_{B_{-j} \setminus B_{-j+1}} e^{-q(H(\bar{\eta})) + t} \, d\bar{\eta} \leq \sum_{j=1}^{T} e^{-\delta j} \sum_{t=j}^{T} \int_{B_{-j-1} \setminus B_{-j-1+1}} e^{-q(H(\bar{\eta}))} \, d\bar{\eta} $$

$$ \leq \sum_{j=1}^{T} e^{-\delta j} \int_{B_{-2T}} e^{-q(H(\bar{\eta}))} \, d\bar{\eta} , $$

since $B_{-j-t} \setminus B_{-j-t+1}$ are disjoint for distinct $t$, fixed $j$. Transforming $\bar{\eta} \to \bar{\eta} \cdot \bar{\eta}$, this.
\( \bar{n} - 2T \), one can estimate the last integral by \( \psi(2TH_0) \leq CT^q \) as before, and so the last sum is also \( O(T^q) \).

Altogether then, we conclude
\[
\sum_{t=1}^{T} I(B,t) \leq CT^q.
\]

Because of Lemma 2.2, this estimate remains valid if we replace summation in \( t \) by integration \( dt \). If we let \( F = \{ t : I(B,t) \leq C t^{q - 1} \} \) and choose \( C \) large enough, it is clear that \( I(B,t) = o(t^q) \) in \( F \) and that \( F \) is as dense at \( \infty \) as claimed.

Finally, to find an \( F \) which works for all \( U \) simultaneously, we repeat this construction as \( U \) describes a neighborhood basis at \( eM \), choosing the values of \( \varepsilon \) suitably. The proof of Theorem 5.2 is complete.

Notice that we actually proved that
\[
\int_0^T \psi(tH_0)\varphi_0(h_\tau)^{-1}P_0(h_\tau, km)dt \int_0^T \psi(tH_0)dt
\]
is an approximate identity as \( T \rightarrow \infty \). Since this makes it possible to reconstruct \( f \) from \( P_0f \), we incidentally also get a proof of the fact that the value \( \lambda = 0 \) is simple without using the general criterion of Helgason [5, Theorem 6.1].

**Proof of Theorem 5.1.** We only indicate at which points this proof differs from that of Theorem 4.1, leaving the details to the reader. Assume first \( u = P_0\mu, \mu \geq 0 \) a measure, and take \( \alpha > 0 \). As before, we need only care about the region where \( |H| \sim \log^\alpha \phi \). If, further, \( H \) is in a restricted cone, we know that \( \psi(H) \sim |H|^q \sim \log^\alpha \phi \). Now the (b) conditions are proved as in Section 4, by means of Theorem 3.4 instead of Theorem 3.1.

Conversely, let \( u \) satisfy (b), say, in the restricted domain corresponding to \( S' \subset S \). As in the deduction of (4.8), we have
\[
\int_{\bar{n}}^{T} t^{-1} dt \int_{S'} I_0(tH_0) dH_0 \leq CT^r.
\]
This implies that \( I_0(tH_0) \leq C \) in "most of" the set \( \{(t,H_0) : T \leq t \leq 2T, H_0 \in S'\} \) for every large \( T \) and some \( C \). But then one can find an \( H_0 \in S' \) for which the same inequality holds for most \( t \) in \([2^j, 2^{j+1}]\) for infinitely many values of \( j \). Hence, there is a sequence \( t_j \rightarrow \infty \) contained in the set \( F \) of Theorem 5.2 and such that \( I(t_jH_0) \) is bounded as \( j \rightarrow \infty \). This is all we need to apply the reasoning at the end of the proof of Theorem 4.1, and the proof is complete.
6. The case when $\sigma$ is between 0 and $2\varrho$.

We say that the maximum theorem holds for a $p>1$ and a $\lambda$, $\Re i\lambda \in \mathfrak{a}_+^*$ or $\lambda=0$, if
\[
u^*(kM) \equiv \sup \{|u(k \exp H)|/\varphi_\lambda(\exp H) : H \in \mathfrak{a}_+\} \in L^p(K/M)
\]
whenever $u = P_\lambda f$ and $f \in L^p(K/M)$. This is true for all such $p$ and $\lambda$ when $r=1$ (see Michelson [10, Sec. 3]). For $r>1$, the maximum theorem holds for $p$ large enough, at least when $i\lambda=\varrho$ (see Lindahl [8]). The following result generalizes a theorem of Lohoué and Rychener [9, Proposition 1].

**Theorem 6.1.** Let $p>1$ and $\Re i\lambda \in \mathfrak{a}_+^*$ or $\lambda=0$, and assume $\sigma \in +\mathfrak{a}_+^* \cup (-\mathfrak{a}_+^*)$. If the maximal theorem holds for these $p$ and $\lambda$, then conditions (a), (b$_2$), and (b$_3$) are equivalent.

To prove (a) $\Rightarrow$ (b$_2$), one estimates $u$ by means of $\nu^*$. The details are left to the reader (see also [9]). For the converse implications, the corresponding proofs given in Sections 4 and 5 carry over without change.

However, (a) does not in general imply (b$_1$) under the hypotheses of Theorem 6.1. To get a counterexample, consider a bi-disk $U^2$, $U$ being the noneuclidean unit disk, and write each coordinate $z_i \in U$ as $(r_i \cos \theta_i, r_i \sin \theta_i)$, $-\pi < \theta_i \leq \pi$, $i=1,2$. Then $dm_\sigma$ is essentially the product of the measures $r_i(1-r_i)^{-1-s_i}dr_id\theta_i$, $i=1,2$, and we let $0<s_i<1$, which means choosing $\sigma$ strictly "between" 0 and $2\varrho$. Given $p \geq 1$ and $\varepsilon>0$, choose
\[
\nu(\theta_1, \theta_2) = \nu(\theta_1) = |\theta_1|^{-1/p} \log_*^{-1+(1+\varepsilon)/p}|\theta_1|,
\]
which is an $L^p$ function on the boundary $\partial U \times \partial U$. If
\[
v(z_1, z_2) = (1-r_1)^{s_1/p}(1-r_2)^{s_2/p}P_\lambda f/\varphi_\lambda
\]
and $i\lambda \in \mathfrak{a}_+^*$, it is easily seen that
\[
v \geq (1-r_1)^{s_1/p}(1-r_2)^{s_2/p}f(\max (|\theta_1|, 1-r_1))/C.
\]
Let $0<\varepsilon' < 1-s_1$ and $\alpha > 0$. Suppose
\[
(1-r_1)^{1-s_1-\varepsilon'} < (1-r_2)^{s_2}\alpha^{-p} < 1
\]
so that
\[
\log_* (1-r_1)^{s_1}(1-r_2)^{s_2}\alpha^{-p} \sim \log_* (1-r_1).
\]
If in addition
\[
1-r_1 < |\theta_1| < (1-r_1)^{s_1}(1-r_2)^{s_2}\alpha^{-p}\log_*^{-1-\varepsilon}(1-r_1)/C,
\]
it follows from (6.1) that \( v > \alpha \). For \( r_2 \) fixed, we integrate \( r_1 (1 - r_1)^{-1 - s_1} dr_1 d\theta_1 \) over the set of \( (r_1, \theta_1) \) defined by (6.2) and (6.3), getting at least
\[
(1 - r_2)^{2\alpha - p} \log^\varepsilon (1 - r_2)^{s_2 \alpha - p})/C.
\]
Integrating now in \( r_2 \) and \( \theta_2 \), we see that \( m_\sigma \{ v > \alpha \} = \infty \), so that \( v \notin A^1_{p, \sigma} \).

Next, we give examples showing that Theorem 6.1 is false for \( p = 1 \) and \( \alpha \) "between" 0 and 2\( \theta \). When \( \sigma = 0 \), the function \( u = P_\alpha = \varphi_\alpha \) does not satisfy any (b). For other \( \sigma \), we consider only the ordinary Poisson kernel \( P \) in the unit disk, or, more conveniently, the upper half-plane \( \mathbb{R}_+^2 = \{(x, t) : t > 0\} \). We choose measures in \( 0 \leq x \leq 1 \) and estimate their Poisson integrals in \( \mathbb{R}_+^2 \) near this interval. If \( \sigma = s \cdot 2\theta \), \( 0 < s < 1 \), we have \( e^{-\sigma} \sim t^s \) and \( dm_\sigma \sim t^{-s-1} dx dt \) here. For \( s = 1 \), \( \sigma = 2\theta \), consider the Dirac measure \( \delta_\theta \). It is easily verified that \( tP\delta_\theta \sim t^2/ (x^2 + t^2) \) is not in \( A_{1, \sigma} \). And when \( 0 < s < 1 \), we use measures of Cantor type, carried by Cantor sets of ration \( 2^{-x} \), \( x = 1/(1-s) > 1 \), constructed as follows. Choose two 1st step intervals \([0, 2^{-x}) \) and \([1 - 2^{-x}, 1] \), thus situated at the ends of \([0, 1] \), and then four 2nd step intervals, each of length \( 2^{-2x} \), at the ends of the two 1st step intervals. Continuing in this way, we get at the \( n \)th step \( 2^n \) intervals of length \( 2^{-nx} \). There exists a measure \( \mu \) such that each of these \( n \)th step intervals has measure \( 2^{-n} \). It is easily verified that at points \((x, t)\) with \( t \sim 2^{-nx} \) and \( x \) in an \( n \)th step interval, we have \( t^s P\mu(x, t) \sim 1 \). Hence, \( m_\sigma \{ t^s P\mu(x, t) \sim 1 \} = \infty \), and we are done.

REFERENCES

