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FINITELY GENERATED Ext ALGEBRAS

GERSON LEVIN

Introduction.

It was conjectured by the author in [6] that the algebra $\mathrm{Ext}_R(k,k)$ of a local ring R is finitely generated.

This was demonstrated for several classes of local rings in [7]. However, the conjecture is false. J.-E. Roos [13] has given an example of a local ring R such that $\mathrm{Ext}_R(k,k)$ is not finitely generated.

In this paper we ask the following question. If $S \to R$ is a homomorphism of local rings and $\mathrm{Ext}_S(k,k)$ is finitely generated, under what conditions is $\mathrm{Ext}_R(k,k)$ finitely generated?

If $S \to R$ is a Golod homomorphism [8], it turns out that it is sufficient that $\mathrm{Ext}_S(R,k)$ be a finitely generated left $\mathrm{Ext}_S(k,k)$-module. (Theorem 4.3). Using this criterion, it will be shown in the following cases, that if $\mathrm{Ext}_S(k,k)$ is finitely generated, so is $\mathrm{Ext}_R(k,k)$.

a) $R=S/(x)$ where x is a non-zero divisor in $\mathfrak{m}^2$.
b) $R=S/(0:\mathfrak{m})$ where S is a Gorenstein ring.
c) $R=S/\mathfrak{m}^n$ for sufficiently large n.
d) S is a Golod ring and $S \to R$ is a Golod homomorphism.
e) S is a complete intersection and $S \to R$ is a Golod homomorphism.
f) S is a complete intersection and $S \to A$ and $A \to R$ are Golod homomorphisms.

For what local rings R does $\mathrm{Ext}_R(k,k)$ have finite global dimension?

Roos has shown that $\mathrm{Ext}_R(k,k)$ has global dimension one if and only if $\mathfrak{m}^2=0$. Furthermore, in his counter-example, $\mathrm{Ext}_R(k,k)$ has global dimension three and $\mathfrak{m}^3=0$. In Theorem 5.10 we show that if $\mathrm{Ext}_R(k,k)$ has finite global dimension, then R is artinian.

Also included is a section (section 7) on how Roos' example can be used to give a counter-example to another question in the homology of local rings.

NOTATION. We will abbreviate Hom_R and $\otimes_R$ as just Hom and $\otimes$. If A is a graded R-module, $\bar{A}$ denotes the elements of positive degree. If A is an augmented R-algebra with augmentation $\varepsilon\colon A \to k$,

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$$I(A) = \operatorname{Ker} \varepsilon$$
$$\tilde{Z}(A) = Z(A) \cap I(A)$$
$$\tilde{H}(A) = \tilde{Z}(A)/B(A) = \operatorname{Ker}(H(A) \xrightarrow{\varepsilon_*} k)$$

and

$$Q(A) = I(A)/(I(A))^2 ,$$

the indecomposable elements of A. As usual one defines the graded module

$$(sA)_i = A_{i-1}, \quad i \geqq 1$$
$$(sA)_0 = 0 .$$

1. The action of $\operatorname{Ext}_R(M, N)$ as natural transformations.

In this section, R may be any commutative ring and M, N, P any R-modules. Then $\operatorname{Ext}_R(M, N)$ acts as natual transformations

$$\operatorname{Tor}^R(M, -) \to \operatorname{Tor}^R(N, -) .$$

(In fact, Mehta [12] has shown that if R is a complete, local noetherian ring, then every such natural transformation has this form.)

The action is the following. We identify

$$\operatorname{Tor}^R(M, P) \cong H(X \otimes P)$$

where X, ε_M is a projective resolution of M. Let Y, ε_N be a projective resolution of N. Then an element $\sigma \in \operatorname{Ext}_R^n(M, N)$ is represented by a cocycle $f \in \operatorname{Hom}(X_n, N)$ and, by the comparison theorem, there is a mapping of complexes $F: X \to Y$ of degree $-n$ such that $\varepsilon_N F = f$. Define the left action

$$L_\sigma : \operatorname{Tor}_i^R(M, P) \to \operatorname{Tor}_{i-n}^R(N, P)$$

by

$$L_\sigma = (F \otimes 1_P)_* : H_i(X \otimes P) \to H_{i-n}(Y \otimes P) .$$

It can be checked that L_σ is independent of the choice of f and F and is natural in P. We will also define the right action of σ. Let T be the twist mapping on $X \otimes_R Y$, i.e.,

$$T(x \otimes y) = (-1)^{(\deg x)(\deg y)} y \otimes x .$$

Since the augmentation $\varepsilon_N : Y \to N$ induces an isomorphism

$$(1 \otimes \varepsilon_N)_* : H(X \otimes_R Y) \xrightarrow{\cong} H(X \otimes_R N)$$

we can define isomorphisms

$$C_{M,N}\colon \operatorname{Tor}^R(M,N) \xrightarrow{\cong} \operatorname{Tor}^R(N,M)$$

by

$$C_{M,N} = (1\otimes\varepsilon_M)_* T(1\otimes\varepsilon_N)_*^{-1}\,.$$

Now, for an element $\sigma \in \operatorname{Ext}_R^n(M,N)$ the right action

$$R_\sigma\colon \operatorname{Tor}_i^R(P,M) \to \operatorname{Tor}_{i-n}^R(P,N)$$

is defined by

$$R_\sigma = C_{N,P} L_\sigma C_{P,M}\,.$$

REMARK. If $F\colon A \to B$ and $G\colon C \to D$ are mappings of complexes, one defines

$$F\otimes G\colon A\otimes C \to B\otimes D$$

by

$$(F\otimes G)(a\otimes c) = (-1)^{(\deg F)(\deg c)} F(a)\otimes G(c)\,.$$

With this definition, $F\otimes G$ is also a mapping of complexes and

$$T(F\otimes G) = (G\otimes F)T\,.$$

THEOREM 1.1. *If $\sigma \in \operatorname{Ext}_R^n(M,N)$ is represented by a cocycle $f \in \operatorname{Hom}_R(X_n, N)$, then the left action $L_\sigma\colon \operatorname{Tor}^R(M,P) \to \operatorname{Tor}^R(N,P)$ is given by*

$$L_\sigma = C_{P,N}(1\otimes f)_* T(1\otimes\varepsilon_P)_*^{-1}\,.$$

PROOF. Let X, ε_M; Y, ε_N; and W, ε_P be projective resolutions of M, N and P, respectively, and, as above, let F be a mapping of complexes $F\colon X \to Y$ such that $\varepsilon_N F = f$. Then

$$\begin{aligned}
&C_{P,N}(1\otimes f)_* T(1\otimes\varepsilon_P)_*^{-1}\\
&\quad= (1\otimes\varepsilon_P)_* T(1\otimes\varepsilon_N)_*^{-1}(1\otimes\varepsilon_N F)_* T(1\otimes\varepsilon_P)_*^{-1}\\
&\quad= (1\otimes\varepsilon_P)_* T(1\otimes F)_* T(1\otimes\varepsilon_P)_*^{-1}\\
&\quad= (1\otimes\varepsilon_P)_* (F\otimes 1)_* (1\otimes\varepsilon_P)_*^{-1} \qquad \text{(by the above remark)}\\
&\quad= (F\otimes 1_P)_* = L_\sigma
\end{aligned}$$

by definition.

It is now easy to calculate the right action.

THEOREM 1.2. *If* $\sigma \in \mathrm{Ext}_R^n(M,N)$ *is represented by a cocycle* $f \in \mathrm{Hom}_R(X_n, N)$, *then the right action*

$$R_\sigma\colon \mathrm{Tor}^R(P,M) \to \mathrm{Tor}^R(P,N)$$

is given by

$$R_\sigma = (1\otimes f)_*(1\otimes\varepsilon_M)_*^{-1}\,.$$

PROOF. By definition,

$$\begin{aligned} R_\sigma &= C_{N,P}L_\sigma C_{P,M} \\ &= C_{N,P}C_{P,N}(1\otimes f)_*T(1\otimes\varepsilon_P)_*^{-1}C_{P,M} \\ &= (1\otimes f)_*(1\otimes\varepsilon_M)_*^{-1}\,. \end{aligned}$$

2. The Yoneda product.

For modules M,N,P there is a pairing

$$\varphi\;:\;\mathrm{Ext}_R(N,P)\otimes\mathrm{Ext}_R(M,N) \to \mathrm{Ext}_R(M,P)$$

called the Yoneda product and defined as follows. Let X,ε_M and Y,ε_N be projective resolutions of M and N, respectively, and let $\sigma \in \mathrm{Ext}_R(N,P)$ and $\tau \in \mathrm{Ext}_R(M,N)$ be represented by cocycles $f \in \mathrm{Hom}(Y,P)$ and $g \in \mathrm{Hom}(X,N)$, respectively. Let G be a mapping of complexes $G\colon X \to Y$ such that $\varepsilon_N G = g$. Then define

$$\varphi(\sigma\otimes\tau) = \mathrm{cls}\,(fG)\,.$$

Note that if W,ε_P is a projective resolution of P and F is a mapping of complexes $F\colon Y \to W$ such that $\varepsilon_P F = f$, then

$$F\circ G\colon X \to W$$

satisfies $\varepsilon_P(F\circ G) = fG$ so that

$$L_{\varphi(\sigma\otimes\tau)} = L_\sigma\circ L_\tau\,.$$

If we set $P=N=M$, the Yoneda product makes $\mathrm{Ext}_R(N,N)$ an associative algebra, and with just $P=N$, it makes $\mathrm{Ext}_R(M,N)$ a left $\mathrm{Ext}_R(N,N)$-module.

From this point on, R will be a noetherian, local ring with maximal ideal $\mathfrak{m}$ and residue field $k=R/\mathfrak{m}$. All modules are assumed to be finitely generated.

As defined by Assmus [2], $\mathrm{Tor}_R(k,k)$ has the structure of a Hopf algebra and $\mathrm{Tor}_R(M,k)$ that of a right $\mathrm{Tor}_R(k,k)$-co-module. In both cases, the product is the usual ψ-product of [3] and the co-product may be defined as follows. Let X,ε_M and Y,ε_k be free resolutions of M and k respectively and let

$$h\colon \operatorname{Tor}_R(M,k) \to \operatorname{Tor}_R(M,k)\otimes\operatorname{Tor}_R(k,k)$$

be the composite

$$\begin{aligned}\operatorname{Tor}_R(M,k) \cong H(X\otimes k) &\xrightarrow{(1\otimes\varepsilon_k)_*^{-1}} H(X\otimes Y) \to H(X\otimes k\otimes Y\otimes k) \to \\ &\xrightarrow{\alpha^{-1}} H(X\otimes k)\otimes H(Y\otimes k) \\ &\cong \operatorname{Tor}_R(M,k)\otimes\operatorname{Tor}_R(k,k)\end{aligned}$$

where α is the Künneth isomorphism which is induced by sending cycles $z_1 \in X\otimes k$ and $z_2 \in Y\otimes k$ into the cycle $z_1\otimes z_2$ in $X\otimes k\otimes Y\otimes k$. The coproduct is then

$$\Delta = (1\otimes C)h$$

where

$$C = C_{k,k}\colon \operatorname{Tor}_R(k,k) \to \operatorname{Tor}_R(k,k)\,.$$

It can be shown that C is the conjugation on the Hopf algebra $\operatorname{Tor}_R(k,k)$. In particular, for $M=k$, one obtains the coproduct on $\operatorname{Tor}_R(k,k)$. Let ε denote the augmentation of the Hopf algebra $\varepsilon\colon \operatorname{Tor}_R(k,k) \to k$. Then the mapping, which sends $\sigma \in \operatorname{Ext}_R(M,k)$ into εL_σ, gives an isomorphism

$$\operatorname{Ext}_R(M,k) \cong \operatorname{Hom}(\operatorname{Tor}^R(M,k),k)\,.$$

Naturally, one wants to know how the dual of the coproduct on $\operatorname{Tor}^R(M,k)$ compares with the Yoneda product on $\operatorname{Ext}_R(M,k)$.

THEOREM 2.1. *The coproduct*

$$\Delta\colon \operatorname{Tor}^R(M,k) \to \operatorname{Tor}^R(M,k)\otimes\operatorname{Tor}^R(k,k)$$

is dual to the opposite of the Yoneda product

$$\varphi\colon \operatorname{Ext}_R(k,k)\otimes\operatorname{Ext}_R(M,k) \to \operatorname{Ext}_R(M,k)$$

i.e.,

$$\Delta^* = \varphi T\,.$$

PROOF. It must be shown that

$$\Delta^*(\sigma\otimes\tau) = (-1)^{mn}\varphi(\tau\otimes\sigma)$$

where m and n are the degrees of σ and τ or i.e., that

$$(\varepsilon L_\sigma\otimes\varepsilon L_\tau)\Delta = (-1)^{mn}\varepsilon L_{\varphi(\tau\otimes\sigma)}\,.$$

Let f be a cocycle in $\operatorname{Hom}(X,k)$ representing σ. Then

$$
\begin{aligned}
(\varepsilon L_\sigma \otimes \varepsilon L_\tau)\Delta &= (\varepsilon L_\sigma \otimes \varepsilon L_\tau)(1 \otimes C)h \\
&= \varepsilon L_\tau C(\varepsilon L_\sigma \otimes 1)h \\
&= (-1)^{mn} \varepsilon L_\tau C(1 \otimes \varepsilon L_\sigma)Th \\
&= (-1)^{mn} \varepsilon L_\tau C(1 \otimes f)_* T(1 \otimes \varepsilon_k)_*^{-1} \\
&= (-1)^{mn} \varepsilon L_\tau L_\sigma \\
&= (-1)^{mn} L_{\varphi(\tau \otimes \sigma)} \qquad \text{by Theorem 1.1 .}
\end{aligned}
$$

3. Distinguished subalgebras of the minimal resolution.

Let Y, ε_k be a minimal algebra resolution of k. In [10], Löfwall considers differential graded subalgebras U of Y having the following properties,

a) Y is free as a graded U-module, i.e., there is a free graded R-module W with $W_0 = R$ and $Y \cong U \otimes W$ as graded U modules.

b) For any R-homomorphism $f: U \to k$, there is a mapping of complexes $F: U \to U$ such that $\varepsilon_k F = f$.

We will call any such subalgebra U of Y a distinguished subalgebra of the minimal resolution.

If U is a distinguished subalgebra of Y, then a Yoneda product can be defined on $\mathrm{Hom}\,(U, k)$ as in section 2, i.e., if $f, g \in \mathrm{Hom}\,(U, k)$, let $F: U \to U$ be a mapping of complexes with $\varepsilon_k F = f$ and define

$$\varphi(g \otimes f) = gF .$$

Löfwall shows (Proposition 2.3, p. 27 of [10]) that with this product, $\mathrm{Hom}\,(U, k)$ is a Hopf algebra and, in fact, there is an exact sequence of Hopf algebras

$$k \to \mathrm{Hom}\,(W, k) \to \mathrm{Hom}\,(Y, k) \to \mathrm{Hom}\,(U, k) \to k .$$

Here are some examples.

1) Let $S \to R$ be a small homomorphism of local rings. Following Avramov [2], a homomorphism of local rings with the same residue field k is *small* if the induced map

$$\mathrm{Tor}^S\,(k, k) \to \mathrm{Tor}^R\,(k, k)$$

is injective. Then if X is a minimal algebra resolution of k as an S-module, the algebra $U = R \otimes_S X$ is distinguished.

Avramov proves condition a) [2, Theorem 3.1, p. 24], and since

$$\mathrm{Hom}_R\,(U, k) = \mathrm{Hom}_S\,(X, k) \cong \mathrm{Ext}_S\,(k, k) ,$$

condition b) is just the comparison theorem. Clearly, the Yoneda product on $\mathrm{Hom}_R(U,k)$ coincides with the Yoneda product on $\mathrm{Ext}_S(k,k)$ defined in section 2.

2) The subalgebra $U=Y^r$ of Y obtained by adjoining to R all variables of degree $\leqq r$ is distinguished. Condition a) is immediate in this case. To check b), let $U=R\langle T_1,\ldots,T_s\rangle$. Using [6, Lemma 1.3.2, p. 16] it follows that each of the derivations J_i associated with the adjunction of T_i may be extended to a derivation on U. On the other hand, the augmentation $\varepsilon\colon U \to k$ induces a surjection

$$\varepsilon^*\colon R[J_1,\ldots,J_s] \to \mathrm{Hom}_R(U,k)$$

where $R[J_1,\ldots,J_s]$ denotes the free algebra on $J_1,\ldots,J_s$. This proves b).

3) Any differential graded, skew-commutative algebra satisfying

i) If $du \in \mathfrak{m}^2U$, then $u \in \mathfrak{m}U$ or $u \in U_0$;
ii) $\tilde{Z}(U) \subset \mathfrak{m}^2U+BU$

is a distinguished subalgebra of Y. [10, Proposition 2.3].

Let U be a distinguished subalgebra of Y, and M an R-module. Then we can define a right action of $\mathrm{Hom}(U,k)$ on $H(M\otimes U)$ as follows. For $f \in \mathrm{Hom}(U,k)$, let F be a mapping of complexes $F\colon U \to U$ such that $\varepsilon_k F=f$ and define

$$R'_f\colon H(M\otimes U) \to H(M\otimes U)$$

by

$$R'_f = (1\otimes F)_* .$$

With this action, $\mathrm{Hom}(H(M\otimes U),k)$ becomes a right $\mathrm{Hom}(U,k)$-module with multiplication

$$\psi\colon \mathrm{Hom}(H(M\otimes U),k)\otimes\mathrm{Hom}(U,k) \to \mathrm{Hom}(H(M\otimes U),k)$$

defined by

$$\psi(g\otimes f) = gR'_f .$$

If U is the distinguished subalgebra of example 1, then

$$\mathrm{Hom}(U,k) \cong \mathrm{Ext}_S(k,k)$$

and

$$\mathrm{Hom}(H(M\otimes U),k) \cong \mathrm{Ext}_S(M,k)$$

so we can compare ψ with the Yoneda product

$$\varphi\colon \mathrm{Ext}_S(k,k)\otimes\mathrm{Ext}_S(M,k) \to \mathrm{Ext}_S(M,k) .$$

THEOREM 3.1. Let $S \to R$ be a small homomorphism of local rings, and let M be an R-module. Suppose that X, ε_k and Y, ε_k are minimal resolutions of k over S and R respectively and that P, ε_M is a free resolution of M over S. Let U be the distinguished subalgebra

$$U = R \otimes_S X \subset Y$$

and α the isomorphism

$$\alpha\colon \operatorname{Tor}^S(M,k) = H(P \otimes_S k) \to H(M \otimes_S X) = H(M \otimes_R U)$$

defined by

$$\alpha = (\varepsilon_M \otimes 1)_*(1 \otimes \varepsilon_k)_*^{-1} .$$

Then the diagram

$$\begin{array}{ccc} \operatorname{Ext}_S(M,k) \otimes \operatorname{Ext}_S(k,k) & \xrightarrow{\varphi T(1 \otimes C^*)} & \operatorname{Ext}_S(M,k) \\ \Big\uparrow {\scriptstyle \alpha^* \otimes 1} & & \Big\uparrow {\scriptstyle \alpha^*} \\ \operatorname{Hom}(H(M \otimes U), k) \otimes \operatorname{Hom}(U,k) & \xrightarrow{\psi} & \operatorname{Hom}(H(M \otimes U), k) \end{array}$$

is commutative.

PROOF. Let $g \in \operatorname{Hom}(H(M \otimes U), k)$ and $f \in \operatorname{Hom}(U,k)$. We must show that

$$\alpha^*(\psi(g \otimes f)) = \varphi T(1 \otimes C^*)(\alpha^* g \otimes f)$$

or i.e.,

$$\psi(g \otimes f)\alpha = \varphi T(g\alpha \otimes fC) .$$

By definition

$$\psi(g \otimes f) = gR'_f$$

and one checks easily that

$$R'_f = \alpha R_f \alpha^{-1} = \alpha(1 \otimes f)_*(1 \otimes \varepsilon_k)_*^{-1}\alpha^{-1}$$

by Theorem 1.2. So

$$\begin{aligned} \psi(g \otimes f)\alpha &= g\alpha(1 \otimes f)_*(1 \otimes \varepsilon_k)_*^{-1} \\ &= (g\alpha \otimes f)h = (g\alpha \otimes f)(1 \otimes C)\Delta \\ &= (g\alpha \otimes fC)\Delta . \end{aligned}$$

But

$$(g\alpha \otimes fC)\Delta = \varphi T(g\alpha \otimes fC)$$

by Theorem 2.1.

4. Golod algebras.

Let Y, ε_k be a minimal algebra resolution of k. A distinguished subalgebra U of Y is called a Golod *algebra* if all the Massey products in $\tilde{H}(U)$ vanish. (See [8] for definitions). A small homomorphism $S \to R$ of local rings is a *Golod homomorphism* if $R \otimes_S X$ is a Golod algebra, where X is a minimal algebra resolution of k over S.

If U is a Golod algebra, then by [9, Theorem 13] there is a free R-module V such that as R-modules,

$$Y \cong U \otimes T(V)$$

and such that $d(V) \subset U$, inducing an isomorphism

$$\begin{array}{ccc} S\colon V \otimes k & \xrightarrow{\cong} & \tilde{H}(U) \\ \uparrow & & \uparrow \\ V & \xrightarrow{d} & \tilde{Z}(U) \end{array} \tag{4.1}$$

It was proved in [2, Theorem 3.5] and in [10, Corollary 2.4] that there is an exact sequence of Hopf algebras

$$k \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to k$$

where as algebras $\mathscr{A} \cong T(s\operatorname{Hom}(\tilde{H}(U), k))$, the free algebra on $s\operatorname{Hom}(\tilde{H}(U), k)$, $\mathscr{B} \cong \operatorname{Ext}_R(k,k)$ with the Yoneda product and $\mathscr{C} \cong \operatorname{Hom}(U,k)$ with the Yoneda product. Since $\mathscr{B}$ is a Hopf algebra, $\mathscr{B}$ is free as a right $\mathscr{A}$-module.

Associated with this sequence is a spectral sequence [3, Theorem 6.1, p. 349]

$$\operatorname{Tor}^{\mathscr{C}}_{p,*}(\operatorname{Tor}^{\mathscr{A}}_{q,*}(k,k),k) \underset{p}{\Longrightarrow} \operatorname{Tor}^{\mathscr{B}}_{p+q,*}(k,k)\,.$$

Since $\mathscr{A}$ is a free algebra,

$$\operatorname{Tor}^{\mathscr{A}}_{q,*}(k,k) = 0 \quad \text{for } q \neq 0, 1\,,$$

so the spectral sequence degenerates into a long exact sequence

$$\begin{aligned} \ldots \to \operatorname{Tor}^{\mathscr{C}}_{p,*}(k,k) &\to \operatorname{Tor}^{\mathscr{C}}_{p-2,*}(\operatorname{Tor}^{\mathscr{A}}_{1,*}(k,k),k) \to \\ &\to \operatorname{Tor}^{\mathscr{B}}_{p-1,*}(k,k) \to \operatorname{Tor}^{\mathscr{C}}_{p-1,*}(k,k) \to \ldots \end{aligned} \tag{4.2}$$

In particular, for $p=2$, one obtains the exact sequence

$$\operatorname{Tor}^{\mathscr{C}}_{2,*}(k,k) \to Q\mathscr{A} \otimes_{\mathscr{C}} k \to Q\mathscr{B} \to Q\mathscr{C} \to 0\,. \tag{4.3}$$

We now take a closer look at the structure of $Q\mathscr{A}$ as a right $\mathscr{C}$-module. Since $\mathscr{A}$ is a sub-Hopf algebra of $\mathscr{B}$, $\mathscr{B}$ is free as a right $\mathscr{A}$-module and there are isomorphisms

$$\operatorname{Tor}^{\mathscr{A}}_{1,*}(k,k) \to \operatorname{Tor}^{\mathscr{B}}_{1,*}(k, \mathscr{B} \otimes_{\mathscr{A}} k) = \operatorname{Tor}^{\mathscr{B}}_{1,*}(k, \mathscr{C})\,.$$

Since $\mathrm{Tor}_{1,*}^{\mathscr{B}}(k,\mathscr{C})$ is naturally a right $\mathscr{C}$-module, $Q\mathscr{A}$ becomes a right $\mathscr{C}$-module via the diagram

$$\begin{array}{ccc} Q\mathscr{A} = I\mathscr{A}/(I\mathscr{A})^2 & \xrightarrow{\cong} & \mathscr{B}(I\mathscr{A})/(I\mathscr{B})(I\mathscr{A}) \\ \downarrow\cong & & \downarrow\cong \\ \mathrm{Tor}_{1,*}^{\mathscr{A}}(k,k) & \xrightarrow{\cong} & \mathrm{Tor}_{1,*}^{\mathscr{B}}(k,\mathscr{C}) \end{array}$$

Since $\mathscr{A}\cong T(s\,\mathrm{Hom}\,(\tilde{H}(U),k))$, there is an isomorphism of graded vector spaces

$$\sigma\colon \mathrm{Hom}\,(\tilde{H}(U),k) \to Q\mathscr{A}$$

defined as follows. Let p_U and p_V denote respectively the projections $Y\to U$ and $Y\to V$. Then it is easy to see that

$$p_V^*(\mathrm{Hom}\,(V,k)) \subset I\mathscr{A}$$

inducing an isomorphism

$$\tau\colon \mathrm{Hom}\,(V,k) \to Q\mathscr{A}$$

and σ is the composite

$$\mathrm{Hom}\,(\tilde{H}(U),k) \xrightarrow{S^*} \mathrm{Hom}\,(V,k) \xrightarrow{\tau} Q\mathscr{A}\ .$$

We compute the multiplication

$$\gamma\colon Q\mathscr{A}\otimes\mathscr{C} \to Q\mathscr{A}$$

by computing $\gamma(\tau g\otimes f)$, for $g\in \mathrm{Hom}\,(V,k)$ and $f\in \mathrm{Hom}\,(U,k)$. In [14], Sjödin gives an explicit formula for lifting elements $g\in\mathrm{Hom}\,(V,k)$ to mappings of complexes $G\colon Y\to Y$. Namely, let $\tilde{g}\colon V\to R$ lift g and then for $x\in V$, $y\in Y$, $u\in U$, define

$$G(y\otimes x) = (-1)^{(\deg y)(\deg x)}y\tilde{g}(x), \qquad \text{and} \qquad G(u) = 0\ .$$

Then G is a mapping of complexes $Y=U\oplus(Y\otimes V)\to Y$ lifting g. If $v\in V$, $G(v)\in R$ or $G(v)=0$. Hence for $f'\in I\mathscr{B}$ and $g\in\mathrm{Hom}\,(V,k)$, $\varphi(f'\otimes\tau g)=f'G$, and thus all elements of $(I\mathscr{B})(I\mathscr{A})$ must vanish on V. Then the restriction $\varrho\colon \mathrm{Hom}\,(Y,k)\to\mathrm{Hom}\,(U,k)$ induces a map

$$\mathscr{B}I\mathscr{A}/(I\mathscr{B})I\mathscr{A} \to \mathrm{Hom}\,(V,k)$$

which makes the diagram

$$\begin{array}{ccc} \mathrm{Hom}\,(V,k) & \xrightarrow{\tau} & Q\mathscr{A} \\ & \nwarrow & \downarrow\cong \\ & & \mathscr{B}I\mathscr{A}/(I\mathscr{B})(I\mathscr{A}) \end{array}$$

commute. Hence

$$\gamma(\tau g \otimes f) = \tau\varrho\varphi(p_{\tilde{V}}^* g \otimes p_{\tilde{U}}^* f) .$$

Now, recall from section 3 that Hom $(\tilde{H}(U), k)$ is also a right $\mathscr{C}$-module via the multiplication ψ.

THEOREM 4.1. *The isomorphism*

$$\sigma\colon \mathrm{Hom}\,(\tilde{H}(U), k) \to Q\mathscr{A}$$

is an isomorphism of right $\mathscr{C}$-modules.

PROOF. We will show that

$$S^*\psi = \varrho\varphi(p_{\tilde{V}}^* S^* \otimes p_{\tilde{U}}^*)$$

from which it follows that

$$\sigma\psi = \tau S^*\psi = \tau\varrho\varphi(p_{\tilde{V}}^* S^* \otimes p_{\tilde{U}}^*) = \gamma(\tau S^* \otimes 1) = \gamma(\sigma \otimes 1)$$

as desired.

Let $g \in$ Hom $(\tilde{H}(U), k)$ and $f \in$ Hom (U, k). Sjödin's argument in [14] can be modified to show that there is a mapping of complexes $F\colon Y \to Y$ such that $\varepsilon F = f$, $F(U) \subset U$ and $F(V) \subset U \oplus V$. Namely since U is a Golod algebra, there is a mapping of complexes $F\colon U \to U$ such that $\varepsilon F = f$. Then because of the isomorphism S; $V \otimes k \to \tilde{H}(U)$, F can be extended to $F\colon Y \to Y$ such that $F(V) \subset U \oplus V$.

Let $v \in V$ and put $F(v) = u + v'$, $u \in U$, $v' \in V$. Then

$$\varphi(p_{\tilde{V}}^* S^* g \otimes p_{\tilde{U}}^* f)(v) = gSp_V F(v) = gS(v') .$$

On the other hand

$$(S^*\psi(g \otimes f))(v) = gF_* S(v)$$

so to finish the proof we need only note that the diagram

$$\begin{array}{ccc} V & \xrightarrow{F} & U \oplus V \\ \downarrow d & & \downarrow d \\ \tilde{Z}(U) & \xrightarrow{F} & \tilde{Z}(U) \\ \downarrow & & \downarrow \\ \tilde{H}(U) & \xrightarrow{F_*} & \tilde{H}(U) \end{array}$$

commutes because F is a mapping of complexes.

THEOREM 4.2. *Let U be a distinguished Golod algebra. Then* $\mathrm{Ext}_R(k, k)$ *is a finitely generated algebra if* Hom (U, k) *is a finitely generated algebra and* Hom $(\tilde{H}(U), k)$ *is a finitely generated right* Hom (U, k)*-module. Furthermore if*

$\mathrm{Hom}\,(U, k)$ is finitely generated and finitely presented, then $\mathrm{Ext}_R\,(k, k)$ is a finitely generated algebra if and only if $\mathrm{Hom}\,(\tilde{H}(U), k)$ is a finitely generated $\mathrm{Hom}\,(U, k)$-module.

PROOF. The proof is now immediate from exact sequence (4.3) and Theorem 4.1.

We can apply this to both of the examples of distinguished subalgebras given in section 3. It will be applied to Example 2 in section 7.

THEOREM 4.3. *Let $S \to R$ be a Golod homomorphism. Then $\mathrm{Ext}_R\,(k, k)$ is a finitely generated algebra if $\mathrm{Ext}_S\,(k, k)$ is a finitely generated algebra and if $\mathrm{Ext}_S\,(R, k)$ is a finitely generated left $\mathrm{Ext}_S\,(k, k)$-module. Furthermore, if $\mathrm{Ext}_S\,(k, k)$ is finitely generated and finitely presented, then $\mathrm{Ext}_R\,(k, k)$ is a finitely generated algebra if and only if $\mathrm{Ext}_S\,(R, k)$ is a finitely generated left $\mathrm{Ext}_S\,(k, k)$-module.*

PROOF. Let X be a minimal algebra resolution of k over S. Then with $U = R \otimes_S X$, U is a distinguished Golod algebra. But

$$\mathrm{Hom}\,(\tilde{H}(U), k) \cong I\,\mathrm{Ext}_S\,(R, k)\,.$$

By Theorem 3.1 with $M = R$, $\mathrm{Hom}\,(\tilde{H}(U), k)$ is a finitely generated right $\mathrm{Ext}_S\,(k, k)$-module via the product ψ if and only if $I\,\mathrm{Ext}_S\,(R, k)$ is a finitely generated left $\mathrm{Ext}_S\,(k, k)$-module via the product φ. Now Theorem 4.3 follows from Theorem 4.2.

As an easy application of Theorem 4.3, consider the case where $R = S/(x)$, x a non-zero divisor in $\mathfrak{m}^2$. Then $S \to R$ is a Golod homomorphism [8, Theorem 3.7] and $\mathrm{Ext}_S^i\,(R, k) = 0$ for $i > 1$. Thus $\mathrm{Ext}_S\,(R, k)$ is certainly finitely generated as an $\mathrm{Ext}_S\,(k, k)$-module, so $\mathrm{Ext}_R\,(k, k)$ is a finitely generated algebra if $\mathrm{Ext}_S\,(k, k)$ is.

5. Examples of finitely generated Ext algebras.

First we state some easy lemmas about the Yoneda product. See [11, Chapter 3] for proofs.

LEMMA 5.1. *If $0 \to A \to B \to C \to 0$ is a short exact sequence of R-modules, then the connecting homomorphism*

$$\partial\,:\; s\,\mathrm{Ext}_R\,(A, k) \to \mathrm{Ext}_R\,(C, k)$$

is a homomorphism of left $\mathrm{Ext}_R\,(k, k)$-modules.

LEMMA 5.2. *For any homomorphism $f: A \to B$ of R-modules, the induced homomorphism*

$$\mathrm{Ext}_R(B,k) \to \mathrm{Ext}_R(A,k)$$

is a homomorphism of left $\mathrm{Ext}_R(k,k)$*-modules.*

LEMMA 5.3. *The isomorphisms*

$$\mathrm{Ext}_R(A \oplus B, k) \cong \mathrm{Ext}_R(A,k) \oplus \mathrm{Ext}_R(B,k)$$

are isomorphisms of left $\mathrm{Ext}_R(k,k)$*-modules.*

THEOREM 5.4. *Let $\mathfrak{a}$ be an ideal contained in the socle of R. Then $I\,\mathrm{Ext}_R(R/\mathfrak{a},k)$ is a free left $\mathrm{Ext}_R(k,k)$-module on r generators $r = \dim_k \mathfrak{a}$.*

PROOF. By Lemmas 5.1 and 5.3, there are $\mathrm{Ext}_R(k,k)$-isomorphisms

$$I\,\mathrm{Ext}_R(R/\mathfrak{a},k) \cong \mathrm{Ext}_R(\mathfrak{a},k) \cong \mathfrak{a} \otimes \mathrm{Ext}_R(k,k)\,.$$

THEOREM 5.5. *If $\mathfrak{a}$ is an ideal contained in $(0:\mathfrak{m})$ and $R \to R/\mathfrak{a}$ is a small homomorphism, then*

$$Q_i(\mathrm{Ext}_{R/\mathfrak{a}}(k,k)) \cong Q_i(\mathrm{Ext}_R(k,k)) \quad \text{for } i \neq 2$$

and

$$\mathrm{Tor}_{i*}^{\mathrm{Ext}_{R/\mathfrak{a}}(k,k)}(k,k) \cong \mathrm{Tor}_{i*}^{\mathrm{Ext}_R(k,k)}(k,k) \quad \text{for } i > 1\,.$$

In particular, $\mathrm{Ext}_{R/\mathfrak{a}}(k,k)$ *is finitely generated if and only if* $\mathrm{Ext}_R(k,k)$ *is finitely generated and* $\mathrm{Ext}_{R/\mathfrak{a}}(k,k)$ *is finitely presented if and only if* $\mathrm{Ext}_R(k,k)$ *is finitely presented.*

PROOF. It was proved in [8, Theorem 3.9] that for $\mathfrak{a} \subset (0:\mathfrak{m})$, $R \to R/\mathfrak{a}$ is Golod if and only if it is small. By Theorems 3.1 and 4.1, the composite

$$\beta = \sigma(\alpha^*)^{-1}\colon I\,\mathrm{Ext}_R(R/\mathfrak{a},k) \to Q\mathscr{A}$$

satisfies

$$\beta\varphi T(1 \otimes C^*) = \gamma(\beta \otimes 1)$$

and since $I\,\mathrm{Ext}_R(R/\mathfrak{a},k)$ is a free left $\mathrm{Ext}_R(k,k)$-module, $Q\mathscr{A}$ is a free right $\mathrm{Ext}_R(k,k)$-module, so

$$\mathrm{Tor}_{i*}^{\mathrm{Ext}_R(k,k)}(Q\mathscr{A},k) = 0$$

for $i > 0$. Now the result follows from the exact sequence (4.2).

THEOREM 5.6. *If R is a local Gorenstein ring with socle $\mathfrak{a}$, then* $\operatorname{Ext}_R(k,k)$ *is finitely generated if and only if* $\operatorname{Ext}_{R/\mathfrak{a}}(k,k)$ *is finitely generated.*

PROOF. It is a result of the author and L. Avramov [9, Theorem 2.9] that for R a local Gorenstein ring, $R \to R/\mathfrak{a}$ is a Golod homomorphism. Apply Theorem 5.5.

In Theorem 5.5, $I \operatorname{Ext}_R(R/\mathfrak{a}, k)$ is generated by elements of degree one as a left $\operatorname{Ext}_R(k,k)$-module. In fact,

$$I \operatorname{Ext}_R(R/\mathfrak{a}, k) \cong \operatorname{Ext}_R(\mathfrak{a}, k)$$

which is generated by its elements of degree zero.

LEMMA 5.7. *For any R-module M,* $\operatorname{Ext}_R(M,k)$ *is generated as a left* $\operatorname{Ext}_R(k,k)$*-module by its elements of degree zero if and only if the induced homomorphism*

$$\operatorname{Ext}_R(M,k) \to \operatorname{Ext}_R(\mathfrak{m}M,k)$$

is zero.

PROOF. The elements of degree zero generate $\operatorname{Ext}_R(M,k)$ if and only if the Yoneda product

$$\varphi\colon \operatorname{Ext}^i_R(k,k) \otimes \operatorname{Ext}^0_R(M,k) \to \operatorname{Ext}^i_R(M,k)$$

is surjective for each $i \geqq 0$. By Lemmas 5.1 and 5.2, the exact sequence

$$0 \to \mathfrak{m}M \to M \to M/\mathfrak{m}M \to 0$$

yields a commutative diagram

$$\begin{array}{ccc}
\operatorname{Ext}^i_R(k,k) \otimes \operatorname{Ext}^0_R(M/\mathfrak{m}M,k) & \xrightarrow{\varphi'} & \operatorname{Ext}^i_R(M/\mathfrak{m}M,k) \\
\downarrow u & & \downarrow \\
\operatorname{Ext}^i_R(k,k) \otimes \operatorname{Ext}^0_R(M,k) & \xrightarrow{\varphi} & \operatorname{Ext}^i_R(M,k) \\
\downarrow & & \downarrow v \\
\operatorname{Ext}^i_R(k,k) \otimes \operatorname{Ext}^0_R(\mathfrak{m}M,k) & \longrightarrow & \operatorname{Ext}^i_R(\mathfrak{m}M,k) .
\end{array}$$

By Lemma 5.3, φ' is an isomorphism. Since u is always an isomorphism, φ is surjective if and only if $v = 0$.

THEOREM 5.8. *Suppose that $R \to R/\mathfrak{a}$ is a Golod homomorphism such that the induced homomorphism*

$$I \operatorname{Ext}_R(R/\mathfrak{a},k) \to I \operatorname{Ext}_R(R/\mathfrak{m}\mathfrak{a},k)$$

is zero. Then

$$Q_i(\mathrm{Ext}_{R/\mathfrak{a}}(k,k)) \cong Q_i(\mathrm{Ext}_R(k,k))$$

for $i \neq 2$.

PROOF. By Lemma 5.7, $\mathrm{Ext}_R(\mathfrak{a},k)$ is generated by its elements of degree zero, so

$$Q_i(I\,\mathrm{Ext}_R(R/\mathfrak{a},k)) = 0$$

for $i>1$. The result then follows from exact sequence (4.3).

We obtain as a corollary a result of Sjödin [14].

THEOREM 5.9. *If R is a regular local ring, then for any integer $n \geqq 2$, $\mathrm{Ext}_{R/\mathfrak{m}^n}(k,k)$ is generated as an algebra by its elements of degree* 1 *and* 2.

PROOF. Since $\mathrm{Ext}_R(k,k)$ is the exterior algebra on $\mathrm{Ext}^1{}_R(k,k)$,

$$Q_i(\mathrm{Ext}_R(k,k)) = 0$$

for $i \neq 1$, by Theorem 5.8, it is sufficient to prove that

$$I\,\mathrm{Ext}_R(R/\mathfrak{m}^n,k) \to I\,\mathrm{Ext}_R(R/\mathfrak{m}^{n+1},k) \tag{5.1}$$

is zero. But if K is the Koszul complex of R,

$$H(\mathfrak{m}^nK) = \frac{\mathfrak{m}^nK \cap Z(K)}{\mathfrak{m}^nB(K)}$$

and (5.1) just says that

$$\mathfrak{m}^{n+1}K \cap Z(K) \subset \mathfrak{m}^nB(K)$$

which is well-known for regular local rings.

THEOREM 5.10. *For any local ring R there is an integer n_0 such that for $n \geqq n_0$*

$$Q_i(\mathrm{Ext}_{R/\mathfrak{m}^n}(k,k)) \cong Q_i(\mathrm{Ext}_R(k,k))$$

for $i \neq 2$.

PROOF. In [8, Theorem 3.15], it was shown that there is an integer r_0 such that for $n \geqq r_0$

$$I\,\mathrm{Ext}_R(R/\mathfrak{m}^n,k) \to I\,\mathrm{Ext}_R(R/\mathfrak{m}^{n+1},k)$$

is zero. It follows that $R \to R/\mathfrak{m}^{n+1}$ is Golod. Put $n_0 = r_0 + 1$ and apply Theorem 5.8.

Note that in this case, $I\,\mathrm{Ext}_R\,(R/\mathfrak{a}, k)$ is not free, but it is possible to give a minimal resolution of it as a left $\mathrm{Ext}_R\,(k,k)$-module. Namely, it was shown in [9, Theorem 4.9] that for any R-module M there is an integer n_0 such that for $n \geqq n_0$

$$\mathrm{Ext}_R\,(\mathfrak{m}^n M, k) \to \mathrm{Ext}_R\,(\mathfrak{m}^{n+1} M, k)$$

is zero. Then for $n \geqq n_0$ one obtains exact sequences

$$0 \to \mathrm{Ext}_R^i\,(\mathfrak{m}^{n+1}M, k) \to \mathrm{Ext}_R^{i+1}\,(\mathfrak{m}^n M/\mathfrak{m}^{n+1}M, k) \to \mathrm{Ext}_R^{i+1}(\mathfrak{m}^n M, k) \to 0$$

for any $i \geqq 0$. Since $\mathrm{Ext}_R\,(\mathfrak{m}^n M/\mathfrak{m}^{n+1}M, k)$ is a free $\mathrm{Ext}_R\,(k,k)$-module, composing these exact sequences yields a free resolution

$$\ldots \to F_2 \to F_1 \to F_0 \to \mathrm{Ext}_R\,(\mathfrak{m}^n M, k) \to 0$$

where

$$F_i = s^i\, \mathrm{Ext}_R\,(\mathfrak{m}^{n+i}M/\mathfrak{m}^{n+i+1}M, k)\,.$$

This is a minimal resolution because

$$dF_i \subset \bar{F}_{i-1} = (I\,\mathrm{Ext}_R\,(k,k))\,\mathrm{Ext}_R^0\,(\mathfrak{m}^{n+i-1}M/\mathfrak{m}^{n+i}M, k)\,.$$

We obtain some easy consequences of this resolution.

THEOREM 5.11. *For any local ring R, if the algebra* $\mathrm{Ext}_R\,(k,k)$ *has finite global dimension, then R is artinian.*

PROOF. Assume that $\mathfrak{m}^n \neq 0$ for all n. Then for sufficiently large n, the above resolution shows that $\mathrm{Ext}_R\,(R/\mathfrak{m}^n, k)$ has infinite projective dimension.

If $C = C_{i,j}$ is a bigraded vector space over k, let the Poincaré series of C,

$$P(x,y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (\dim_k C_{i,j}) x^i y^j\,.$$

THEOREM 5.12. *For any R-module M, there is an integer n_0 such that for $n \geqq n_0$, the Poincaré series of*

$$\mathrm{Tor}_{i,j}^{\mathrm{Ext}_R\,(k,k)}\,(k, \mathrm{Ext}_R\,(\mathfrak{m}^n M, k))$$

is a rational function.

PROOF. Because of the resolution given above

$$P(x,y) = \sum_{i=0}^{\infty} \dim_k\,(\mathfrak{m}^{n+i}M/\mathfrak{m}^{n+i+1}M) x^i y^i$$

By the Hilbert theory, $\dim_k(\mathfrak{m}^{n+i}M/\mathfrak{m}^{n+i+1}M)$ is a polynomial in i for large i so $P(x,y)$ is a rational function.

6. Noetherian Ext algebras.

Let R be a complete intersection, i.e., suppose $R=S/(a_1,\ldots,a_r)$ where S is a regular local ring and $a_1,\ldots,a_r$ is an S-sequence in $\mathfrak{m}_S^2$. Gulliksen [4] has shown that for any R-modules M,N, $\operatorname{Ext}_R(M,N)$ may be regarded as a module over a polynomial ring $S[x_1,\ldots,x_r]$ and as such is a noetherian module. In [12], Mehta has shown that there is a homomorphism from the free R-module

$$\sum_{i=1}^{r} Re_i \cong \operatorname{Ext}_S^1(R,R) \xrightarrow{\theta} \operatorname{Ext}_R^2(M,M)$$

such that the action of x_i above is just left multiplication by $\theta(e_i)$ with the Yoneda product. Hence $\operatorname{Ext}_R(M,N)$ is also noetherian as a left $\operatorname{Ext}_R(M,M)$-module. In particular, $\operatorname{Ext}_R(M,M)$ is a left noetherian ring. For $M=k$, this can be reversed.

THEOREM 6.1. *Let R be a local ring such that $\operatorname{Ext}_R(k,k)$ is a left noetherian ring. Then for any R-module M, $\operatorname{Ext}_R(M,k)$ is noetherian as a left $\operatorname{Ext}_R(k,k)$-module. Consequently, $\operatorname{Ext}_A(k,k)$ is a finitely generated algebra for any Golod homomorphism $R\to A$. (In general, $\operatorname{Ext}_A(k,k)$ will not be noetherian.)*

PROOF. We first prove the result for modules M of finite length by induction on $\ell(M)$. The case $\ell=1$ is obvious.

Let

$$0\to M'\to M\to M''\to 0$$

be an exact sequence of R-modules where M' and M'' have smaller length than M. Then $\operatorname{Ext}_R(M',k)$ and $\operatorname{Ext}_R(M'',k)$ are noetherian by induction and the long exact sequence shows that $\operatorname{Ext}_R(M,k)$ is noetherian.

For the general case, as noted in section 5, there is an integer n_0 such that for $n\geqq n_0$

$$\operatorname{Ext}_R(\mathfrak{m}^nM,k)\to\operatorname{Ext}_R(\mathfrak{m}^{n+1}M,k)$$

is zero. Then the connecting homomorphism

$$s\operatorname{Ext}_R(\mathfrak{m}^nM/\mathfrak{m}^{n+1}M,k)\to\operatorname{Ext}_R(\mathfrak{m}^nM,k)\to 0$$

is a surjective homomorphism of $\operatorname{Ext}_R(k,k)$-modules. Since (Lemma 5.3),

$\mathrm{Ext}_R(\mathfrak{m}^n M/\mathfrak{m}^{n+1}M, k)$ is a finitely generated free $\mathrm{Ext}_R(k,k)$-module, $\mathrm{Ext}_R(\mathfrak{m}^n M, k)$ is noetherian.

Now consider the short exact sequence

$$0 \to \mathfrak{m}^n M \to M \to M/\mathfrak{m}^n M \to 0 .$$

Since $M/\mathfrak{m}^n M$ has finite length, $\mathrm{Ext}_R(M/\mathfrak{m}^n M, k)$ is noetherian and then the long exact sequence shows that $\mathrm{Ext}_R(M,k)$ is noetherian.

In particular, if $R \to A$ is a Golod homomorphism, $\mathrm{Ext}_R(A,k)$ is noetherian, hence finitely generated as a $\mathrm{Ext}_R(k,k)$-module. Since $\mathrm{Ext}_R(k,k)$ is left noetherian, the ideal $I\,\mathrm{Ext}_R(k,k) \cong s\,\mathrm{Ext}_R(\mathfrak{m},k)$ is finitely generated so $\mathrm{Ext}_R(k,k)$ is a finitely generated algebra, and by Theorem 4.3, $\mathrm{Ext}_A(k,k)$ is a finitely generated algebra.

In general, $\mathrm{Ext}_A(k,k)$ will not be noetherian, e.g. if R is regular and $A = R/\mathfrak{m}^2$, $\mathrm{Ext}_A(k,k)$ is a free algebra which is non-noetherian if $\dim \mathfrak{m}/\mathfrak{m}^2 > 1$. In fact, it is not known whether there are any non-complete intersections R for which $\mathrm{Ext}_R(k,k)$ *is noetherian.*

Corollary 6.2. *If R is a complete intersection, then* $\mathrm{Ext}_A(k,k)$ *is finitely generated for every Golod homomorphism $R \to A$.*

One can ask what happens to the Ext algebra under a composite of Golod homomorphisms.

Theorem 6.3. *If $R \to A$ is a Golod homomorphism and M is an A-module such that* $\mathrm{Ext}_R(M,k)$ *is noetherian as a left* $\mathrm{Ext}_R(k,k)$*-module, then* $\mathrm{Ext}_A(M,k)$ *is finitely generated as a left* $\mathrm{Ext}_A(k,k)$*-module.*

Proof. From [9, Theorem 1.5] one has an exact sequence of complexes

$$0 \to A \otimes_R X \to Y \to Y \otimes_A V \to 0$$

which splits as graded modules, where X is a minimal resolution of k over R, Y is a minimal resolution of k over A and V is a free graded A-module with differential zero. This yields an exact sequence

$$0 \to M \otimes_R X \to M \otimes_A Y \to (M \otimes_A Y) \otimes_A V \to 0$$

and

$$\ldots \to \mathrm{Tor}_R(M,k) \xrightarrow{\beta} \mathrm{Tor}_A(M,k) \xrightarrow{\alpha} \mathrm{Tor}_A(M,k) \otimes_A V \to \ldots$$

and an exact sequence of dual vector spaces

$$\ldots \to \mathrm{Ext}_A(M,k)\otimes V^* \xrightarrow{\alpha^*} \mathrm{Ext}_A(M,k) \xrightarrow{\beta^*} \mathrm{Ext}_R(M,k) \to \ldots .$$

The image of β^* is a sub $\mathrm{Ext}_R(k,k)$-module of $\mathrm{Ext}_R(M,k)$, hence finitely generated. Thus it is sufficient to show that α^* is just the product ψ which by Theorem 3.1, with $S=R=A$, is $\varphi T(1\otimes C^*)$.

As in the proof of Theorem 4.1, one lifts $g \in V^*$ to a mapping of complexes $G\colon Y \to Y$ defined by

$$G(x+y\otimes v) = yg(v)$$

for $x \in A\otimes_R X$, $y \in Y$, $v \in V$. Then for $f \in \mathrm{Ext}_A(M,k) \cong \mathrm{Hom}(H(M\otimes_A Y),k)$,

$$\begin{aligned}\psi(f\otimes g) &= f(1_M\otimes G)\\ &= (f\otimes g)\alpha = \alpha^*(f\otimes g)\,.\end{aligned}$$

Corollary 6.4. *If* $\mathrm{Ext}_R(k,k)$ *is noetherian* (e.g. *if R is a complete intersection*) *and $R \to A$ and $A \to B$ are Golod homomorphisms, then* $\mathrm{Ext}_B(k,k)$ *is a finitely generated algebra.*

Proof. By Theorem 6.1, $\mathrm{Ext}_R(A,k)$ and $\mathrm{Ext}_R(B,k)$ are noetherian $\mathrm{Ext}_R(k,k)$-modules so $\mathrm{Ext}_A(k,k)$ is a finitely generated algebra. By Theorem 6.3, $\mathrm{Ext}_A(B,k)$ is a finitely generated $\mathrm{Ext}_A(k,k)$-module, so $\mathrm{Ext}_B(k,k)$ is a finitely generated algebra by Theorem 4.3.

7. A counter-example.

The methods of this paper may be used to provide a negative answer to a question raised by a theorem of Gulliksen. Let Y be a minimal algebra resolution of k. As in section 3, example 2, Y^r denotes the subalgebra of Y obtained by adjoining all variables of degree $\leqq r$. It was shown in section 3 that Y^r is a distinguished subalgebra.

In [5], Gulliksen has shown that if Y^r is a Golod algebra, then the Poincaré series of R is a rational function. Thus the question arises: Is it true that for every local ring R, some Y^r is a Golod algebra? In the light of Gulliksen's theorem a positive answer would prove Kaplansky's conjecture that the Poincaré series is always a rational function. The answer, however, is no, and as a counter-example we may take the example of Roos [13].

Theorem 7.1. *If some Y^r is a Golod algebra, then* $\mathrm{Ext}_R(k,k)$ *is finitely generated.*

Proof. Let $J_1,\ldots,J_p$ be the derivations associated with the variables of even degree $\leqq r$. These induce homomorphisms

$$J_{i*}\colon H(Y^r) \to H(Y^r)$$

defining a homomorphism

$$\operatorname{Hom}(\tilde{H}(Y^r), k) \otimes T(J_1, \ldots, J_p) \to \operatorname{Hom}(\tilde{H}(Y^r), k)$$

(as noted in section 3), where $T(J_1, \ldots, J_p)$ is the free algebra on $J_1, \ldots, J_p$. From [5] it follows that $\operatorname{Hom}(\tilde{H}(Y^r), k)$ is noetherian as a right $T(J_1, \ldots, J_p)$-module.

Sending $J_i \to \varepsilon J_i$ defines a ring homomorphism

$$T(J_1, \ldots, J_p) \xrightarrow{f} \operatorname{Hom}(Y^r, k)\,.$$

Since $J_i\colon Y^r \to Y^r$ extends εJ_i, left multiplication by J_i on $\operatorname{Hom}(\tilde{H}(Y^r), k)$ is the same as left multiplication by $f(J_i)$. Thus $\operatorname{Hom}(\tilde{H}(Y^r), k)$ is also noetherian as a right $\operatorname{Hom}(Y^r, k)$-module. Then by Theorem 4.2, $\operatorname{Ext}_R(k, k)$ is finitely generated.

In the example of Roos [13], $\operatorname{Ext}_R(k, k)$ is not finitely generated, so no Y^r can be Golod.

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BROOKLYN COLLEGE OF THE CITY UNIVERSITY OF NEW YORK
BROOKLYN NEW YORK 11210
U.S.A.