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ON MODULES WITH LE-DECOMPOSITION

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Introduction.

Following [2] we call a module A an Le-*module* provided its endomorphism ring is local, and a module M has an Le-*decomposition* if it is isomorphic to a direct sum of Le-modules.

In the first section of this paper we will study modules with an Le-decomposition that complements direct summands. It turns out that this is the case precisely when the Le-decomposition has the finite exchange property.

This motivates the main theorem of the last section, which states that the finite exchange property of a module implies that idempotents lift modulo the Jacobson radical of its endomorphism ring, thereby generalizing a theorem of Harada [5, Theorem 3].

The terminology is standard with a few exceptions. We refer the reader to the book of Anderson and Fuller [1]. We consider left modules over an associative ring R with an identity. A ring R is local if the non-units in R form an ideal. The symbol $\oplus$ is used for *internal sum.* If M is a module, we let J_M denote the Jacobson radical of $\operatorname{End} M$. All maps between modules are R-homomorphisms.

1. Modules with Le-decomposition.

DEFINITION [3]. A module M has the *exchange property* if for any module Ω and for any decomposition

$$\Omega = M' \oplus L = \oplus_I N_i$$

with $M' \cong M$, there are submodules $N_i' \subset N_i$ such that

$$\Omega = M' \oplus (\oplus_I N_i') .$$

The module M has the *finite exchange property* if this holds whenever the index set I is finite.

It is well known ([3], [7]) that the finite exchange, the exchange and in fact [2, Theorem 3] also the mutual exchange properties coincide for

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indecomposable modules. The indecomposable modules with the exchange property are exactly the Le-modules.

For decomposable modules, however, it is not known whether the exchange and finite exchange properties coincide, so the distinction in the definition above is meaningful.

However, it is known that a finite sum of Le-modules has the exchange property. A simple proof of this is given in [2, Remark p. 397].

Let $M = \oplus_I A_i$ be an Le-decomposition. Let $\varepsilon_i\colon A_i \to M$ and $\pi_j\colon M \to A_j$ be the canonical injections and projections. For any $f \in \operatorname{End} M$, let

$$f_{ij} = \pi_j f \varepsilon_i : A_i \to A_j .$$

Harada and Sai [4] define J'_M to be the set of $f \in \operatorname{End} M$ for which f_{ij} is not an isomorphism for any $i, j \in I$. They show that J'_M is independent of the Le-decomposition and that J'_M is a two-sided ideal in $\operatorname{End} M$ which contains J_M.

The following proposition is also due to them, and we omit its proof.

PROPOSITION 1. *Let* $M = \oplus_I A_i$ *be an* Le-*decomposition. The following properties are equivalent*:

(1) *The* Le-*decomposition complements direct summands.*

(2) $J'_M = J_M$.

REMARK. If $M = \oplus_I A_i$ is an Le-decomposition, it is possible to show that $\operatorname{End} M/J'_M$ is a von Neumann regular ring. If the Le-decomposition complements direct summands we therefore have, by Proposition 1, that $\operatorname{End} M/J_M$ is a von Neumann regular ring.

Now we are prepared to state and prove the following results:

THEOREM 2. Let $M = \oplus_I A_i$ *be an* Le-*decomposition. Let* $S = \operatorname{End} M$ *and let* J_M *be the Jacobson radical of* S. *The following properties are equivalent*:

(1) *The* Le-*decomposition complements direct summands.*

(2) M *has the finite exchange property.*

(3) *For all* $f \in S$, *there is an idempotent* $e \in S$ *such that* $Sf + J_M = Se + J_M$.

(4) S/J_M *is von Neumann regular and idempotents lift modulo* J_M.

PROOF. (1) $\Rightarrow$ (4): By Proposition 1 and the above remark together with [5, Theorem 3]. (3) $\Rightarrow$ (2): By [9, Theorem 2 and 3].

(4) $\Rightarrow$ (3): Let $f \in S$. Since S/J_M is von Neumann regular there is a $g \in S$ such that $\bar{f} = \bar{f}\bar{g}\bar{f}$ where the bar denotes element of S/J_M. This implies that $\bar{g}\bar{f}$ is an idempotent in S/J_M, and by the hypothesis there is an idempotent $e \in S$ such that $\bar{g}\bar{f} = \bar{e}$. Hence $Sf + J_M = Se + J_M$.

(2) $\Rightarrow$ (1): Let $(0) \neq K$ be a direct summand of M and choose a subset $J \subseteqq I$ maximal with respect to $(\oplus_J A_j) \cap K = (0)$.

Let π be the projection of M on $\oplus_{I \setminus J} A_i$ along $\oplus_J A_j$. Then the restriction $\pi | K$ of π to K is a monomorphism. We will show that $\pi(K)$ is a direct summand of M.

Let i: $\pi(K) \to \oplus_{I \setminus J} A_i$ be the inclusion map. This is an R-monomorphism and we have to show that it splits. Since $\pi \in S$ and M has the finite exchange property, there exist, by [6, Theorem 1], $g, h \in S$ such that

$$(1) \qquad h(1-\pi)(1-g\pi) = 1-g\pi \qquad (\text{and } g\pi g = g)\,.$$

Multiplying (1) on the right by π, we get, since π is an idempotent of S, that $\pi = (1-\pi+h\pi)g\pi$. Hence $(1-h+h\pi)gi = 1_{\pi(K)}$ and $\pi(K)$ is a direct summand of $\oplus_{I \setminus J} A_i$. Therefore $\pi(K)$ is a direct summand of M, say $M = \pi(K) \oplus M'$.

Using the modular law we get

$$(2) \qquad \oplus_{I \setminus J} A_i = \pi(K) \oplus N\,,$$

where $N = (\oplus_{I \setminus J} A_i) \cap M'$.

Let p be the projection of $\oplus_{I \setminus J} A_i$ on $\pi(K)$ along N in (2), and consider the decomposition

$$(3) \qquad M = \pi(K) \oplus N \oplus (\oplus_J A_j)\,.$$

It is easy to see that the projection of M on $\pi(K)$ in (3) is $p\pi$ and that the restriction $p\pi | K$ of $p\pi$ to K is an isomorphism. By [1, Proposition 5.5], we therefore have

$$(4) \qquad M = K \oplus N \oplus (\oplus_J A_j)\,.$$

We claim that $N = (0)$. (See also [1, p. 291] for a special case). Suppose $N \neq (0)$. By the K.R.S.A.-Theorem 4 in [2], there is a direct summand N' of N, say $N = N' \oplus N''$ such that N' is an Le-module. From (4) we get

$$(5) \qquad (K \oplus (\oplus_J A_j) \oplus N'') \oplus N' = M = \oplus_I A_i\,,$$

where $\oplus_I A_i$ is the given Le-decomposition of M.

Since, by the Azumaya-theorem, an Le-decomposition complements *maximal* direct summands, there is by the K.R.S.A.-theorem an $i_0 \in I$ such that

$$(6) \qquad K \oplus (\oplus_J A_j) \oplus N'' \oplus A_{i_0} = M\,.$$

It is obvious that $i_0 \notin J$ and that $K \cap (\oplus_{J \cup \{i_0\}} A_j) = (0)$. This contradicts the maximality of J; hence $N = (0)$ and this completes the proof of the theorem.

REMARK. The property (3) in the above theorem shows that the endomorphism ring of a module with an Le-decomposition that complements direct summands is included in the class of exchange rings defined by Warfield in [9, Theorem 3].

Monk [6] has shown that the property (3) above is not a necessary condition for a module to have the finite exchange property.

2. The finite exchange property and lifting of idempotents.

The implication (4) $\Rightarrow$ (2) in Theorem 2 is always true. See [9, Remark after Theorem 3]. What about the converse? In this section we will show that the finite exchange property of a module M implies that idempotents in $\operatorname{End} M/J_M$ lift to $\operatorname{End} M$, so that the only question is the von Neumann regularity of $\operatorname{End} M/J_M$.

In the proof we use the following lemma, which is easily proved:

LEMMA 3. *Let R be a ring and let $e \in R$ be an idempotent of R. Then for each $x \in R$, $t = e + (1-e)xe$ is also an idempotent.*

THEOREM 4. *Let M be a left R-module. Let $S = \operatorname{End} M$ and let J_M be the Jacobson radical of S.*

If M has the finite exchange property, then idempotents in S/J_M can be lifted to S.

PROOF. Let $f \in S$ be such that $f - f^2 \in J_M$. Since M has the finite exchange property, we have by [6, Theorem 1] that there exist $g, h \in S$ such that

$$h(1-f)(1-gf) = 1-gf \quad \text{and} \tag{1}$$

$$gfg = g\,. \tag{2}$$

Multiplying (1) on the right by f we get that $f = gff + h(1-f)(1-gf)f$. In the following we let, for convenience, a bar over the endomorphisms denote elements of S/J_M. An easy computation gives, using the fact that $\bar{f}^2 = \bar{f}$, that

$$\bar{f} = \bar{g}\bar{f} + \bar{h}(\bar{1}-\bar{f})\bar{g}\bar{f}\,. \tag{3}$$

By (2) gf is an idempotent of S, and therefore also $t = gf + (1-gf)(f-1)gf$ is an idempotent of S by Lemma 3. Standard computations, noting that $(\bar{1}-\bar{f})\bar{f} = \bar{0}$, give that $\bar{t} = \bar{f}$ which completes the proof of the theorem.

COROLLARY 5. *Let E be an injective R-module. Then idempotents in* $\operatorname{End} E/J_E$ *lift to* $\operatorname{End} E$.

PROOF. Injective modules have the exchange property [8]; hence the finite exchange property.

Another proof of Corollary 5 can be found in [1, p. 312].

As mentioned above the injective modules have the exchange property. On the other hand there exist injective modules which have no Le-decomposition [1, Theorem 25.6]. Therefore Theorem 4 does generalize the following result of Harada [5, Theorem 3]:

COROLLARY 6. (*Harada*). *Let* $M=\oplus_I A_i$ *be an* Le-*decomposition. If* $J_M=J'_M$ *then idempotents in* $\operatorname{End} M/J_M$ *lift to* $\operatorname{End} M$.

PROOF. Proposition 1 and Theorem 2 show that $J_M=J'_M \Leftrightarrow M$ has the finite exchange property.

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