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TRANSLATION INVARIANT SUBSPACES OF WEIGHTED l^p AND L^p SPACES

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1.

We shall introduce weights on Z and R, such that the corresponding weighted l^p and L^p spaces have the family of translations to the right, or the family of arbitrary translations, as operators. Our object is to investigate whether there exist translation invariant subspaces, apart from the most obvious ones. In this section we give a survey, present some new results and raise problems. Proofs are given in Section 2 and 3.

For Z, our discussion can be interpreted in terms of invariant or hyperinvariant subspaces for bilateral (or unilateral) weighted shifts on l^p. For the connection between these concepts and ours we refer to Shields [10] or to Gellar and Herrero [7].

In our terminology, a subspace is always assumed to be closed.

A. Z *and right translations.*

$w=\{w_n\}_{-\infty}^{\infty}$ is a decreasing sequence of positive numbers, and $1 \leqq p \leqq \infty$. $l^p(w)$ is the Banach space of complex-valued sequences $c=\{c_n\}_{-\infty}^{\infty}$ with

$$cw = \{c_n w_n\}_{-\infty}^{\infty} \in l^p(\mathsf{Z}) ,$$

and with the norm of c defined as the l^p norm of cw. $c_0(w)$ is the closed subspace of $l^\infty(w)$ of sequences c for which $c_n w_n \to 0$, as $|n| \to \infty$. By the monotonicity of w, (right) translation T, defined by

$$Tc = \{c_{n-1}\}_{-\infty}^{\infty} ,$$

is an operator on every $l^p(w)$ and on $c_0(w)$. A subspace of $l^p(w)$ or $c_0(w)$ is called invariant, if it is invariant under T.

For every $m \in \mathsf{Z} \cup \{-\infty\} \cup \{\infty\}$,

$$l^p(w,m) = \{c \in l^p(w) : \ c_n=0, \ n<m\}$$

is an invariant subspace of $l^p(w)$, $1 \leqq p < \infty$. The spaces $c_0(w,m)$ are defined analogously and are invariant subspaces of $c_0(w)$. We call all these subspaces

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the standard invariant subspaces. Are all invariant subspaces standard? In terms of bilateral weighted shifts: is the corresponding shift unicellular?

There are two well-known cases where non-standard invariant subspaces are easy to construct. If

$$\varliminf_{n\to\infty} \frac{\log w_n}{n} > -\infty\,, \tag{1}$$

there exist non-standard invariant subspaces in all spaces considered, for instance the subspace of all c with

$$c_n = e^{-an}, \quad n<0\,,$$

for some fixed a with real part smaller than the left member of (1). If

$$\varliminf_{n\to\infty} \frac{\log w_n}{n} > -\infty\,, \tag{2}$$

a non-standard invariant subspace in anyone of the spaces is formed by the elements c with

$$\sum_0^\infty c_n e^{bn} = 0\,,$$

for some fixed b with real part smaller than the left member of (2). In these cases, the structure of the family of invariant subspaces is in general very complicated and largely unknown. A special case, when the knowledge is complete, is the case when $l^p(w)=l^2$ (Helson and Lowdenslager [8]).

The following theorem shows that exponential behavior of w at infinity, as in (1) or (2), is not the critical magnitude for existence of non-standard invariant subspaces, at least not if we have similar growth conditions both at $-\infty$ and ∞.

THEOREM 1. *$l^p(w)$, $1\leqq p<\infty$, and $c_0(w)$ have non-standard invariant subspaces, if*

$$\varliminf_{|n|\to\infty} \left(\frac{\log w_n}{n} + \log|n|\right) > -\infty\,. \tag{3}$$

On the other hand, we have the following theorem, which gives an affirmative answer to Question 22, p. 109 in [10].

THEOREM 2. *Let $\log w$ be convex, for $n\leqq 0$, and concave, for $n\geqq 0$, and let*

$$\sum_{n\neq 0} w_n^{1/n} < \infty\,. \tag{4}$$

If either

$$\overline{\lim_{n \to -\infty}} \frac{\log w_n}{n^2} > \log 3 , \tag{5}$$

or

$$\underline{\lim}_{n \to \infty} \frac{\log w_n}{n^2} < -\log 3 , \tag{6}$$

then all invariant subspaces of $l^p(w)$, $1 \leqq p < \infty$, *and* $c_0(w)$, *are standard.*

The regularity assumption on w in Theorem 2 is in the nature of things. Thus it is possible to find weight sequences w, with $w_n \to 0$ arbitrarily rapidly, as $n \to 0$, and such that $l^p(w,0)$ contains non-standard invariant subspaces. This has been shown by N. K. Nikolskiĭ [9, § 3.2, Theorem 4]. On the other hand, regularity conditions on w together with the negation of (2) suffice to prevent the existence of non-standard invariant subspaces which are included in $l^p(w,0)$. A sufficient regularity condition is given for $l^p(w)$ in [9, § 3.2. Theorem 2]. It is fulfilled if $\{\log w_n - \alpha \log n\}$ is concave for large positive values of n, for some $\alpha > (p-1)/p$, if $1 < p < \infty$, and for $\alpha = 0$, if $p = 1$.

As for the structure of the family of invariant subspaces in $l^p(w,0)$, if (2) holds, we shall here only refer to Remark 4 of [5], which gives the complete structure, for $p = 1$, in the so called quasi-analytic case, if w is regular enough, then giving an affirmative anwer to a problem which was posed, for $p = 2$, as Question 17 p. 103 in [10]. The result in [5] can be easily extended to arbitrary p, $1 \leqq p < \infty$.

B. **Z** *and arbitrary translations.*

$w = \{w_n\}_{-\infty}^{\infty}$ is now a positive sequence, satisfying

$$0 < \inf \frac{w_{n+1}}{w_n} \leqq \sup \frac{w_{n+1}}{w_n} < \infty . \tag{7}$$

$l^p(w)$, $c_0(w)$ and T are defined as in A. In this situation, both T and T^{-1} are operators on $l^p(w)$, $1 \leqq p < \infty$, and $c_0(w)$, and a subspace is called invariant if it is invariant under T and T^{-1}. $\{0\}$ and the whole space are the trivial invariant subspaces. For which w do we know that $l^p(w)$, $1 \leqq p < \infty$, or $c_0(w)$ have non-trivial invariant subspaces? The question is closely linked to the problem of the existence of non-trivial hyperinvariant subspaces for bilateral weighted shifts.

We refer to [7] and [10] for various results. A proof in [7] has been modified by Atzmon, giving the result [2, Theorem 5.1] that

$$\sum_{-\infty}^{\infty} \frac{|\log w_n|}{1+n^2} < \infty ,$$

suffices for existence of non-trivial invariant subspaces. As for the general problem, no w is known for which $l^p(w)$, or $c_0(w)$, has only trivial invariant subspaces.

C. R *and right translations.*

w is a decreasing positive function on R, and $1 \leqq p \leqq \infty$. $L^p(w)$ is the Banach space of complex-valued functions f on R with $fw \in L^p(\mathsf{R})$, and with the norm of f defined as the L^p norm of fw. $C_0(w)$ is the closed subspace of $L^\infty(w)$ of continuous functions f, for which $f(x)w(x) \to 0$, as $|x| \to \infty$.

For every $a \geqq 0$, (right) translation T_a, defined by

$$T_a f(x) = f(x-a), \qquad x \in \mathsf{R} ,$$

is an operator in these spaces. A subspace of $L^p(w)$, $1 \leqq p < \infty$, or $C_0(w)$, is called invariant if it is invariant under the family $\{T_a\}$. The invariant subspaces

$$L^p(w,a) = \{f \in L^p(w) : f(x)=0, \quad \text{a.e.}, \quad x<a\} ,$$

$a \in \mathsf{R} \cup \{\infty\} \cup \{-\infty\}$, $1 \leqq p < \infty$, and the subspaces $C_0(w,a)$, defined similarly, are the *standard* invariant subspaces. Are there any non-standard invariant subspaces?

As in A, we can find non-standard invariant subspaces if

$$\varliminf_{x \to -\infty} \frac{\log w(x)}{x} > -\infty , \tag{8}$$

or

$$\varliminf_{x \to \infty} \frac{\log w(x)}{x} > -\infty . \tag{9}$$

As for two-sided conditions, we have the following result.

THEOREM 3. *$L^p(w)$, $1 \leqq p < \infty$, and $C_0(w)$ have non-standard invariant subspaces, if*

$$\int_{-\infty}^{\infty} \frac{d(\log w(x))}{1+x^2} > -\infty . \tag{10}$$

This result is less restrictive than Theorem 1 as for the order of magnitude of w at infinity, since it holds if $|\log w(x)| = O(|x|^\alpha)$, $\alpha < 2$. On the other hand, we have no correspondence to Theorem 2, for no w is known for which $L^p(w)$, or

$C_0(w)$, has only standard invariant subspaces. Turning to the structure of the family of invariant subspaces included in $L^p(w, 0)$, $1 \leqq p < \infty$, and $C_0(w, 0)$, our knowledge is very incomplete. Assuming that $\log w$ is concave on $\mathbf{R}^+$, and that (9) is not valid, one does not know a single w, for which it is known whether or not all invariant subspaces in $L^p(w, 0)$ or $C_0(w, 0)$ are standard. For partial results connected with this challenging open problem we refer to Allan [1], Bade and Dales [3], Dales and McClure [4], and Domar [6].

D. $\mathbf{R}$ *and arbitrary translations.*

w is now a positive Lebesgue measurable function on $\mathbf{R}$ with

$$w(x+y) < Cw(x),$$

$x \in \mathbf{R}$, $y \in [-1, 1]$, for some positive constant C. $L^p(w)$, $C_0(w)$, and T_a are defined as in case C, now with $a \in \mathbf{R}$. Every T_a is an operator on $L^p(w)$, $1 \leqq p < \infty$, and on $C_0(w)$. A subspace is invariant, if it is invariant under $\{T_a\}$. $\{0\}$ and the whole space are the trivial invariant subspaces. In contrast to case B, the basic problem is easy to solve.

THEOREM 4. *For every w, $L^p(w)$, $1 \leqq p < \infty$, and $C_0(w)$ have non-trivial invariant subspaces.*

To conclude this section, we shall discuss the relation between $\mathbf{Z}$ and $\mathbf{R}$ as for these problems. To every $w = \{w_n\}_{-\infty}^{\infty}$, we can associate a step-function v on $\mathbf{R}$, defined by

$$v(x) = w_n, \quad x \in [n, n+1[\,.$$

Starting with case A (or B), we can, for instance by using Hahn–Banach's theorem, map the family of invariant subspaces of $l^p(w)$ (or $c_0(w)$) injectively into the family of invariant subspaces of $L^p(v)$ (or $C_0(v)$). This means that existence of a non-standard (non-trivial) subspace in the discrete case implies existence of a non-standard (non-trivial) subspace in the corresponding continuous case. It would be desirable to have an example showing that the converse is not generally true.

2.

This section contains the proofs of Theorems 1, 3, and 4, together with a remark on Theorem 1, motivated by the method of its proof.

PROOF OF THEOREM 1. (3) implies that there is a $\lambda > 0$ such that

$$w_n \geqq \frac{\lambda^n}{n!},$$

if $n>0$, and

$$w_n \leqq \frac{|n|!}{\lambda^{|n|}}$$

if $n<0$. Starting from $l^p(w)$ or $c_0(w)$, we observe that a bounded linear functional on the space is given by the mapping

$$c \to \langle b, c\rangle = \sum_{-\infty}^{\infty} b_n c_n ,$$

where $b=\{b_n\}_{-\infty}^{\infty}$ belongs to $l^{p/p-1}(1/w)$ or $l^1(1/w)$. Hence there exists a complex number $a \neq 0$, such that $d=\{d_n\}_{-\infty}^{\infty}$, defined by

$$d_n = \begin{cases} \dfrac{a^{1-n}}{(1-n)!}, & n \leqq 1 \\ 0, & n > 2 \end{cases}$$

is in the space, and such that $b=\{b_n\}_{-\infty}^{\infty}$ defined by

$$b_n = \begin{cases} 0, & n<0 \\ \dfrac{(-a)^n}{n;}, & n \geqq 0 \end{cases}$$

defines a bounded linear functional. Then

$$\langle b, T^m d\rangle = \sum_{n=0}^{m+1} \frac{a^{m+1-n}}{(m+1-n)!} \frac{(-a)^n}{n!} ,$$

for $m \geqq 0$. But the sum to the right vanishes, since it is the coefficient of z^{m+1} in the power series expansion of

$$e^{az} \cdot e^{-az} = 1 .$$

This means that the translates $\{T^m d : m \geqq 0\}$ span a proper invariant subspace. This subspace is non-standard, since $d_n \neq 0$, for $n<0$.

REMARK. Different non-standard invariant subspaces can be obtained either by varying a, or by starting instead from a translate of d, or — if the growth of $|\log w_n|$ at infinity is sufficiently restricted — by using other entire functions without zeros. It would be interesting to know to what extent all these invariant subspaces can be used to describe the family of all invariant subspaces.

The proofs of Theorems 3 and 4 are based on a device which has been used earlier by Vretblad [11].

PROOF OF THEOREM 3. If w_0 is a weight function such that w/w_0 is bounded above and below by positive constants, the identity mappings of $L^p(w)$ to $L^p(w_0)$, and of $C_0(w)$ to $C_0(w_0)$ are isomorphisms which leave every T_a, $a \geqq 0$, invariant. Hence it is no loss of generality to assume that $w(x) = 1$, if x belongs to some neighborhood of 0. (10) shows that Poisson's formula, applied to the negative measure $d(\log w(x))$ on the x-axis, gives a negative harmonic function u in the open upper halfplane Π^+. u is real part of a unique analytic function f in Π^+, such that $f(z)$ tends to 0, as $z \to 0$. Put

$$F(z) = U(x,y) + iV(x,y) = -\int_\gamma f(z)\,dz\,,$$

where γ is any path from 0 to z in Π^+, and where U and V are real. Since

$$(11) \qquad U(x+a,y) - U(x) = -\int_x^{x+a} u(t,y)\,dt\,,$$

$(x,y) \in \Pi^+$, $a \in \mathsf{R}$, U increases as function of x, for every fixed $y > 0$. (11) shows that U is locally bounded at the x-axis, and converges to $-\log w(x)$, except when x is a discontinuity point of w.

Put

$$e^{F(z)} = G(z)\,.$$

Then

$$(12) \qquad \left|\frac{G(z-a)}{G(z)}\right| = \exp\{U(x-a,y) - U(x,y)\} \leqq 1\,,$$

for $z \in \Pi^+$, $a \geqq 0$. G is locally bounded at the x-axis, and converges, as $y \to +0$, almost everywhere to a function g with $|g| = 1/w$. (12) implies that $T_a g/g \in H^\infty(\mathsf{R})$, for $a \geqq 0$.

For $1 \leqq p \leqq \infty$, the set

$$K = \{gh : h \in H^p(\mathsf{R})\}$$

is contained in $L^p(w)$, since $ghw \in L^p(\mathsf{R})$. The set is isomorphic to $H^p(\mathsf{R})$, hence it is a subspace of $L^p(w)$. It is invariant, since $a \geqq 0$ implies that

$$T_a(gh) = T_a g T_a h = g(T_a g/g) T_a h = gk\,,$$

where $k \in H^p(\mathsf{R})$. Obviously K is a non-standard invariant subspace of $L^p(w)$. Putting $\chi_t(x) = e^{itx}$, $t \in \mathsf{R}$, $x \in \mathsf{R}$, we obtain more generally that

$$K(t) = \{\chi_t f : f \in K\}$$

and

$$\overline{K(t)} = \{\chi_{-t}\bar{f} : f \in K\}$$

are non-standard invariant subspaces, for every $t \in \mathbf{R}$.

In the case when $p=\infty$, it is easy to see that the subspaces $K(t) \cap C_0(w)$ and $\overline{K(t)} \cap C_0(w)$ contain non-zero functions, and hence give non-standard invariant subspaces of $C_0(w)$.

Proof of Theorem 4. It is no loss of generality to assume that $w(0)=1$, and that $\log w$ has a bounded and continuous derivative. With the same construction as in the proof of Theorem 3 we obtain a harmonic function u in Π^+, and a harmonic function U in Π^+ with boundary values $-\log w(x)$ everywhere. u is this time bounded, and thus (11) shows that

$$(x, y) \to U(x+a, y) - U(x, y)$$

is bounded in Π^+, for every $a \in \mathbf{R}$. Defining G and g as before, we obtain that $T_a g/g \in H^\infty(\mathbf{R})$, for every $a \in \mathbf{R}$, and the arguments in the final part of the proof of Theorem 3 go through.

3.

To prepare for the proof of Theorem 2, let us first remark that the form of the bounded linear functionals on $l^p(w)$ and $c_0(w)$ was stated in the beginning of the proof of Theorem 1. It is easy to realize that if L is an invariant subspace of $l^p(w)$, $1 \leqq p < \infty$, such that every $c \in L$ is included in some $l^p(w, m(c))$, $m(c) \neq -\infty$, and every $b \in L^\perp$ is included in some $l^p(w, n(b))^\perp$, $n(b) \neq \infty$, then L is a standard invariant subspace, and that the analogous property holds for $c_0(w)$. Hence Theorem 2 is the direct corollary of the following theorem.

Theorem 5. *Let $w=\{w_n\}_{-\infty}^{\infty}$ satisfy the assumptions of Theorem 2. Let $c=\{c_n\}_{-\infty}^{\infty}$ and $b=\{b_n\}_{-\infty}^{\infty}$ be not identically vanishing sequences, such that the sequences $\{c_n w_n\}$ and $\{b_n w_n^{-1}\}$ are bounded. Suppose that*

$$\sum_{-\infty}^{\infty} b_n c_{n-m} = 0, \qquad m>0. \tag{13}$$

Then there is a $p \in \mathbf{Z}$ such that $b_n=0$, $n>p$, and $c_n=0$, $n<p$.

Remark. (4) and the assumed regularity of w imply that

$$\sum_{-\infty}^{\infty} \frac{w_{n+1}}{w_n} < \infty .$$

Hence the series in (13) converges absolutely, for $m>0$.

PROOF OF THEOREM 5. Let us first consider the case when there is an integer $N>0$ such that $c_n=0$, for $n<-N$. Then $T^N c$ is in $l^1(w,0)$, and since the assumptions on w imply that $l^1(w,0)$ contains no non-standard invariant subspace (cf. the discussion after Theorem 2), $T^N c$ and its right translates span a standard invariant subspace. b is in the annihilator, and the conclusion follows easily. Changing the role of b and c, we can prove the theorem similarly, if there is an integer $N>0$ such that $b_n=0$, if $n>N$.

Thus it is enough to show that we obtain a contradiction, if $c_n \neq 0$ for infinitely many negative n, while $b_n \neq 0$ for infinitely many positive n.

Since the sequences $\{w_n\}$ and $\{1/w_{-n}\}$ give equivalent cases of the theorem, we can assume that (5) holds. Without loss of generality we can assume that

$$|c_n| \leqq w_n^{-1} \leqq 1 \quad \text{for } n \leqq 0\,, \tag{14}$$

$$|b_n| \leqq w_n \leqq 1 \quad \text{for } n \geqq 0\,. \tag{15}$$

We define the positive sequence $u=\{u_n\}$ in the following way: $u_0=1$, $\{\log u_n\}_{-\infty}^{0}$ is the largest convex minorant of $\{-\log|c_n|\}_{-\infty}^{0}$, and $\{\log u_n\}_0^{\infty}$ is the smallest concave majorant of $\{\log|b_n|\}_0^{\infty}$.

Then u is a decreasing sequence, and by (14) and (15) it satisfies

$$u_n \geqq w_n, \quad n \leqq 0\,,$$
$$u_n \leqq w_n, \quad n \geqq 0\,.$$

These inequalities and (4) show that

$$\sum_{n \neq 0} u_n^{1/n} < \infty\,,$$

and hence, by the regularity of $(u_n)_{-\infty}^{\infty}$,

$$\sum \frac{u_{n+1}}{u_n} < \infty\,. \tag{16}$$

It follows from the construction of u that

$$|c_n| \leqq u_n^{-1}, \quad n \in \mathbf{Z}\,, \tag{17}$$

$$|b_n| \leqq u_n, \quad n \in \mathbf{Z}\,, \tag{18}$$

and that the set $E \subseteq \mathbf{Z}$ of all n for which

$$u_{n+1}u_{n-1} \neq u_n^2\,, \tag{19}$$

is unbounded above and below, and has the property that

$$|c_n| = u_n^{-1}, \quad \text{if } n \in (-\mathbf{Z}^+) \cap E$$

$$|b_n| = u_n, \quad \text{if } n \in \mathbf{Z}^+ \cap E\,.$$

By (16) and the regularity properties of $u, u_{n+1}/u_n$ decreases monotonically to 0, if $n<0$, $n \to -\infty$, and if $n \geqq 0$, $n \to \infty$.

Let λ and μ, $\lambda<\mu$, be strictly included between the two members of (5). (5) implies that

$$u_n > e^{\mu n^2},$$

for an infinite sequence of negative integers n. Since the second difference of the sequence $\{\mu n^2\}$ takes the constant value $2\mu > 2\lambda$, we must have arbitrarily large positive integers m, such that

$$\log u_{-m-1} - 2\log u_{-m} + \log u_{-m+1} \geqq 2\lambda\,.$$

The last inequality means that for such integers m

$$\frac{u_{-m}}{u_{-m+1}} e^{\lambda} \leqq \frac{u_{-m-1}}{u_{-m}} e^{-\lambda}\,. \tag{20}$$

The definition of E, by (19), shows that $-m \in E$. Furthermore, by the monotonicity properties of u_{n+1}/u_n, there exists, if m is large enough, a $p \in E$ such that

$$\frac{u_{p-1}}{u_p} < \frac{u_{-m-1}}{u_{-m}} e^{-\lambda} \leqq \frac{u_p}{u_{p+1}}\,. \tag{21}$$

Since $-m \in E$, $p \in E$, we have $|c_{-m}| = u_{-m}^{-1}$, $|b_p| = u_p$. Thus (13), with m substituted to $m+p$, gives

$$\frac{u_p}{u_{-m}} = \left|\sum_{n \neq 0} b_{n+p} c_{n-m}\right|.$$

By (17) and (18) we obtain

$$\frac{u_p}{u_{-m}} \leqq \sum_{n \neq 0} \frac{u_{n+p}}{u_{n-m}}, \tag{22}$$

and we shall show that this gives a contradiction, if $q = m+p$ is large enough.

The right hand member of (22) can be written

$$\sum_{n<-p} + \sum_{\substack{-p \leqq n \leqq m \\ n \neq 0}} + \sum_{n<m} \frac{u_{n+p}}{u_{n-m}} = S_1 + S_2 + S_3\,.$$

(20) and (21) give

$$\frac{u_{p+1}}{u_{-m+1}} < e^{-\lambda}\frac{u_p}{u_{-m}},$$

and

$$\frac{u_{p-1}}{u_{-m-1}} < e^{-\lambda}\frac{u_p}{u_{-m}},$$

and since $\log u_{p+n}/u_{m-n}$ is concave for $n \in [-p, m]$, these inequalities give

$$\frac{u_{p+n}}{u_{-m+n}} \leqq e^{-|n|\lambda}\frac{u_p}{u_{-m}}, \tag{23}$$

$n \in [-p, m]$. Hence

$$S_2 \leqq \sum_{n \neq 0} e^{-|n|\lambda}\frac{u_p}{u_{-m}} = C\frac{u_p}{u_{-m}},$$

where

$$C = \sum_{n \neq 0} e^{-|n|\lambda} < \sum_{n \neq 0} 3^{-|n|} = 1 .$$

Thus, to obtain a contradiction in (22), it is enough to show that

$$u_{-m}u_p^{-1}S_1 \quad \text{and} \quad u_{-m}u_p^{-1}S_3$$

tend to 0, as $q = m+p \to \infty$. We prove this for the second of these products. The first is treated similarly.

Taking $n = m$ in (23), we find that

$$u_{m+p} = \frac{u_{m+p}}{u_0} \leqq e^{-|m|\lambda}\frac{u_p}{u_{-m}} \leqq \frac{u_p}{u_{-m}} .$$

Using this inequality, we obtain

$$u_{-m}u_p^{-1}S_3 \leqq \sum_{n>m} \frac{u_{n+p}}{u_{n-m}u_{m+p}} = \sum_{n>0} \frac{u_{n+q}}{u_n u_q} . \tag{24}$$

By the concavity of $\{\log u_n\}_0^\infty$,

$$\frac{u_{n+q}}{u_q} \leqq \frac{u_{n+1}}{u_1},$$

for $q \geqq 0$. Hence the terms in

$$\sum_{n>0} \frac{u_{n+q}}{u_n u_q}$$

are dominated by a constant times the corresponding terms in the convergent series (16). Furthermore, for fixed n,

$$\lim_{q\to\infty} \frac{u_{n+q}}{u_n u_q} \leqq \lim_{q\to\infty} \frac{u_{q+1}}{u_q} = 0 .$$

Hence the right hand member of (24) tends to 0, as $q \to \infty$, proving that

$$u_{-m} u_p^{-1} S_3 \to 0 ,$$

as $q \to \infty$.

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