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ORDER BOUNDED OPERATORS AND TENSOR PRODUCTS OF BANACH LATTICES

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Introduction.

In this paper we study relations between order bounded operators and the concepts of p-convexity and p-concavity and study the structure of the m- and ℓ-tensor products of Banach lattices, originally introduced by Schaefer [13]. Since it follows from Gordon and Lewis [1] and Schütt [14] that if X and Y are Banach lattices then neither $X \otimes_\pi Y$ nor $X \otimes_\varepsilon Y$ have local unconditional structure in general the two tensor products of Schaefer are different from the usual tensor products between Banach spaces.

In section 1 of the paper we prove some basic results on the connection between convexity and concavity in Banach lattices and order bounded operators, results which will be used frequently in section 2. We prove e.g. that if X is a p-convex Banach lattice, which is weakly sequentially complete, then every operator which has p-summing adjoint, is normable by X. This result is used to solve a problem on unconditional bases of $L_p(0,1)$, $1<p<2$ which has left over in [11].

In section 2 of the paper we investigate the basic properties of the tensor products mentioned above. Among other things we describe the tensor products in case the involved lattices are Köthe function spaces and this result shows to some extent that these tensor products are the most natural for Banach lattices.

We also show that if E is a Banach space and X is an order continuous Banach lattice then the dual of the m-tensor product $E \otimes_m X$ is naturally isomorphic to $E^* \otimes_m X^*$ provided E^* has the Radon–Nikodym property.

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Included in the section are also some results concerning permanence properties with respect to type, cotype, convexity and concavity. Finally we show that if X and Y are order continuous Banach lattices then the identity is an isomorphism between the m-tensor product $X \otimes_m Y$ and the ℓ-tensor product $X \otimes_\ell Y$ if and only if either both X and Y are L_p-spaces $1 \leqq p < \infty$ or both spaces are $c_0(\Gamma)$-spaces. This result also yields that if X and Y are order continuous rearrangement invariant function spaces on $[0,1]$, so that $X \otimes_m Y$ is rearrangement invariant on $[0,1] \times [0,1]$, then there is a p, $1 \leqq p < \infty$ such that up to renorming both X and Y are equal to $L_p(0,1)$.

Section 3 of the paper is devoted to studying when the m-tensor product of a Banach space E and a Banach lattice X has the uniform approximation property (u.a.p.). X is said to have the order u.a.p., if X has the u.a.p. and the approximating operators can be chosen to satisfy a certain order theoretical inequality (Definition 3.5) which somehow gives control of the modulus of the operators. We show that if E has the u.a.p. and X has the order u.a.p. then $E \otimes_m X$ has the u.a.p.

This is then used to prove an important result on the u.a.p. in Banach lattices, namely that for superflexive lattices the order u.a.p. is equivalent to the u.a.p. Let us here point out that it is still unknown whether the positive u.a.p. is equivalent to the u.a.p. for Banach lattices (even if we assume superflexivity).

We end the section by proving a result on the Grothendieck uniform approximation property (introduced in [2]) of $E \otimes_m X$ similar to the result above but without the "order" assumption.

0. Notation and preliminaries.

In this paper we shall use the terminology and notation commonly used in Banach space theory and the theory of Banach lattices, as it appears in [8] and [9]. All vector spaces are assumed to be over the reals unless otherwise stated.

If E and F are Banach spaces then we denote the space of all bounded operators from E to F by $B(E,F)$ and we let B_E denote the unit ball of E.

If $(\Omega, \mathcal{M}, \mu)$ is a measure space, X a Banach space and $1 \leqq p \leqq \infty$, then $L_p(\mu, X)$ is the space of all equivalence classes of measurable functions $f\colon \Omega \to X$, for which $\int \|f\|^p \, d\mu < \infty$ ($\operatorname{ess\,sup} \|f\| < \infty$ if $p = \infty$).

Let X be a Banach lattice and $x_1, x_2, \ldots, x_n \in X$ and let $\mathcal{H}_n$ denote the space of all continuous, 1-homogeneous real functions of $\mathbb{R}^n$ equipped with the topology of uniform convergence on compacta (under which $\mathcal{H}_n$ is a Banach lattice). It follows from the results in [9, section I.d] that there is a unique bounded linear operator $\tau\colon \mathcal{H}_n \to X$, which preserves the lattice operations and so that the image of the ith coordinate function by τ is equal to x_i for $i \leqq n$. If $f \in \mathcal{H}_n$, then we denote $\tau(f)$ by $f(x_1, x_2, \ldots, x_n)$, like in [9]. The calculus of 1-

homogeneous expression in Banach lattices was first developed by Krivine [5].

We recall that if $1 \leqq p < \infty$, then X is called p-convex (p-concave), if there is a constant $K \geqq 1$ so that

$$(*)\qquad \left\| \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p} \right\| \leqq K \left(\sum_{j=1}^{n} \|x_j\|^p \right)^{1/p} \quad \text{for all } x_1, x_2, \ldots, x_n \in X$$

$$\left(K^{-1} \left(\sum_{j=1}^{n} \|x_j\|^p \right)^{1/p} \leqq \left\| \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p} \right\| \quad \text{for all } x_1, x_2, \ldots, x_n \in X \right).$$

The smallest constant K which can be used in (*) is called the p-convexity constant (p-concavity constant) of X.

We also recall that if E is a Banach space, then an operator $T \in B(E, X)$ is called order bounded, if $T(B_E)$ is an order bounded subset of X. The space of all order bounded operators from E to X shall be denoted by $\mathscr{B}(E, X)$. If $T \in \mathscr{B}(E, X)$ then the order bounded norm $\|T\|_m$ is defined by

$$\|T\|_m = \inf\{\|z\| \mid |Tx| \leqq z\|x\| \quad \text{for all } x \in E\}.$$

$\mathscr{B}(E, X)$ is a Banach space under the norm $\|\cdot\|_m$, [11, Chapter IV].

If F is another Banach space and $T \in B(E, F)$, then T is called normable by X ([11]), if for all $S \in B(F, X)$ $ST \in \mathscr{B}(E, X)$. We define the norm $s_X(T)$ by

$$s_X(T) = \sup\{\|ST\|_m \mid S \in B(F, X);\ \|S\| \leqq 1\}.$$

Under the norm s_X the space $\mathscr{S}_X(E, F)$ of all operators from E to F, which are normable by X, is a Banach operator ideal ([11, Chapter IV]).

1. p-convexity, p-concavity and order bounded operators.

In this section we shall prove a few basic results on the connection between convexity and concavity in Banach lattices and order bounded operators, results which will be used frequently in the sequel.

Throughout the section, E and F will denote Banach spaces and X a Banach lattice. If $e_1, e_2, \ldots, e_n \in E$, then the function $f: \mathsf{R}^n \to \mathsf{R}$ defined by

$$f(t_1, t_2, \ldots, t_n) = \left\| \sum_{j=1}^{n} t_j e_j \right\| \quad \text{for all } t_1, t_2, \ldots, t_n \in \mathsf{R}$$

belongs to $\mathscr{H}_n$ and therefore the expression $\|\sum_{j=1}^{n} x_j e_j\|_E$ represents an element in X. (We shall omit the index E on the norm in E, if there is no possibility for misunderstandings.)

We start with a lemma, which can be proved as in [9, Remark after Theorem I.d.1] and the proof is therefore left to the reader.

1.1. LEMMA. *If* $x_1, x_2, \ldots, x_n \in X$ *and* $e_1, e_2, \ldots, e_n \in E$, *then*

$$\left\| \sum_{j=1}^{n} x_j e_j \right\| = \sup \left\{ \left| \sum_{j=1}^{n} x^*(e_j) x_j \right| \;\middle|\; x^* \in E^*, \|x^*\| \leqq 1 \right\}.$$

Lemma 1.1 shows in particular that if $T \in \mathscr{B}(E^*, X)$ is of the form $T = \sum_{j=1}^n x_j \otimes y_j$ where $(x_j)_{j=1}^n \subseteqq E$, $(y_j)_{j=1}^n \subseteqq X$ then

$$\|T\|_m = \left\| \left\| \sum_{j=1}^{n} y_j x_j \right\|_E \right\|_X,$$

a fact which will be useful in the sequel.

Let $1 \leqq p < \infty$ and let $(\Omega, \mathscr{M}, \mu)$ be a measure space. If $f \in L_p(\mu, X)$ is a simple function, say $f = \sum_{j=1}^n 1_{A_j} x_j$, where $(x_j)_{j=1}^n \subseteqq X$ and $(A_j)_{j=1}^n$ is a set of mutually disjoint measurable sets, then we put

$$\left(\int |f|^p \, d\mu \right)^{1/p} = \left(\sum_{j=1}^{n} \mu(A_j) |x_i|^p \right)^{1/p}.$$

We can now prove the following proposition.

1.2. PROPOSITION. *Let* X *be p-convex,* $1 \leqq p < \infty$ *and let* $f \in L_p(\mu, X)$. *If* $(s_n) \subseteqq L_p(\mu, X)$ *is a sequence of simple functions with* $\int \|f - s_n\|^p \, d\mu \to 0$ *for* $n \to \infty$, *then the sequence* $(\int |s_n|^p \, d\mu)^{1/p}$ *converges to a limit, which only depends on* f *and* p. *This limit is denoted* $(\int |f|^p \, d\mu)^{1/p}$ *and its satisfies the inequality*

$$\text{(i)} \qquad \left\| \left(\int |f|^p \, d\mu \right)^{1/p} \right\| \leqq K \left(\int \|f\|^p \, d\mu \right)^{1/p}$$

where K *is the p-convexity constant for* X.

PROOF. If $s \in L_p(\mu, X)$ is a simple function, then it follows directly from the definition and the p-convexity of X that

$$\text{(1)} \qquad \left\| \left(\int |s|^p \, d\mu \right)^{1/p} \right\| \leqq K \left(\int \|s\|^p \, d\mu \right)^{1/p}.$$

Hence if (s_n) is a sequence of simple functions with the stated properties, (1) gives that $((\int |s_n|^p \, d\mu)^{1/p})$ is a Cauchy sequence in X and hence convergent. If the limit is denoted by $(\int |f|^p \, d\mu)^{1/p}$ then (1) gives that (i) holds. Since $((\int |s_n|^p \, d\mu)^{1/p})$ is convergent for any sequence (s_n) with the stated properties the limit does not depend on (s_n).

REMARK. Let $(f_j)_{j=1}^n \subseteq L_p(\mu)$ and $(x_j)_{j=1}^n \subseteq X$; and put

$$f(t) = \sum_{j=1}^{n} f_j(t)x_j \qquad t \in \Omega .$$

The expression $(\int |f|^p \, d\mu)^{1/p}$ is 1-homogeneous and therefore defines an element in X, also when X is not p-convex. By approximating the f_j's by simple functions it is readily verified that $(\int |f|^p \, d\mu)^{1/p}$ agrees with the "integral" in Proposition 1.2 in case X is p-convex. If X is p-concave, then it is easy to see that

$$\left(\int \|f\|^p \, d\mu\right)^{1/p} \leqq K \left\| \left(\int |f|^p \, d\mu\right)^{1/p} \right\|$$

where f is as above and K is the p-concavity constant of X.

1.3. THEOREM. *Let X be weakly sequentially complete and p-convex with the bounded approximation property. Then every $T \in B(E,F)$ with p-summing adjoint is normably by X. Furthermore, there exists a constant K so that $s_X(T) \leqq K\pi_p(T^*)$ for all $T \in B(E,F)$ with T^* p-summing.*

PROOF. From [11, Theorem 4.16] and [9, Theorem I.c.4], it follows that it is enough to show that T^* is (x_n)-summing for every normalized sequence $(x_n) \subseteq X$ consisting of mutually disjoint elements with a constant independent of (x_n). Hence let (x_n) be such a sequence and we may without loss of generality assume that $x_n \geqq 0$ for all $n \in \mathsf{N}$. Since T^* is p-summing there is a probability measure μ on B_F^{**} so that

$$\|T^*y^*\| \leqq \pi_p(T^*)\left[\int |y^{**}(y^*)|^p \, d\mu(y^{**})\right]^{1/p} \tag{1}$$

for all $y^* \in F^*$.

If $y_1^*, y_2^*, \ldots, y_k^* \in F^*$ and K denotes the p-convexity constant of X, then by Proposition 1.2

$$\begin{aligned}
\left\| \sum_{j=1}^{k} \|T^*y_j^*\| x_j \right\| &\leqq \pi_p(T^*) \left\| \left(\int \left| \sum_{j=1}^{k} y^{**}(y_j^*)x_j \right|^p d\mu(y^{**})^{1/p} \right\| \\
&\leqq K\pi_p(T^*) \left(\int \left\| \sum_{j=1}^{k} y^{**}(y_j)x_j \right\|^p d\mu(y^{**})\right)^{1/p} \\
&\leqq K\pi_p(T^*) \sup\left\{ \left\| \sum_{j=1}^{k} y^{**}(y_j)x_j \right\| \;\middle|\; y^{**} \in F^{**}, \|y^{**}\| \leqq 1 \right\}.
\end{aligned} \tag{2}$$

(2) shows that T^* is (x_n)-summing with a constant less than or equal to $K\pi_p(T^*)$.

The theorem has the following interesting corollary, which should be compared with [11, Theorem 5.9]:

1.4. COROLLARY. *Assume that X is p-concave, $1 \leqq p < \infty$. If $T \in \mathscr{B}(E, X)$, then T^* is normable by any p-convex, weakly sequentially complete Banach lattice Y with the bounded approximation property.*

PROOF. If $T \in \mathscr{B}(E, X)$, then T admits a factorization

$$\begin{tikzcd}
E \arrow[rr, "T"] \arrow[dr, "T_1"'] & & X \\
& C(S) \arrow[ur, "T_2"'] &
\end{tikzcd}$$

where S is compact, $\|T_1\| \leqq 1$ and $T_2 \geqq 0$. Since X is p-concave, T_2 is p-summing [9], but then T and hence T^{**} is p-summing, and the result follows from Theorem 1.3.

ADDED IN PROOF. It has been proved in [18] and independently in [19] that Theorem 1.3 and Corollary 1.4 hold without the assumption that X has the bounded approximation property.

As a corollary of Theorem 1.3 we can give a complete answer to problem 6.7 of [11].

If (x_n) is an unconditional basis of X then we say that an operator is normable by (x_n), if it is normable, when X is equipped with the Banach lattice structure defined by (x_n).

1.5. COROLLARY. *Let (x_n) be an unconditional basis for $L_p(0, 1)$, $1 < p \leqq 2$, and let $T \in B(E, F)$. Then T is normable by (x_n) if and only if T^* is p-summing.*

PROOF. By a result of Johnson, Maurey, Schechtman, and Tzafriri [3] (see also [9, Theorem I.d.7]) $L_p(0, 1)$, $1 < p \leqq 2$ is p-convex, when $L_p(0, 1)$ is equipped with the lattice structure defined by (x_n). Hence if T^* is p-summing T is normable by (x_n) by Theorem 1.3. The "only if" part follows from [11, Proposition 6.5].

The final result of this section is

1.6. PROPOSITION. *Let $p_0 = \sup \{p \mid X \text{ is } p\text{-convex}\}$. Every operator which is normable by X has p_0-summing adjoint.*

PROOF. By a result of Krivine [5], X contains $(l_{p_0}^n)$ uniformly on disjoint elements. If $T \in \mathscr{S}_X(E,F)$, then by [11], T is normable by l_{p_0}, and hence T^* is p_0-summing.

2. Tensor products of Banach lattices.

In this section we wish to study the m-and ℓ-products of Banach lattices introduced by Schaefer [13] in further detail. These tensor products are again Banach lattices, if the two factors are, and we shall here investigate the basic properties of them with respect to permanence, etc. and their mutual relation.

In the sequel we let X and Y be two Banach lattices and E a Banach space. We start by recalling the definition of the two tensor products.

We can consider the algebraic tensor product $E \otimes X$ as a subspace of $\mathscr{B}(E^*, X)$ and likewise we may consider $X \otimes E$ as a subspace of

$$\mathscr{B}^*(X^*, E) = \{T \in B(X^*, E) \mid T^* \in \mathscr{B}(E^*, X)\},$$

which is a Banach space under the norm $\|T\|_\ell = \|T^*\|_m$ for all $T \in \mathscr{B}^*(X^*, E)$. It is readily verified that both $\|\cdot\|_m$ and $\|\cdot\|_\ell$ are crossnorms on $E \otimes X$ respectively $X \otimes E$. We introduce the following definition:

2.1. DEFINITION. The m-tensor product $E \otimes_m X$ is the closure of $E \otimes X$ in $\mathscr{B}(E^*, X)$ and the ℓ-tensor product $X \otimes_\ell E$ is the closure of $X \otimes E$ in $\mathscr{B}^*(X^*, E)$.

REMARK. Clearly, the map $T \to T^*$ defines an isometry of $E \otimes_m X$ onto $X \otimes_\ell E$ and hence it may seem artificial to introduce two tensor products. However, if X and Y are Köthe function spaces and one wants to define a lattice tensor product between X and Y, then there are two natural candidates "X with image in Y" and "Y with image in X". As we shall see later these two situations correspond precisely to the two tensor products defined above. Also it seems to us that several results of this section become more clear (especially in case $X = Y$) when both tensor products are defined.

It follows from Schaefer [13 IV Theorem 7.2] that both $X \otimes_m Y$ and $X \otimes_\ell Y$ are Banach lattices under their respective norms and the canonical order between operators.

He also proved that

$$\mathscr{B}(X, Y) \quad \text{and} \quad \mathscr{B}^*(X, Y) = \{T \in B(X, Y) \mid T^* \in \mathscr{B}(Y^*, X^*)\}$$

with the norm $\|T\|_\ell = \|T^*\|_m$ for $T \in \mathscr{B}^*(X, Y)$ are Banach lattices, provided that there is a contractive positive projection of Y^{**} onto Y, [13 IV Theorem 4.3].

Theorem 9.4 in chapter IV of [13] shows that $(E \otimes_m X)^*$ is naturally isometric to (lattice isometric to in case E is a Banach lattice) $\mathscr{B}(E, X^*)$; if $S \in \mathscr{B}(E, X^*)$ and $T \in E \otimes X$ then the duality is given by $\langle S, T \rangle = \text{trace}\,(S^*T)$. An analoguous statement holds for $X \otimes_\ell E$.

We start by examining the space $E \otimes_m X$ in case X is a Köthe function space on a probability space $(\Omega, \mathscr{M}, \mu)$ ([9]). In that case $X(E)$ denotes the space of all measurable functions $f: \Omega \to E$ for which $\|f(\cdot)\|_E \in X$. If $f \in X(E)$, then we put $\|f\|_{X(E)} = \| \|f(\cdot)\|_E \|_X$. This is readily seen to be a norm on $X(E)$ turning it into a Banach space. $X(Y)$ is readily seen to be a Banach lattice, and if X and Y are Köthe function spaces on probability spaces $(\Omega_1, \mathscr{M}_1, \mu_1)$ respectively $(\Omega_2, \mathscr{M}_2, \mu_2)$ then it is easy to see that $X(Y)$ can be identified with the space of those $\mu_1 \times \mu_2$-measurable functions f, for which $f(t, \cdot) \in Y$ and the function $t \to \|f(t, \cdot)\|_Y$ belongs to X. When this space is equipped with the norm $\| \|f(\cdot, \cdot)\|_Y \|_X$ the identification is a lattice isometry.

In [11] it is proved that if X is a Köthe function space, then $\mathscr{B}(E, X)$ can be identified with the space of certain ω^*-scalary measurable functions with image in E^*. The following theorem shows that for elements in the m-tensor product we get strongly measurable functions, therefore it also seems natural to consider this tensor product in full generality.

2.2. Theorem. *Let $(\Omega, \mathscr{M}, \mu)$ be a probability space and let X be a Köthe function space on $(\Omega, \mathscr{M}, \mu)$ with order continuous norm. For every $T \in E \otimes_m X$ there is a $\varphi_T \in X(E)$ so that for all $x^* \in E^*$*

(i) *$(Tx^*)(t) = \langle x^*, \varphi_T(t) \rangle$ for almost all $t \in \Omega$.*

(ii) *The map $T \to \varphi_T$ is an isometry of $E \otimes_m X$ onto $X(E)$ (a lattice isometry if E is a Banach lattice).*

Proof. Let $T \in E \otimes X$, say $T = \sum_{j=1}^n x_j \otimes f_j$, where $(x_j)_{j=1}^n \subseteq E$ and $(f_j)_{j=1}^n \subseteq X$. Define $\varphi_T \in X(E)$ by

$$(1) \qquad \varphi_T(t) = \sum_{j=1}^{n} f_j(t) x_j \quad \text{for almost all } t \in \Omega\,.$$

Clearly we get for all $x^* \in E^*$

$$(2) \qquad (Tx^*)(t) = \langle x^*, \varphi_T(t) \rangle \quad \text{for almost all } t \in \Omega\,.$$

(1) and (2) give immediately that φ_T is uniquely determined by T, which shows that $T \to \varphi_T$ is linear. From (2) we get immediately that $\|T\|_m = \|\varphi_T\|_{X(E)}$ and therefore $T \to \varphi_T$ can be extended to an isometry from $E \otimes_m X$ into $X(E)$. We have to show that (2) holds for all of $E \otimes_m X$, but if $T \in E \otimes_m X$ we can define $S \in \mathscr{B}(E^*, X)$ by

(3) $$(Sx^*)(t) = \langle x^*, \varphi_T(t)\rangle, \qquad x^* \in E^*, \text{ almost all } t \in \Omega .$$

If $(T_n) \subseteq E \otimes X$ with $T_n \to T$, then

$$\|T_n - S\|_m = \|\varphi_T - \varphi_{T_n}\|_{X(E)} = \|T - T_n\|_m ,$$

so that $T = S$.

Let $\varphi \in X(E)$, since X is order continuous we can find a sequence of simple functions $(\varphi_n) \subseteq X(E)$ with $\|\varphi - \varphi_n\|_{X(E)} \to 0$ for $n \to \infty$. Define for all n

(4) $$(T_n x^*)(t) = \langle x^*, \varphi_n(t)\rangle, \qquad x^* \in E^*, \text{ almost all } t \in \Omega$$

(5) $$(Tx^*)(t) = \langle x^*, \varphi(t)\rangle, \qquad x^* \in E^*, \text{ almost all } t \in \Omega .$$

Clearly $T_n \to T$ in $\mathscr{B}(E^*, X)$ so that $T \in E \otimes_m X$. It is obvious that $T \to \varphi_T$ preserves the lattice operations.

Since $Y \otimes_\ell X$ is lattice isometric to $X \otimes_m Y$ by the map $T \to T^*$ the remarks above and Theorem 2.2 show that if X and Y are Köthe function spaces over the probability spaces $(\Omega_1, \mathscr{M}_1, \mu_1)$, respectively $(\Omega_2, \mathscr{M}_2, \mu_2)$ then we may identify $Y \otimes_\ell X$ with the space pf those $\mu_1 \times \mu_2$-measurable functions f: $\Omega_1 \times \Omega_2 \to \mathsf{R}$ for which $f(\cdot, s) \in X$ for almost all $s \in \Omega_2$ and the function $s \to \|f(\cdot, s)\|_X$ belongs to Y. The norm of an element f in the space is given by $\| \|f(\cdot,\cdot)\|_X\|_Y$.

We now wish to characterize $X \otimes_m Y$ in case the order in X is defined by an unconditional basis (e_n) with unconditional constant 1. We start with the lemma:

2.3. Lemma. *For every $T \in X \otimes_m Y$ there is a unique sequence $(x_n) \subseteq Y$ so that $T = \sum_{n=1}^\infty e_n \otimes x_n$, where the series converges unconditionally in $X \otimes_m Y$.*

Proof. For all $n \in \mathsf{N}$ we define

(1) $$Z_n = \operatorname{span}\{e_k \otimes x \mid k \leqq n,\ x \in Y\} ,$$

(2) $$Z = \operatorname{span}\{e_k \otimes x \mid k \in \mathsf{N},\ x \in Y\} .$$

Clearly $\bar{Z} = X \otimes_m Y$, and since (e_n) is unconditional we have for all sequences $(x_j) \subseteq X$ and all $n \leqq m$

(3) $$\left\|\sum_{j=1}^n e_j \otimes x_j\right\|_m = \left\|\left\|\sum_{j=1}^n |x_j| e_j\right\|_X\right\|_Y \leqq \left\|\left\|\sum_{j=1}^m x_j e_j\right\|_X\right\|_Y = \left\|\sum_{j=1}^m e_j \otimes x_j\right\|_m .$$

(3) shows that if $P_n: Z \to Z_n$ is defined by

(4) $$P_n\left(\sum_j e_j \otimes x_j\right) = \sum_{j=1}^n e_j \otimes x_j \qquad \text{for all } \sum_j e_j \otimes x_j \in Z ,$$

then P_n is a projection of norm one and therefore it extends uniquely to a projection of $X \otimes_m Y$ onto Z_n, also denoted P_n.

Clearly $P_{n+1}P_n = P_n$ for all $n \in \mathsf{N}$. Since $P_n\mathscr{U} \to \mathscr{U}$ for all $\mathscr{U} \in Z$, it follows that also $P_n\mathscr{U} \to \mathscr{U}$ for all $\mathscr{U} \in X \otimes_m Y$. If $\mathscr{U} \in X \otimes_m Y$ we can find a sequence $(x_n) \subseteq Y$ so that $e_n \otimes x_n = (P_n - P_{n-1})\mathscr{U}$, $n > 1$ and $e_1 \otimes x_1 = P_1\mathscr{U}$. It follows from the above that

$$\mathscr{U} = \sum_{n=1}^{\infty} e_n \otimes x_n .$$

The unconditional convergence of the series follows from (3). It is clear that (x_n) is uniquely determined by $\mathscr{U}$.

We define

$$Y(X)^{\sim} = \left\{ (x_n) \subseteq Y \;\middle|\; \sup_n \left\| \left\| \sum_{j=1}^{n} |x_k| e_k \right\|_X \right\|_Y < \infty \right\},$$

where the norm of an element $(x_n) \in Y(X)^{\sim}$ is defined by $\sup_n \| \|\sum_{k=1}^{n} |x_k| e_k\|_X \|_Y$. Under this norm $Y(X)^{\sim}$ is a Banach lattice. We denote by $Y(X)$ the closure in $Y(X)^{\sim}$ of the sequences which are eventually zero. $Y(X)$ is readily seen to be a Banach latice. We can now prove

2.4. THEOREM. *If the order in X is induced by an unconditional basis (e_n) with constant one, then there is a lattice isometry I of $Y(X)$ onto $X \otimes_m Y$ so that*

(i) $$If = \sum_{n=1}^{\infty} e_n \otimes f(n) \quad \text{for all } f \in Y(X) .$$

PROOF. Let $f \in Y(X)$ and define $f_k \in Y(X)$ for all k by $f_k(n) = f(n)$, $n \leq k$ and $f_k(n) = 0$, $n > k$. It is easy to see that $f_k \to f$ in $Y(X)$. If $k \leq m$, then

(1) $$\|If_m - If_k\|_m = \left\| \left\| \sum_{n=1}^{m} |f_m(n) - f_k(n)| e_n \right\|_X \right\|_Y = \|f_m - f_k\|_{Y(X)} .$$

(1) shows that the series $\sum_{n=1}^{\infty} e_n \otimes f(n)$ is convergent in $X \otimes_m Y$ so that I is well-defined.

If $f \in Y(X)$ then

$$\|If\|_m = \lim_{k \to \infty} \left\| \sum_{n=1}^{k} e_n \otimes f(n) \right\|_m = \lim_{k \to \infty} \left\| \left\| \sum_{n=1}^{k} |f(n)| e_n \right\|_X \right\|_Y = \|f\|_{Y(X)}$$

so that I is an isometry.

If $T \in X \otimes_m Y$, then we can find a sequence $(x_n) \subseteq Y$ so that $T = \sum_{n=1}^{\infty} e_n \otimes x_n$, but again

$$\|T\|_m = \lim_k \left\| \left\| \sum_{n=1}^{k} |x_n| e_n \right\|_X \right\|_Y$$

so that $(x_n) \in Y(X)^{\sim}$. However, since $\sum_{n=1}^{k} e_n \otimes x_n$ converges to T it is clear that $(x_n) \in Y(X)$, and hence I is onto. It is readily seen that I is a lattice isomorphism.

In [9] the space $Y(l_p)$ was defined and this space has a lot of applications in the field. As a special case of Theorem 2.4 we can state:

2.5. COROLLARY. *The space $Y(l_p)$, $1 \leqq p < \infty$ is lattice isometric to $l_p \otimes_m Y$.*

$E^* \otimes_m X$ can be considered as a subspace of $\mathscr{B}(E, X)$ and our next theorem gives a necessary and sufficient condition for equality.

2.6. THEOREM. *$E^* \otimes_m X = \mathscr{B}(E, X)$ for all X with order continuous norm if and only if E^* has the Radon–Nikodym property* (RNP).

PROOF. The proof is similar to that of Theorem 4.19 of [11].

Assume that E^* has RNP and that X is order continuous and let $A \in \mathscr{B}(E, X)$. Then there is a compact set K and operators $S \in B(E, C(K))$ and $T \in B(C(K), X)$ with $\|S\| \leqq 1$ and $T \geqq 0$ so that $A = TS$.

Since X is order continuous T is weakly compact [11, Proposition 2.4], and therefore T^* is weakly compact as well. Hence by the characterization of weakly subsets of $C(K)^*$ we can find a positive measure $\mu \in C(K)^*$ with $T^*X^* \subseteqq L_1(\mu)$. ($L_1(\mu)$ is here considered as a subspace of $C(K)^*$ via the Radon–Nikodym theorem.) Since E^* has RNP there is a $g \in L_\infty(\mu, E^*)$ so that

$$(1) \qquad S^*f = \int fg\, d\mu \quad \text{for all } f \in L_1(\mu)$$

which gives for every $x \in E$

$$(2) \qquad (Sx)(t) = \langle g(t), x \rangle \quad \text{for almost all } t \in K\,.$$

We can now find a sequence of E^*-valued simple functions so that $g_n \to g$ in μ-measure and $\|g_n(t)\| \leqq 2\|g(t)\|$ for almost all $t \in K$. We define $S_n \colon E \to L_\infty(\mu)$ by

$$(3) \qquad (S_n x)(t) = \langle g_n(t), x \rangle \quad \text{for all } x \in E \text{ and almost all } t \in K\,.$$

By the weak compactness of T we get that $T^{**}(C(K)^{**}) \subseteqq X$ and since $T^*X^* \subseteqq L_1(\mu)$ it follows that we may consider T^{**} as an operator from $L_\infty(\mu)$

to X. Hence we can define $A_n\colon E \to X$ by $A_n = T^{**}S_n$ for all $n \in \mathbf{N}$. Since g_n is simple it is readily verified that $A_n \in E^* \otimes X$ for every $n \in \mathbf{N}$.

If $x \in E$, $\|x\| \leqq 1$ and $y^* \in X^*$, $y^* \geqq 0$, $\|y^*\| \leqq 1$ we get the following estimate:

$$(4)\qquad |\langle y^*, Ax - A_n x\rangle| = |\langle y^*, T^{**}(Sx - S_n x)\rangle| = |\langle T^* y^*, Sx - S_n x\rangle| \leqq \int (T^* y^*)(t) \|g(t) - g_n(t)\|\, d\mu(t) = \langle y^*, T^{**}(\|g(\cdot) - g_n(\cdot)\|)\rangle$$

and therefore

$$(5)\qquad \|A - A_n\|_m \leqq \|T^{**}(\|g(\cdot) - g_n(\cdot)\|)\| \quad \text{for all } n \in \mathbf{N}.$$

The proof will be complete, if we show that the right hand side of (5) tends to 0. To this end let $\varepsilon > 0$ be arbitrary. Since T^* is weakly compact we can find a $\delta > 0$ so that for all measurable $C \subseteq K$ we have

$$(6)\quad \mu(C) \leqq \delta \Rightarrow 3\|S\| \int_C |T^* y^*|\, d\mu \leqq \varepsilon \quad \text{for all } y^* \in X^*,\ \|y^*\| \leqq 1.$$

For every $n \in \mathbf{N}$ we put

$$(7)\qquad C_n = \{t \in K \mid \|g(t) - g_n(t)\| > \varepsilon\}$$

and determine n_0 so that $\mu(C_n) \leqq \delta$ for all $n \geqq n_0$. Hence for $n \geqq n_0$:

$$(8)\qquad \|T^{**}(\|g(\cdot) - g_n(\cdot)\|)\| = \sup_{\|y^*\| \leqq 1} \left| \int (T^* y^*)(t) \|g(t) - g_n(t)\|\, d\mu(t) \right| \leqq \varepsilon + \varepsilon \sup_{\|y^*\| \leqq 1} \int_{K \setminus C_n} |(T^* y^*)(t)|\, d\mu(t) \leqq \varepsilon(\|T\| + 1),$$

and we have proved what we wanted.

Assume next that $E^* \otimes_m X = \mathscr{B}(E, X)$ for all X with order continuous norm. Let $(\Omega, \mathscr{S}, \mu)$ be a finite measure space and $T \in B(L_1(\mu), E^*)$ with $\|T\| = 1$. If $I\colon L_\infty(\mu) \to L_1(\mu)$ denotes the formal identity the operator $S = IT^*|_E$ is an order bounded operator and therefore by assumption $S \in E^* \otimes_m L_1(\mu)$. By Theorem 2.2 there is a $g \in L_1(\mu, E^*)$ so that

$$(9)\qquad (Sx)(t) = \langle g(t), x\rangle \quad \text{for all } x \in E \text{ and almost all } t \in \Omega.$$

Since $|Sx| \leqq 1$ for all $x \in E$, $\|x\| \leqq 1$ we get that $\|g(t)\| \leqq 1$ for almost all $t \in \Omega$, so that $g \in L_\infty(\mu, E^*)$. Hence for all $f \in L_1(\mu)$ and all $x \in E$

$$(10)\qquad \langle Tf, x\rangle = \langle f, T^* x\rangle = \int f(t)(T^* x)(t)\, d\mu(t) = \int f(t)\langle g(t), x\rangle\, d\mu(t) = \left\langle \int f(t) g(t)\, d\mu(t), x \right\rangle$$

so that

$$Tf = \int fg\,d\mu \quad \text{for all } f \in L_1(\mu)\,, \tag{11}$$

which shows that E^* has RNP.

The following corollary is a generalization of the fact that $L_p(\mu, E)^* = L_q(\mu, E^*)$ for all measures μ and all $1 \leqq p < \infty$, $p^{-1} + q^{-1} = 1$ if and only if E^* has RNP.

2.7. Corollary. If X^* is order continuous and E^* has RNP then $(E \otimes_m X)^* = E^* \otimes_m X^*$. Especially if both E and X are reflexive, then $E \otimes_m X$ is reflexive as well.

Proof. We have $(E \otimes_m X)^* = \mathscr{B}(E, X^*)$ and hence the statement follows immediately from Theorem 2.6.

2.8. Remark. The condition X^* order continuous is essential. Indeed, if $X = L_\infty(\mu)$, then if E^* has the RNP

$$(E \otimes_m L_1(\mu))^* = \mathscr{B}(E, L_\infty(\mu)) = L_\infty(\mu, E) \neq E^* \otimes_m L_\infty(\mu)\,.$$

Now we turn our attention to investigate, what can be said about the type or convexity of $E \otimes_m X$ and $X \otimes_m Y$ in case we know the type or convexity of E, X, and Y. We have the following two results.

2.9. Theorem

(i) *If E is of type p, $1 \leqq p \leqq 2$ and X is p-convex and q-concave for some $q < \infty$, then $E \otimes_m X$ is of type p.*

(ii) *If E is of cotype p, $2 \leqq p < \infty$ and X is p-concave, then $E \otimes_m X$ is of cotype p.*

Proof. Assume (i). We wish to show that there exists a constant K, so that for all finite sets $\{T_1, T_2, \ldots, T_n\} \subseteqq E \otimes_m X$ we have

$$\left(\int_0^1 \left\| \sum_{k=1}^n r_k(t) T_k \right\|_m^q dt \right)^{1/q} \leqq K \left(\sum_{k=1}^n \|T_k\|_m^p \right)^{1/p} \tag{1}$$

where (r_k) denotes the sequence of Rademacher functions on $[0, 1]$. Since E is of type p, there is a constant $K_{p,q}$ so that for all finite sets $\{x_j^k \mid 1 \leqq j \leqq m, 1 \leqq k \leqq n\} \subseteqq E$ and all $\{t_j^k \mid 1 \leqq j \leqq m, 1 \leqq k \leqq m\} \subseteqq \mathbb{R}$ we have

$$\left(\int_0^1 \left\| \sum_{k=1}^n r_k(t) \sum_{j=1}^m t_j^k x_j^k \right\|^q dt \right)^{1/q} \leqq K_{p,q} \left(\sum_{k=1}^n \left\| \sum_{j=1}^m t_j^k x_j^k \right\|^p \right)^{1/p}. \tag{2}$$

Since each side is 1-homogeneous we get from the remark just after Theorem 1.2 that

$$\left(\int_0^1 \left\|\sum_{k=1}^n r_k(t) \sum_{j=1}^m y_j^k x_j^k\right\|^q dt\right)^{1/q} \leqq K_{p,q}\left(\sum_{k=1}^n \left\|\sum_{j=1}^m y_j^k x_j^k\right\|^p\right)^{1/p} \tag{3}$$

for all finite sets $\{y_j^k \mid 1 \leqq j \leqq m,\ 1 \leqq k \leqq n\} \subseteq X$. Of continuity reasons it is enough to prove (1) when $T_1, T_2, \ldots, T_n \in E \otimes X$, and hence let

$$T_k = \sum_{j=1}^m x_j^k \otimes y_j^k, \quad 1 \leqq k \leqq n\,,$$

where $\{x_j^k \mid j \leqq m,\ k \leqq n\} \subseteq E$, $\{y_j^k \mid j \leqq m,\ k \leqq n\} \subseteq X$. If C_p denotes the p-convexity constant and C_q the q-concavity constant of X we get using the remark after Theorem 1.2 and (3):

$$\begin{aligned}
&\left(\int_0^1 \left\|\sum_{k=1}^n r_k(t) \sum_{j=1}^m x_j^k \otimes y_j^k\right\|_m^q dt\right)^{1/q} \\
&= \left(\int_0^1 \left\|\left\|\sum_{k=1}^n r_k(t) \sum_{j=1}^m y_j^k x_j^k\right\|_E\right\|_X^q dt\right)^{1/q} \\
&\leqq C_q \left\|\left(\int_0^1 \left\|\sum_{k=1}^n r_k(t) \sum_{j=1}^m y_j^k x_j^k\right\|_E^q dt\right)^{1/q}\right\|_X \\
&\leqq K_{p,q} C_q \left\|\left(\sum_{k=1}^n \left\|\sum_{j=1}^m y_j^k x_j^k\right\|_E^p\right)^{1/p}\right\|_X \\
&\leqq K_{p,q} C_p C_q \left(\sum_{k=1}^n \left\|\left\|\sum_{j=1}^m y_j^k x_j^k\right\|_E\right\|_X^p\right)^{1/p} \\
&= K_{p,q} C_p C_q \left(\sum_{k=1}^n \left\|\sum_{j=1}^m x_j^k \otimes y_j^k\right\|_m^p\right)^{1/p}.
\end{aligned} \tag{4}$$

The statement on cotype p can be proved in a similar manner.

REMARK. One could hope that the theorem above was true, if we just assumed that X was of type p. As was pointed out to the authors by B. Maurey it is not so in general. Indeed, assume that $l_p \otimes_m X$ is of type p, $1 \leqq p \leqq 2$ with constant K, let (e_k) denote the unit vector basis of l_p and let $(x_k)_{k=1}^n \subseteq X$. Then by Lemma 2.3

$$\begin{aligned}
\left\|\left(\sum_{k=1}^n |x_k|^p\right)^{1/p}\right\| &= \left\|\sum_{k=1}^n e_k \otimes x_k\right\|_m \\
&= \left(\int_0^1 \left\|\sum_{k=1}^n r_k(t) e_k \otimes x_k\right\|_m^p dt\right)^{1/p} \leqq K\left(\sum_{k=1}^n \|x_k\|^p\right)^{1/p}
\end{aligned}$$

which shows that X is p-convex.

2.10. THEOREM. *If Y is order continuous and X and Y are p-convex $1 \leqq p < \infty$, then $X \otimes_m Y$ is p-convex. The analoguous statement holds for p-concavity.*

PROOF. We prove only the first statement, since the other one can be proved similarly.

Since Y is order continuous we may assume without loss of generality that Y is a Köthe function space on a measure space $(\Omega, \mathcal{M}, \mu)$ [9, I.b.14].

Hence let $(f_j)_{j=1}^k \subseteq Y(X)$. Since X is p-convex there is a constant K_1 so that

$$(1) \qquad \left\| \left(\sum_{j=1}^k |f_j(t)|^p \right)^{1/p} \right\| \leqq K_1 \left(\sum_{j=1}^k \|f_j(t)\|^p \right)^{1/p} \quad \text{for almost all } t \in \Omega .$$

Since Y is p-convex with constant K_2 say, then by Theorem 2.2 and (1)

$$\left\| \left(\sum_{j=1}^k |f_j|^p \right)^{1/p} \right\|_m = \left\| \left\| \left(\sum_{j=1}^k |f_j(\cdot)|^p \right)^{1/p} \right\|_X \right\|_Y$$
$$\leqq K_1 K_2 \left(\sum_{j=1}^k \|f_j\|_{Y(X)}^p \right)^{1/p} .$$

REMARK. A slightly more complicated proof shows that it is possible to drop the hypothesis "Y order continuous".

We now wish to examine the relation between the m-norm and ℓ-morm in tensor products of Banach lattices.

2.11. THEOREM. *Let X and Y be order continuous. The identity in $X \otimes Y$ can be extended to an isomorphism I of $X \otimes_m Y$ into $X \otimes_\ell Y$ if and only if either*

(i) *there is a p, $1 \leqq p < \infty$ and measures μ and ν, so that X is lattice isomorphic to $L_p(\mu)$, and Y is lattice isomorphic to $L_p(\nu)$,*

or

(ii) *there exist sets Γ_1 and Γ_2 with X lattice isomorphic to $c_0(\Gamma_1)$, and Y lattice isomorphic to $c_0(\Gamma_2)$.*

PROOF. Assume that I is an isomorphism. There is then a constant $K \geqq 1$ so that

$$(1) \qquad K^{-1}\|T^*\|_m \leqq \|T\|_m \leqq K\|T^*\|_m \quad \text{for all } T \in X \otimes_m Y .$$

If $T = \sum_{k=1}^n x_k \otimes y_k \in X \otimes Y$, then (1) gives

$$(2) \qquad K^{-1} \left\| \left\| \sum_{k=1}^n x_k y_k \right\|_Y \right\|_X \leqq \left\| \left\| \sum_{k=1}^n y_k x_k \right\|_X \right\|_Y \leqq K \left\| \left\| \sum_{k=1}^n x_k y_k \right\|_Y \right\|_X .$$

According to Corollary 2.12 of [11], which follows from a result of Zippin [17], it is enough to show that if $(z_k) \subseteq X$ and $(y_k) \subseteq Y$ are normalized sequences each consisting of mutually disjoint elements, then (z_k) and (y_k) are equivalent as basic sequences. If (t_k) is a sequence of scalar and we define $x_k = t_k z_k$, $k \in \mathbf{N}$, then by (2)

$$K^{-1}\left\|\sum_{k=1}^{n} t_k z_k\right\| = K^{-1}\left\|\left\|\sum_{k=1}^{n} x_k y_k\right\|_Y\right\|_X \leqq \left\|\left\|\sum_{k=1}^{n} y_k x_k\right\|_X\right\|_Y \tag{3}$$

$$= \left\|\sum_{k=1}^{n} t_k y_k\right\| \leqq K\left\|\sum_{k=1}^{n} t_k z_k\right\| \quad \text{for all } n \in \mathbf{N},$$

where the equalities in X and Y are obtained using the order continuity of X and Y (see [11, Proposition 2.7). (3) gives that (z_k), and (y_k) are equivalent.

The "if"-part is actually Kwapien's formulation [7] of the Schwartz duality theorem. Indeed, if $1 \leqq p < \infty$ and $T \in L_p(\mu) \otimes_m L_p(\nu)$, then T is p-summing by Kwapien's result and hence T^* is normable by $L_p(\mu)$ and hence order bounded. Since the role of T and T^* can be interchanged we are done. For $c_0(\Gamma_1) \otimes_m c_0(\Gamma_2)$ the statement is trivial, since an operator with image in a $c_0(\Gamma)$-space is order bounded if and only if it is compact.

2.12. Corollary. *Let $(\Omega, \mathcal{M}, \mu)$ be a measure space, let X be order continuous and $1 \leqq p < \infty$. Consider the statements:*

(i) *The identity $I\colon X \otimes_m L_p(\mu) \to X \otimes_\ell L_p(\mu)$ is continuous,*
(ii) *X is p-convex.*

Then (i) $\Rightarrow$ (ii). *If X is weakly sequentially complete and has the bounded approximation property, then* (i) $\Leftrightarrow$ (ii).

Proof. Assume the (i) holds and let $x_1, x_2, \dots, x_n \in X$ be arbitrary. Choose $(y_k)_{k=1}^{\infty} \subseteqq L_p(\mu)$ normalized and mutually disjoint. By assumption the left inequality in (4) of theorem 2.11 holds. Hence

$$\left\|\left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p}\right\| = \left\|\left\|\sum_{k=1}^{n} x_k y_k\right\|_p\right\|_X \leqq K\left(\sum_{k=1}^{n} \|x_k\|^p\right)^{1/p}$$

so that X is p-convex.

Assume next that X is weakly sequentially complete and has the bounded approximation property. If X is p-convex and $T \in X \otimes_m L_p(\mu)$, then T is p-summing, but from Theorem 1.3 it then follows that T^* is X-order bounded. This shows that (i) holds.

It follows from Theorem 2.2 and the remark just before the theorem that if X and Y are Köthe function spaces on $I = [0, 1]$ with the Lebesgue measure, then $X \otimes_m Y$ can be considered as a Köthe function space on I^2. One may ask when this space is rearrangement invariant. The answer is contained in

2.13. COROLLARY. *Let* X and Y *be two order continuous rearrangement invariant function spaces on* $[0,1]$. *If* $X \otimes_m Y = Y(X)$ *is a rearrangement invariant function space on* I^2, *then there is a* p, $1 \leqq p < \infty$ *so that the formal identity operators from* X *and* Y *to* $L_p(0,1)$ *are isomorphisms.*

PROOF. We realize $X \otimes_m Y$ and $X \otimes_\ell Y$ as function spaces on I^2 as in the remarks just before Theorem 2.2. We wish to show that the identity is an isomorphism of $X \otimes_m Y$ onto $X \otimes_\ell Y$.

Let $\varphi \colon I^2 \to I^2$ be defined by $\varphi(t,s) = (s,t)$ for $(t,s) \in I^2$. Clearly φ is measure preserving and hence by assumption $h \circ \varphi \in X \otimes_m Y$ with $\|h \circ \varphi\|_m = \|h\|_m$ for all $h \in X \otimes_m Y$. This means however that

$$\|h\|_m = \| \|h(t,s)\|_s \|_t = \| \|h(s,t)\|_s \|_t = \|h\|_\ell$$

for all $h \in X \otimes_m Y$.

By Theorem 2.11 both X and Y are lattice isomorphic to L_p-spaces (the case c_0 is excluded here). By [9, 2.b.3] it then follows that both X and Y are isomorphic to $L_p(0,1)$ via the identity.

Let us end the section by briefly discussing a third tensor product of Banach lattices.

We recall that if $T \in B(X,Y)$ then T is called regular if T maps order bounded sets to order bounded sets. If Y is order complete the space $R(X,Y)$ of all regular operators from X to Y consists precisely of the space of operators for which the modulus exist. With the norm $\|T\|_r = \| |T| \|$ for $T \in R(X,Y)$ the space is a Banach lattice under the canonical order. $X \otimes Y$ can be considered as a subspace of $R(X,Y^*)^*$ by

$$\left(\sum_{j=1}^n x_j \otimes y_j \right)(S) = \sum_{j=1}^n (Sx_j)(y_j)$$

for $S \in R(X,Y^*)$ and $\sum_{j=1}^n x_j \otimes y_j \in X \otimes Y$.

2.14. DEFINITION. $X \otimes_{\max} Y$ denotes the closure of $X \otimes Y$ in the Banach lattice $R(X,Y^*)^*$ (note that Y^* is always order complete).

2.15. LEMMA. *If* Y *is order complete then the subspace of* $X \otimes_\pi Y$ *consisting of all* $T = \sum_{j=1}^n x_j \otimes y_j$, *where the* y_j*'s are mutually disjoint is dense in* $X \otimes_\pi Y$.

PROOF. Let $T = \sum_{j=1}^n x_j \otimes y_j \in X \otimes Y$ and let $\varepsilon > 0$. Since Y is order complete, we can find $h_1, h_2, \ldots, h_m$ mutually disjoint in Y and scalars a_{ij}, $i \leqq m$, $j \leqq n$ so that

$$\sum_{j=1}^n \|x_j\| \left\| y_j - \sum_{i=1}^m a_{ij} h_i \right\| \leqq \varepsilon \tag{1}$$

and hence

$$\left\| \sum_{j=1}^{n} x_j \otimes y_j - \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) \otimes h_i \right\|_{\pi} \leqq \varepsilon .$$

The following result was also proved in [16].

2.16. THEOREM. *If Y is order continuous then the norm of $X \otimes_{\max} Y$ induced by $R(X, Y^*)^*$ is a cross norm turning $X \otimes_{\max} Y$ into a Banach lattice under the canonical order.*

PROOF. It is trivial that the norm on $X \otimes_{\max} Y$ is a cross norm. We have to show that $X \otimes_{\max} Y$ is a sublattice of $R(X, Y^*)^*$. If $T = \sum_{j=1}^{n} x_j \otimes y_j \in X \otimes Y$ with the y_j's mutually disjoint then by the order continuity of Y we get for $x^* \in X^*$, $x^* \geqq 0$

$$(1) \qquad |T|(x^*) = \sup \left\{ \left\| \sum_{j=1}^{n} y^*(x_j) y_j \right\| \;\middle|\; |y^*| \leqq x^* \right\} = \sum_{j=1}^{n} |x^*(|x_j|)|y_j|$$

hence $|T| = \sum_{j=1}^{n} |x_j| \otimes |y_j|$.

If now $\mathscr{U} \in X \otimes_{\max} \acute{Y}$, then by Lemma 2.15 there is a sequence $(\mathscr{U}_n) \subseteqq X \otimes Y$ of the above form so that $\mathscr{U}_n \to \mathscr{U}$ and therefore $|\mathscr{U}_n| \to |\mathscr{U}|$ and $|\mathscr{U}| \in X \otimes_{\max} Y$.

It can be proved that $(X \otimes_{\max} Y)^* = R(X, Y^*)$ from which it follows that $X \otimes_{\max} Y$ is the maximal tensor product of Banach lattices.

It is easily seen that this tensor product is closely related to the characterization of injective Banach lattices, and it would therefore be interesting to get a description of it.

Note that unlike the other tensor products defined in this paper it is a problem, whether the elements in $X \otimes_{\max} Y$ is in one to one correspondence with operators from X^* to Y unless, of course, X^* or Y has the approximation property.

3. The uniform approximation property for m-tensor products.

In this section we investigate under which conditions m-tensor products have the uniform approximation property. In the sequel we let E denote a Banach space and X an Banach lattice. We start with the following definition:

3.1. DEFINITION. Let φ: $\mathsf{N} \to \mathsf{N}$ be a function and $\lambda \geqq 1$. E is said to have the (λ, φ)-uniform approximation property ((λ, φ)-u.a.p.), if for every n and every n-

dimensional subspace $F \subseteqq E$ there is a bounded operator T on E so that $Tx = x$ for all $x \in F$, $\operatorname{rk} T \leqq \varphi(n)$, $\|T\| \leqq \lambda$.
We shall say that E has the u.a.p., if E has the (λ, φ)-u.a.p. for some $\lambda \geqq 1$ and some function φ. Likewise we shall say that E has the λ-u.a.p., if it has the (λ, φ)-u.a.p. for some function φ.

3.2. Definition. E is said to have the Grothendieck uniform approximation property (G.u.a.p.), if for every $\varepsilon > 0$ and every sequence $\beta = (\beta_n)$ of positive reals with $\beta_n \to 0$ there exist $\lambda = \lambda(\beta, \varepsilon)$ and $\varphi = \varphi(\beta, \varepsilon)$ so that the following holds: For every sequence $(x_n) \subseteqq E$ with $\|x_n\| \leqq \beta_n$ there is a bounded operator T on E, satisfying $\|Tx_n - x_n\| \leqq \varepsilon$ for all n, $\|T\| \leqq \lambda$ and $\operatorname{rk} T \leqq \varphi$.

It is readily seen that the u.a.p. is a kind of local version of the bounded approximation property, while the G.u.a.p. is a local version of the Grothendieck approximation property.

The u.a.p. was originally introduced by Pelczynski and Rosenthal [12] who proved that $C(S)$-spaces, S compact and $L_p(\mu)$-spaces, $1 \leqq p \leqq \infty$ have the $1+\varepsilon$-u.a.p. for all $\varepsilon > 0$. Lindenstrauss and Tzafriri [10] proved that reflexive Orlicz sequence and function spaces have the $(1+\varepsilon)$-u.a.p. and their paper also contains some important results on the u.a.p. for superreflexive spaces (E is called superreflexive, if every ultrapower of E is reflexive. Recall that a Banach lattice X is superreflexive, if and only if it is p-convex and q-concave for some p, $1 < p < \infty$ and some q, $1 \leqq q < \infty$ [9]).

The G.u.a.p. was introduced in [2], which also contains an extensive study of the u.a.p. and its relation to the bounded approximation property. Among other things it is proved that the u.a.p. is a self-dual property.

Basically the only known examples of spaces with the u.a.p. are the ones mentioned above and some spaces constructed from them in a nice manner. An important open problem is whether the classical Lorentz spaces have the u.a.p. There are examples of spaces with a symmetric basic which do not have the u.a.p. [15].

A perturbation argument, [4, Lemma 2.4], easily yields the following Lemma, which will be very useful for us in the sequel.

3.3. Lemma. *Let* $\lambda \geqq 1$. *If there is a function* $\psi\colon \mathbf{N} \times \mathbf{R}_+ \to \mathbf{N}$ *so that for every* $\varepsilon > 0$ *and every n-dimensional subspace* $F \subseteqq E$ *there is an operator* T *on* E *with* $\|T\| \leqq \lambda$, $\operatorname{rk} T \leqq \psi(n, \varepsilon)$ *and* $\|Tx - x\| \leqq \varepsilon \|x\|$ *for all* $x \in F$, *then* E *has the* $(\lambda + \varepsilon)$-u.a.p. *for all* $\varepsilon > 0$.

Remark. Lemma 2.4 of [4] shows that if E has the above property and $(1-\varepsilon)^{-1}\varepsilon n < 1$, then there is a operator S on E of dimension at most $\psi(n, \varepsilon)$ so that S is the identity on F and $\|S\| \leqq ((1-\varepsilon)^{-1}\varepsilon n + 1)\lambda$.

We can now prove

3.4. THEOREM. *Let* $\lambda \geqq 1$. *Then the following statements are equivalent*:

(i) *There is a function* $\varphi\colon \mathbf{N} \times \mathbf{R}_+ \to \mathbf{N}$ *so that for every* $\varepsilon > 0$ *and every* n-*dimensional subspace* $F \subseteq X$ *there is an operator* T *on* X *with* $\|Tx - x\| \leqq \varepsilon \|x\|$ *for all* $x \in F$, $\operatorname{rk} T \leqq \varphi(n, \varepsilon)$ *and*

$$\left\| \bigvee_{j=1}^{n} |Tx_j| \right\| \leqq (\lambda + \varepsilon) \left\| \bigvee_{j=1}^{n} |x_j| \right\|$$

for all n-tuples $(x_1, x_2, \ldots, x_n) \subseteq X$.

(ii) *For every finite dimensional Banach space* E, $E \otimes_m X$ *has the* $(\lambda + \varepsilon)$-u.a.p. *for all* $\varepsilon > 0$.

(iii) *The same as* (i) *with the addition that* $Tx = x$ *for all* $x \in F$.

PROOF. (i) $\Rightarrow$ (ii). Assume (i), let $\varepsilon > 0$ and let E be a k-dimensional Banach space; we choose an Auerbach basis $(e_i)_{i=1}^{k}$ for E (i.e., if $(e_i^*)_{i=1}^{k}$ is the biorthogonal system then $\|e_i\| = \|e_i^*\| = 1$, $i \leqq k$, see [8]). If $F \subseteq E \otimes_m X$ is an n-dimensional subspace with Auerbach basis $(u_j)_{j=1}^{n}$, then we can find $\{x_{ij} \mid i \leqq k, j \leqq n\} \subseteq X$ so that

$$u_j = \sum_{i=1}^{k} e_i \otimes x_{ij} \quad \text{for } j \leqq n\,. \tag{1}$$

Put $X_1 = \operatorname{span}\{x_{ij} \mid i \leqq k, j \leqq n\}$ and let $(f_r^*)_{r=1}^{N}$ be an ε-net in B_{E^*}. Then clearly $(1-\varepsilon)B_{E^*} \subseteq \operatorname{conv}\{f_r^* \mid r \leqq N\}$ and therefore for every $u \in E \otimes_m X$

$$\|u\|_m \leqq \|\sup\{|u(e^*)| \mid e^* \in B_{E^*}\}\| \leqq (1-\varepsilon)^{-1} \left\| \bigvee_{r=1}^{N} |u(f_r^*)| \right\|. \tag{2}$$

By assumption we can now find an operator T on X with $\|Tx - x\| \leqq n^{-1}k^{-1}\varepsilon\|x\|$ for all $x \in X_1$, $\operatorname{rk} T \leqq \varphi(\max(N, kn), n^{-1}k^{-1}\varepsilon)$ and

$$\left\| \bigvee_{r=1}^{N} |Tx_r| \right\| \leqq (\lambda + \varepsilon) \left\| \bigvee_{r=1}^{N} |x_r| \right\| \tag{3}$$

for all N-tuples $(x_r)_{r=1}^{N} \subseteq X$.

We now look on the operator $I \otimes T$ where I denotes the identity on E. By (2) and (3) we get for all $u \in E \otimes_m X$

$$\begin{aligned} \|(I \otimes T)u\|_m = \|T \circ u\|_m &\leqq (1-\varepsilon)^{-1} \left\| \bigvee_{r=1}^{N} |T(u(f_r^*))| \right\| \\ &\leqq (\lambda + \varepsilon)(1-\varepsilon)^{-1} \left\| \bigvee_{r=1}^{N} |u(f_r^*)| \right\| \leqq (\lambda + \varepsilon)(1-\varepsilon)^{-1} \|u\|_m \end{aligned} \tag{4}$$

which shows that $I\otimes T$ is a bounded linear operator on $E\otimes_m X$ with norm less than or equal to $(1-\varepsilon)^{-1}(\lambda+\varepsilon)$. Clearly $\operatorname{rk} T\leqq k\varphi(\max(N,kn),\varepsilon/nk)$. Since $|t_i| \leqq \|\sum_{s=1}^k t_s e_s\|$ for all $(t_i)_{i=1}^k\subseteq \mathsf{R}$ we get that $|x_{ij}|\leqq \|\sum_{s=1}^k x_{sj}e_s\|$ for all $j\leqq n$ and $i \leqq k$ and hence

$$\|x_{ij}\| \leqq \|u_j\|_m \quad \text{for } i\leqq k \text{ and } j\leqq n\,. \tag{5}$$

(2) and (5) give for $j\leqq n$

$$\begin{aligned}\|(I\otimes T)u_j-u_j\|_m &\leqq \sum_{i=1}^k \|e_i\|\,\|Tx_{ij}-x_{ij}\| \\ &\leqq n^{-1}k^{-1}\varepsilon \sum_{i=1}^k \|x_{ij}\| \leqq n^{-1}\varepsilon\|u_j\|_m\,.\end{aligned} \tag{6}$$

Since $(u_j)_{j=1}^n$ was an Auerbach basis, (6) shows that $\|(I\otimes T)u-u\|_m\leqq\varepsilon\|u\|_m$ for all $u\in F$.

Using Lemma 3.3 we have now proved that $E\otimes_m X$ has the $(\lambda+\varepsilon)$-u.a.p. for all $\varepsilon>0$.

(ii) $\Rightarrow$ (iii): Assume (ii), let $\varepsilon>0$ be arbitrary and let $F\subseteq X$ be an n-dimensional subspace. By assumption $l_\infty^n\otimes_m X$ has the $(\lambda+\varepsilon)$-u.a.p. with a dimension function φ, say, and we can therefore find an operator S on $l_\infty^n\otimes_m X$, so that S is the identity on $l_\infty^n\otimes F$, $\|S\|\leqq\lambda+\varepsilon$ and $\operatorname{rk} S\leqq\varphi(n^2)$.

If Γ denotes the group of isometries of l_∞^n onto itself, then we can define:

$$S_0 = (2^n n!)^{-1}\sum_{\gamma\in\Gamma}(\gamma^{-1}\otimes I_X)T(\gamma\otimes I_X) \tag{7}$$

where I_X denotes the identity on X. It is easily verified that $\|S_0\|\leqq\lambda+\varepsilon$, $\operatorname{rk} S_0 \leqq 2^n n!\,\varphi(n^2)$ and that S_0 is still the identity on $l_\infty^n\otimes F$. Since S_0 is invariant under all the isometries $\gamma\otimes I_X$, $\gamma\in\Gamma$ of $l_\infty^n\otimes_m X$ onto itself there is a T: $X\to X$ so that $S_0=I_{l_\infty^n}\otimes T$.

Since $\|S_0\|\leqq\lambda+\varepsilon$ it follows from Lemma 1.1 that if $x_1,x_2,\ldots,x_n\in X$, then

$$\left\|\bigvee_{r=1}^n |Tx_r|\right\| \leqq (\lambda+\varepsilon)\left\|\bigvee_{r=1}^n |x_r|\right\|. \tag{8}$$

Clearly $\operatorname{rk} T\leqq(n-1)!\,2^n\varphi(n^2)$ and $Tx=x$ for all $x\in F$.

(iii) $\Rightarrow$ (i) is trivial.

3.5. Definition. Let $\lambda\geqq 1$. X is said to have the order $(\lambda+)$-u.a.p., if it satisfies one of the equivalent conditions of Theorem 3.4.

It is an open problem, whether the u.a.p. is equivalent to the order u.a.p. for general Banach lattices; all known examples of Banach lattices with the u.a.p. satisfy the conditions of Lemma 3.3 with the operator T positive and therefore also trivially condition (i) of Theorem 3.4. For order complete examples this is normally done by finding a positive controllable operator, which is the identity on a given span of mutually disjoint vectors, and then apply Proposition 3 of [10]; for $C(S)$-spaces by a partition of unity. It turns out that the two concepts are equivalent for superreflexive Banach lattices, as it is seen from

3.6. THEOREM. *If X is a superreflexive Banach lattice with the* u.a.p., *then X has the order* $(1+)$-u.a.p.

PROOF. Let E be a k-dimensional space; we wish to show that $E \otimes X$ has the $(1+\varepsilon)$-u.a.p. for all $\varepsilon > 0$. We can then find a K (depending only on k and X), so that $\|u\|_m \leqq K\|u\|$ for all $u \in E \otimes_m X$ $(=B(E^*, X))$. Since X has the $(1+\varepsilon)$-u.a.p., we can perform the construction of (i) $\Rightarrow$ (ii) without the extra condition on T. All formulas will hold except (3) and (4), but instead we get for $u \in E \otimes_m X$

$$\|(I \otimes T)u\|_m \leqq \|T \circ u\|_m \leqq K\|T\|\,\|u\| \leqq (1+\varepsilon)K\|u\|_m .$$

Hence we have shown that $E \otimes_m X$ has the $(K+\varepsilon)$-u.a.p. for all ε. Since E is finite dimensional and X is superreflexive it is easily verified that $E \otimes_m X$ is superreflexive as well (one can for instance use Theorem 2.9) and hence it has the $(1+\varepsilon)$-u.a.p. for all $\varepsilon > 0$ by [10, Theorem 1].

Recall that if E and F are isomorphic Banach spaces then the Banach–Mazur distance $d(E, F)$ is defined by:

$$d(E,F) = \inf\{\|T\|\,\|T^{-1}\| \mid T \text{ isomorphism of } E \text{ onto } F\} .$$

It is well known that for every n $\log d$ is a metric on the set of all n-dimensional Banach spaces turning it into a compact space.

We can now prove:

3.7. THEOREM. *Let E be a Banach space with the* λ-u.a.p. *If X has the order* μ-u.a.p. *then* $E \otimes_m X$ *has the* $(\lambda\mu+\varepsilon)$-u.a.p. *for all* $\varepsilon > 0$. *If X is superreflexive and has the* u.a.p. *then* $E \otimes_m X$ *has the* $(\lambda+\varepsilon)$-u.a.p. *for all* $\varepsilon > 0$.

PROOF. Assume that E has the (λ, φ)-u.a.p. and let $\varepsilon > 0$.

1°. *X is order complete*: For every $k \in \mathbb{N}$, we can choose an ε-net in the compact space consisting of all Banach spaces of dimension k equipped with the Banach–Mazur distance. Hence for every $n \in \mathbb{N}$, we can find a $\bar{\varphi}(n)$ so that $\{Y_m \mid 1 \leqq m \leqq \bar{\varphi}(n)\}$ is the union of the ε-nets for $k \leqq \varphi(n)$. Having fixed such ε-nets for every natural number the space

$$Y = \sum_{m=1}^{\bar{\varphi}(n)} \oplus Y_m$$

is at most $\varphi(n)\bar{\varphi}(n)$-dimensional and depends only on n and ε (we shall omit subscripts n and ε on Y). Let $F \subseteq E \otimes_m X$ be a subspace of dimension at most n and spanned by $\{u_1, u_2, \ldots, u_n\}$. By Lemma 2.15 and Lemma 3.3 we can without loss of generality assume that there is a $k \in \mathbf{N}$ (which we cannot control), a set $\{e_{ij} \mid 1 \leqq i \leqq k,\ 1 \leqq j \leqq n\} \subseteq E$ and a set $\{x_i \mid 1 \leqq i \leqq k\} \subseteq X$ consisting of mutually disjoint elements so that

$$(1) \qquad u_j = \sum_{i=1}^{k} e_{ij} \otimes x_i, \quad 1 \leqq j \leqq n\,.$$

For every $i \leqq k$, we put $E_i = \operatorname{span}\{e_{ij} \mid 1 \leqq j \leqq n\}$ and by assumption we can then find bounded operators T_i, $i \leqq k$, on E so that

$$(2) \qquad \|T_i\| \leqq \lambda,\ i \leqq k, \quad T_i e_{ij} = e_{ij}, \quad i \leqq k,\ j \leqq n$$

and $\operatorname{rk} T_i \leqq \varphi(n)$ for $i \leqq k$. Further for $i \leqq k$, we let X_i denote the band generated by x_i and let P_i denote the band projection of X onto X_i (see [9, page 10].)

Since $T_i(E)$ is at most $\varphi(n)$-dimensional, we can for every $i \leqq k$ find a $m(i) \leqq \bar{\varphi}(n)$ and an isomorphism S_i of $T_i(E)$ onto $Y_{m(i)}$ with $\|S_i\| \leqq 1+\varepsilon$, $\|S_i^{-1}\| \leqq 1$. We define the operator W_1 of $E \otimes_m X$ into $Y \otimes_m X$ by

$$(3) \qquad W_1 u = \sum_{i=1}^{k} P_i \circ u \circ T_i^* S_i^*, \quad u \in E \otimes_m X$$

where T_i^* is considered as a map from $T_i(E)^*$ to E^*.

We have to show that W_1 is bounded and estimate its norm. If $u \in E \otimes_m X$ is of the form $u = \sum_{m=1}^{s} f_m \otimes z_m$, where $(f_m) \subseteq E$, $(z_m) \subseteq X$, then

$$(4) \qquad W_1 u = \sum_{i=1}^{k} \sum_{m=1}^{s} S_i T_i(f_m) \otimes P_i z_m\,.$$

Since $(P_i z_m) \subseteq X_i$ for all $m \leqq s$ and all $i \leqq k$ and the X_i's are disjoint bands then it follows from Lemma 1.1 that

$$(5) \qquad \left\| \sum_{i=1}^{k} \sum_{m=1}^{s} P_i z_m S_i T_i(f_m) \right\|_Y$$

$$= \sum_{i=1}^{k} \left\| \sum_{m=1}^{s} P_i z_m S_i T_i(f_m) \right\|_Y \leqq \lambda(1+\varepsilon) \sum_{i=1}^{k} \left\| \sum_{m=1}^{s} P_i z_m f_m \right\|_E$$

$$= \lambda(1+\varepsilon) \left\| \sum_{i=1}^{k} \sum_{m=1}^{s} P_i z_m f_m \right\|_E \leqq \lambda(1+\varepsilon) \left\| \sum_{m=1}^{s} z_m f_m \right\|_E$$

where the last inequality is true, because the P_i's are mutually disjoint band projections. (5) gives immediately

$$(6)\qquad \|W_1u\|_m = \left\|\left\|\sum_{i=1}^{k}\sum_{m=1}^{s} P_i z_m S_i T_i f_m\right\|_Y\right\|_X$$
$$\leqq \lambda(1+\varepsilon)\left\|\left\|\sum_{m=1}^{s} z_m f_m\right\|_Y\right\|_X = \lambda(1+\varepsilon)\|u\|_m .$$

By Theorem 3.4 there is a function φ_1 defined on the natural numbers so that $Y\otimes_m X$ has the $(\mu+\varepsilon, \varphi_1)$-u.a.p. (since Y is fixed for each n, $\varphi_1(n)$ depends only on n, ε, μ, and X). Hence there is an operator T on $Y\otimes_m X$ so that $\dim T \leqq \varphi_1(n)$ $\|T\| \leqq \mu+\varepsilon$ and $Tv=v$ for $v \in W_1(F)$.

For every $i\leqq k$, we let Q_i denote the natural norm one projection of Y onto $Y_{m(i)}$ and define W_2: $Y\otimes_m X \to E\otimes_m X$ by

$$(7)\qquad W_2 = \sum_{i=1}^{k} S_i^{-1}Q_i\otimes P_i .$$

By doing calculations similar as the ones which gave the norm of W_1 we get that W_2 is a bounded operator of norm less than or equal to one. If $S = W_2TW_1$, then $\|S\| \leqq \lambda(1+\varepsilon)(\mu+\varepsilon)$, $\operatorname{rk} S \leqq \varphi_1(n)$. Further for $j\leqq n$:

$$Su_j = W_2TW_1u_j = W_2W_1u_j = \sum_{i=1}^{k} S_i^{-1}S_ie_{ij}\otimes x_i = u_j .$$

We have now proved that $E\otimes_m X$ has the $\lambda\mu+\varepsilon$-u.a.p. for all $\varepsilon>0$.

The second statement of the theorem follows immediately from the above and Theorem 3.6.

2°. *The General Case*: If G is a finite dimensional space then by [13, Theorem 7.4] $(G\otimes_m X)^{**} = G\otimes_m X^{**}$. By assumption $G\otimes_m X$ has the $(\mu+\varepsilon)$-u.a.p. for all $\varepsilon>0$ and therefore by the above and [10], $G\otimes_m X^{**}$ has the $(\mu+\varepsilon)$-u.a.p. Thus X^{**} has the order $(\mu+)$-u.a.p., and by the first part $E\otimes_m X^{**}$ has the $(\lambda\mu+\varepsilon)$-u.a.p.

Let now $\varepsilon>0$ and assume that the $(\lambda\mu+\varepsilon)$-u.a.p. dimension function for $E\otimes_m X^{**}$ is φ. If $F\subseteqq E\otimes_m X$ is a finite dimensional subspace then (since $E\otimes_m X$ is canonically isometric to a subspace of $E\otimes_m X^{**}$ by Lemma 1.1) we can find an operator T on $E\otimes_m X^{**}$ which is the identity on F and so that $\|T\| \leqq \lambda\mu+\varepsilon$ $\operatorname{rk} T\leqq \varphi(n)$. By a standard perturbation argument, we may assume that $F\subseteqq E\otimes X$ and $\operatorname{Im} T\subseteqq E\otimes X^{**}$. We choose now a basis $(v_j)_{j=1}^{N}$ of $\operatorname{Im} T$, so that $(v_j)_{j=1}^{n}$ forms a basis for F. We can then find $(f_{ij})_{i=1}^{k_j}\subseteqq E$, $1\leqq j\leqq N$, $(z_{ij})_{i=i}^{k_j}\subseteqq X^{**}$, $1\leqq j\leqq N$, $z_{ij} \in X$ when $i\leqq k_j$ and $j\leqq n$, so that

$$v_j = \sum_{i=1}^{k_j} f_{ij}\otimes z_{ij}, \qquad 1\leqq j\leqq N .$$

Put $G = \operatorname{span}\{f_{ij} \mid 1 \leqq i \leqq k_j\}$. It is readily checked that $G \otimes_m X \subseteqq E \otimes_m X$ and $G \otimes_m X^{**} \subseteqq E \otimes_m X^{**}$ in the canonical manner and that $\operatorname{Im} T \subseteqq G \otimes_m X^{**}$. Applying the principle of local reflexivity on $G \otimes_m X$ we get an operator V: $\operatorname{Im} T \to G \otimes_m X$ with $\|V\| \leqq 1+\varepsilon$ and $Vv_j = v_j$ for $j \leqq n$. The map S: $E \otimes_m X \to E \otimes_m X$ defined S: $E \otimes_m X \to E \otimes_m X$ defined by $Su = VTu$ for $u \in E \otimes_m X$ clearly has the desired properties.

We have also

3.8. THEOREM. *If X has the λ-*u.a.p., *then $l_2 \otimes_m X$ $(= X(l_2))$ has the $(\lambda K_G + \varepsilon)$-*u.a.p. *for all $\varepsilon > 0$, where K_G denotes the Grothendieck constant.*

PROOF. Let $n \in \mathbf{N}$. By [11, Corollary 4.12] (the weakly sequential completeness of X can be avoided for finite dimensional Hilbert spaces) or equivalently by a result of Krivine [9, Theorem 1.f.14], we have for every $u \in l_2^k \otimes_m X$ and every bounded operator T on X:

$$\|T \circ u\|_m \leqq K_G \|T\| \, \|u\|_m \, .$$

Reproducing now the proof of Theorem 3.4 (i) $\Rightarrow$ (ii) assuming only the λ-u.a.p. of X we obtain that the operator $I \otimes T$ there is a bounded operator on $l_2^k \otimes_m X$ of norm less than or equal to λK_G. This gives that $l_2^k \otimes_m X$ has the $(\lambda K_G + \varepsilon)$-u.a.p. for all $\varepsilon > 0$.

The proof of Theorem 3.7 can now be applied on $l_2 \otimes_m X$ just by letting the S_i's from there be isometries of $T_i(E)$ onto $l_2^{\dim T_i(E)}$.

We now turn our attention to the Grothendieck uniform approximation property. It turns out that in this case the analogue of Theorem 3.7 holds without the order assumption of the u.a.p. This is proved in Theorem 3.10, while the next lemma corresponds to Theorem 3.4.

3.9. LEMMA. *If X has the* G.u.a.p. *and E is a k-dimensional space then $E \otimes_m X$ has the* G.u.a.p.

PROOF. Let $\{e_1, e_2, \ldots, e_k\}$ *be an Auerbach basis for E* and assume that X has the G.u.a.p. with uniformity functions $\lambda(\beta, \varepsilon)$ and $\varphi(\beta, \varepsilon)$. For every $i \leqq k$ we let P_i denote the canonical projection of E onto the one dimensional space spanned by e_i and put

$$(1) \qquad Q_i = P_i \otimes I \qquad Z_i = Q_i(E \otimes_m X) \, ,$$

where I is the identity operator on X.

Clearly for evey $i \leqq k$, Q_i is a norm one projection and Z_i is isometric to X.

Let $(\beta_j)\subseteq \mathrm{R}_+$ be a null sequence, $\varepsilon>0$ and let $(u_j)\subseteq E\otimes_m X$ be a sequence with $\|u_j\|\leqq\beta_j$.

Since $\|Q_i u_j\|\leqq\beta_j$ for every $i\leqq k$ and every $j\in\mathrm{N}$ we can by assumption find bounded operators T_i on Z_i so that

$$\|T_iQ_iu_j-Q_iu_j\|_m \leqq k^{-1}\varepsilon, \quad i\leqq k,\ j\in\mathrm{N} \tag{2}$$

$$\|T_i\| \leqq \lambda(\beta,k^{-1}\varepsilon), \quad \mathrm{rk}\,(T_i) \leqq \varphi(\beta,k^{-1}\varepsilon), \quad i\leqq k\,. \tag{3}$$

If we put $T=\sum_{i=1}^k T_iQ_i$, then T is a bounded operator on $E\otimes_m X$ with

$$\|T\| \leqq k\lambda(\beta,k^{-1}\varepsilon), \quad \mathrm{rk}\,(T) \leqq k\varphi(\beta,k^{-1}\varepsilon) \tag{4}$$

furthermore for all $j\in\mathrm{N}$

$$\|Tu_j-u_j\|_m \leqq \sum_{i=1}^k \|T_iQ_iu_j-Q_iu_j\|_m \leqq \varepsilon\,. \tag{5}$$

(4) and (5) show that $E\otimes_m X$ has the G.u.a.p.

3.10. THEOREM. *If E and X have the* G.u.a.p. *then $E\otimes_m X$ has the* G.u.a.p. *as well.*

PROOF. Assume that both E and X have the G.u.a.p. with uniformity functions $\lambda(\beta,\varepsilon)$ and $\varphi(\beta,\varepsilon)$.

1° *Assume X is order complete.* Let $(\beta_j)\subseteq\mathrm{R}_+$ be a sequence tending to zero, let $\varepsilon>0$ and choose $n\in\mathrm{N}$ so that

$$((1+\varepsilon)\lambda(\beta,\varepsilon)+1)\beta_j \leqq \varepsilon \quad \text{for all } j\geqq n\,. \tag{1}$$

Let $(u_j)\subseteq E\otimes_m X$ be a sequence with $\|u_j\|\leqq\beta_j$ for all $j\in\mathrm{N}$. Since obviously it is sufficient to check the G.u.a.p.-conditions on a given dense subset of $E\otimes_m X$ we can by Lemma 2.15 assume that there is a finite set $\{x_i \mid 1\leqq i\leqq k\}\subseteq X$ consisting of mutual disjoint elements so that for every $j\leqq n$, u_j is of the form

$$u_j = \sum_{i=1}^k e_{ij}\otimes x_i \tag{2}$$

where $\{e_{ij} \mid i\leqq k, j\leqq n\}\subseteq E$ is a normalized set.

By assumption we can now find operators T_i, $i\leqq k$ so that

$$\|u_j\|_m\|T_ie_{ij}-e_{ij}\| \leqq \varepsilon, \quad i\leqq k,\ j\leqq n \tag{3}$$

$$\|T_i\| \leqq \lambda(\beta,\varepsilon), \quad \mathrm{rk}\,T_i \leqq \varphi(\beta,\varepsilon), \quad i\leqq k\,. \tag{4}$$

We shall now use the same terminology as in the proof of Theorem 3.7 and construct the space Y and the operators W_1 and W_2 in the same way as there. It is easily checked that

$$\|W_1\| \leqq \lambda(\beta,\varepsilon)(1+\varepsilon)\ . \tag{5}$$

If $1 \leqq j \leqq n$, then by the definition of W_1 and W_2:

$$\begin{aligned} W_2W_1u_j &= \sum_{i=1}^{k} S_i^{-1}Q_iS_iT_ie_{ij}\otimes P_ix_i \\ &= \sum_{i=1}^{k} T_ie_{ij}\otimes x_i\ . \end{aligned} \tag{6}$$

Since the x_i's are mutually disjoint we get from Lemma 1.1, (3) and (6) for $1 \leqq j \leqq n$

$$\begin{aligned} \|W_2W_1u_j - u_j\|_m &= \left\|\sum_{i=1}^{k} (T_ie_{ij}-e_{ij})\otimes x_i\right\|_m \\ &= \left\|\sum_{i=1}^{k} \|Te_{ij}-e_{ij}\|x_i\right\| \leqq \varepsilon\|u_j\|_m^{-1}\left\|\sum_{i=1}^{k} \|e_{ij}\|x_i\right\| \\ &= \varepsilon\|u_j\|_m^{-1}\|u_j\|_m = \varepsilon\ . \end{aligned} \tag{7}$$

If $j > n$, then by (1) and (5)

$$\|W_2W_1u_j - u_j\| \leqq \|W_2\|(\|W_1\|+1)\|u_j\| \leqq \varepsilon\ . \tag{8}$$

The space Y constructed as in Theorem 3.7 depends only on ε and $\varphi(\beta,\varepsilon)$ and hence on β and ε. Therefore by Lemma 3.9, $Y\otimes_m X$ has the G.u.a.p. with uniformity function $\lambda_{\beta,\varepsilon}(\gamma,\delta)$ and $\varphi_{\beta,\varepsilon}(\gamma,\delta)$ only depending on β and ε. Since

$$\|W_1u_j\| \leqq (1+\varepsilon)\lambda(\beta,\varepsilon)\beta_j \tag{9}$$

we can find an operator T on $Y\otimes_m X$ so that

$$\|TW_1u_j - W_1u_j\|_m \leqq \varepsilon \tag{10}$$

$$\|T\| \leqq \lambda_{\beta,\varepsilon}((1+\varepsilon)\lambda(\beta,\varepsilon)\beta_j,\varepsilon) \tag{11}$$

$$\operatorname{rk}(T) \leqq \varphi_{\beta,\varepsilon}((1+\varepsilon)\lambda(\beta,\varepsilon)\beta_j,\varepsilon)\ . \tag{12}$$

Let us denote the right hand sides of (11) and (12) by $\lambda_1(\beta,\varepsilon)$ respectively $\varphi_1(\beta,\varepsilon)$. If we put $S = W_2TW_2$, then by (7), (8), and (10)

$$\|Su_j - u_j\|_m \leqq \|w_2TW_1u_j - W_2W_1u_j\|_m + \|W_2W_1u_j - u_j\|_m \leqq 2\varepsilon\ . \tag{13}$$

Further by (5), (11), and (12)

$$\|S\| \leqq \|T\|\,\|W_1\| \leqq (1+\varepsilon)\lambda(\beta,\varepsilon)\lambda_1(\beta,\varepsilon) \tag{14}$$

$$\operatorname{rk} T \leqq \varphi_1(\beta,\varepsilon)\ . \tag{15}$$

(13), (14), and (15) shows that $E\otimes_m X$ has the G.u.a.p.

The case of general Banach lattice can now be treated in a manner similar to the argument of Theorem 3.7.

3.11. A LIST OF CONCRETE SPACES WITH THE u.a.p.

Let $(\Omega_1, \varphi_1, \mu_1)$ and $(\Omega_2, \varphi_2, \mu_2)$ be measure spaces, $1 \leqq p \leqq \infty$, $1 \leqq q \leqq \infty$ and M and N reflexive Orlicz functions. Further we let S a compact Hausdorff space. The following spaces all have the u.a.p.:

$$L_p(\mu_1, L_q(\mu_2)),\ L_M(\mu_1, L_p(\mu_2)),\ L_p(\mu_1, L_M(\mu_2)),\ L_p(\mu_1, C(S)) \quad \text{for } p < \infty\,,$$

$$L_M(\mu_1, C(S)),\ L_M(\mu_1, L_N(\mu_2)),\ C(S, L_p(\mu_1)),\ C(S, L_M(\mu_1))\,.$$

Most of these statements follow from Theorem 2.2 and Theorem 3.7.

That $L_\infty(\mu_1, L_M(\mu_2))$ follows from the fact that the u.a.p. is self-dual and that $L_M(\mu_2)$ has the RNP. Indeed,

$$L_\infty(\mu_1, L_M(\mu_2)) = \mathscr{B}(L_M(\mu_2)^*, L_\infty(\mu_1)) = (L_M(\mu_2)^* \otimes_m L_1(\mu_1))^*\,.$$

By the Ascoli theorem and the definition of $\otimes_m$ we have $C(S, E) = E \otimes_m C(S)$ for every E, thus taking care of the last statements.

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