MATH. SCAND. 126 (2020), 165–169

AN UPPER BOUND FOR THE REGULARITY OF POWERS OF EDGE IDEALS

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Abstract

For a finite simple graph G we give an upper bound for the regularity of the powers of the edge ideal $I(G)$.

In this note we provide an upper bound for the regularity of the powers of the edge ideal $I(G)$ of a finite simple graph G. Three other upper bounds have been already proved almost the same time in 2018, see [1], [11] and [8]. In general, but not always, the upper bound given here in Theorem 1 is better than that in [1] which is given in terms of the matching number. We are undecided about the comparison with the upper bound in [11] given in terms of the co-chord number. Actually, we could not find an example for which the co-chord number is bigger than the stable number which defines our upper bound. On the other hand, it can be easily seen that the upper bound in [8] which is defined via star packing is always less than or equal to the bound presented here. However, the invariant, namely the maximal cardinality among the stable sets of a graph, which is called the stable or independence number and which defines our bound is a well studied number in combinatorics. To find upper bounds of this number in simple terms of the graph is an important problem in graph theory, see [12] and [13]. In this paper we take advantage of such a bound.

A general lower bound is known by Beyarslan, Hà and Trung, see [2, Theorem 4.5]. By Cutkosky, Herzog and Trung [4] and Kodiyalam [10] it is known that for any graded ideal I in $S = k[x_1, \dots, x_n]$ there exist integers $a > 0$ and $b \geq 0$ such that $\operatorname{reg} I^s = as + b$ for all $s \gg 0$. In the case that I is the edge ideal of a graph, the constant a is equal to 2, so that $\operatorname{reg} I(G)^s = 2s + b$ for $s \gg 0$. This result implies that there exists an integer c with $\operatorname{reg} I(G)^s \leq 2s + c$ for all s.

Received 23 October 2018, in final form 23 February 2019. Accepted 26 February 2019.
DOI: https://doi.org/10.7146/math.scand.a-119227

In the following theorem we will see that one can choose c to be the dimension of the complex $\Delta(G)$ of stable sets of G. Recall that a subset S of the vertex set $V(G)$ of G is called a stable (or independent) set, if no 2-element subset of G is an edge of G.

THEOREM 1. *Let G be a finite simple graph, and let c be the dimension of the stable complex of G. Then*

$$\operatorname{reg} I(G)^s \leq 2s + c, \quad \textit{for all } s.$$

PROOF. The proof depends very much on a result by Jayanthan and Selvaraja [9] for very well covered graphs which says that for any very well covered graph G one has $\operatorname{reg} I(G)^s = s + \nu(G) - 1$ for all $s \geq 1$, where $\nu(G)$ is the induced matching number of G. The same result was proved before by Norouzi, Seyed Fakhari, and Yassemi [11] with an additional assumption.

Here we apply their theorem to the whisker graph G^* of G. Recall that G^* is obtained from G by adding to each vertex of G a leaf. Thus if $V(G) = [n]$, then $I(G^*) = (I(G), x_1y_1, \ldots, x_ny_n) \subset K[x_1, \ldots, x_n, y_1, \ldots, y_n]$. It is obvious that G^* is a very well covered graph. Indeed, since it is the polarization of the ideal $J = (I(G), x_1^2, \ldots, x_n^2)$, it is a Cohen-Macaulay ideal, so that all maximal stable sets of G^* have the same cardinality. On the other hand, the vertices of G form a maximal stable sets of G^*. This shows that all maximal stable sets of G have cardinality $n = |V(G^*)|/2$. Being very well covered means exactly this. Thus by the above mentioned theorem of Jayanthan and Selvaraja we have $\operatorname{reg} I(G^*)^s = 2s + \nu(G^*) - 1$, see [9, Corollary 4.10].

Next we use the restriction lemma as given in [7, Lemma 4.4]: let I be a monomial ideal with multigraded (minimal) free resolution $\mathbb{F}$, and let $\mathbf{c} \in \mathbb{N}_\infty^n$, where $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$. Then the restricted complex $\mathbb{F}^{\leq \mathbf{c}}$, which is the subcomplex of $\mathbb{F}$ for which $(\mathbb{F}^{\leq \mathbf{c}})_i$ is spanned by those basis elements of $\mathbb{F}_i$ whose multidegree is componentwise less than or equal to $\mathbf{c}$, is a (minimal) free resolution of the monomial ideal $I^{\leq \mathbf{c}}$ which is generated by all monomials $\mathbf{x}^{\mathbf{b}} \in I$ with $\mathbf{b} \leq \mathbf{c}$, componentwise.

We choose $\mathbf{c} = (\infty, \ldots, \infty, 0, \ldots, 0) \in \mathbb{N}_\infty^{2n}$ with n components equal to ∞ and n components equal to 0. Then $(I(G^*)^s)^{\leq \mathbf{c}} = I(G)^s$ for all s. Hence the restriction lemma implies that $\operatorname{reg} I(G)^s \leq \operatorname{reg} I(G^*)^s = 2s + \nu(G^*) - 1$ for all s. It remains to be shown that $\nu(G^*) = c + 1$, where $c = \dim \Delta(G^*)$. In fact, since $\ell = x_1 - y_1, \ldots, x_n - y_n$ is a regular sequence and since S/J is isomorphic to $K[x_1, \ldots, x_n, y_1, \ldots, y_n]/I(G^*)$ modulo ℓ, it follows that $\operatorname{reg} J = \operatorname{reg} I(G^*)$. Let σ be the maximal degree of a socle element of S/J, then $\operatorname{reg}(J) = \deg \sigma + 1$. The desired conclusion follows since S/J has a K-basis consisting elements $u + J$, where $u = \prod_{i \in F} x_i$ and $F \in \Delta(G)$.

COROLLARY 2. *Let G be a finite simple graph with n vertices and e edges. Then*

$$\operatorname{reg} I(G)^s \leq 2s + \lfloor 1/2 + \sqrt{1/4 + n^2 - n - 2e} \rfloor - 1, \quad \textit{for all } s.$$

PROOF. For the proof we use Theorem 1 and a famous formula of Hansen [6] who showed that the size of a maximal stable set is bounded by $\lfloor 1/2 + \sqrt{1/4 + n^2 - n - 2e} \rfloor$.

There are many other upper bounds for the size of a maximal stable set of a graph. Well known is the bound given by Kwok which is given as an exercise in [12]. Kwok's upper bound is $n - e/\Delta$, where Δ is the maximal degree of a vertex of G. A survey on the known upper bounds can be found in the thesis of Willis [13].

Even though $f(s) = \operatorname{reg} I^s$ is linear function of s for $s \gg 0$ when I is a graded ideal of the polynomial ring, the initial behaviour of $f(s)$ is not so well understood. In [3], Conca gives some examples for the unexpected behaviour of the function $f(s)$. On the positive side, Eisenbud and Harris [5, Proposition 1.1] showed that for a graded ideal $I \subset S = K[x_1, \dots, x_n]$ with height $I = n$ which is generated in a single degree, say d, one has $f(s) = ds + b_i$ with $b_1 \geq b_2 \geq \cdots$. We will use this result in the proof of the next theorem.

For a monomial ideal $L \subset S$ we denote by L^{pol} the polarization of L, and by S^{pol} the polynomial ring in the variables which are needed to define L^{pol}.

THEOREM 3. *Let G be a finite simple graph with n vertices, and let*

$$J = (I(G), x_1^2, \dots, x_n^2).$$

Then $\operatorname{reg} I(G)^s \leq \operatorname{reg} J^s = \operatorname{reg}(J^{\text{pol}})^s$ *for all s.*

PROOF. The inequality $\operatorname{reg} I(G)^s \leq \operatorname{reg} J^s$ follows from the equality $\operatorname{reg} J^s = \operatorname{reg}(J^{\text{pol}})^s$ and the proof of Theorem 1. Thus it remains to prove these equalities. For $s = 1$, the equality holds, since polarization of an ideal does not change its graded Betti numbers. Now since $J^{\text{pol}} = I(G^*)$, [9] implies that $\operatorname{reg}(J^{\text{pol}})^s - 2s$ is a constant function on s. Since the functions $\operatorname{reg}(J^{\text{pol}})^s - 2s$ and $\operatorname{reg} J^s - 2s$ coincide for $s = 1$, the desired result follows once we have shown that $\operatorname{reg} J^s - 2s$ is also a constant function on s. Indeed we will show that $\operatorname{reg} J^s - \operatorname{reg} J \geq 2(s-1)$ for all $s \geq 1$. Then, together with the result of Eisenbud and Harris, the desired conclusion follows.

In order to prove $\operatorname{reg} J^s - \operatorname{reg} J \geq 2(s-1)$ for all $s \geq 1$, we use the following general fact: let $L \subset S$ be a graded ideal with $\dim S/L = 0$. Then the socle $\operatorname{Soc} S/L \neq 0$ and $\operatorname{reg} L = \max\{\deg f : f \in \operatorname{Soc}(S/L)\}$. Now let $F \in \Delta(G)$ a facet with $|F| = c+1$, and set $u = \prod_{i \in F} x_i$. Then $\operatorname{reg} J = \deg u + 1$. We may assume that x_1 divides u, and consider $w = ux_1^{2(s-1)}$. Let $\mathfrak{m} = (x_1, \ldots, x_n)$. Since $\mathfrak{m}u \in J$ it follows that $\mathfrak{m}w \in J^s$. We show that $w \notin J^s$. Indeed, suppose that $w = w_1 w_2 \cdots w_s$ with $w_i \in J$ for $i = 1, \ldots, s$. Since x_1^{2s} does not divide w, one of the factors, say w_1, must be square free. Since w_1 divides w, it then follows that $w_1 \notin J$, a contradiction.

Since $w \in \operatorname{Soc}(S/J^s)$ and $\deg w = \deg u + 2(s-1)$, we conclude that $\operatorname{reg} J^s \geq \deg w + 1 = \operatorname{reg} J + 2(s-1)$.

It should be noted that the equalities $\operatorname{reg} J^s = \operatorname{reg}(J^{\text{pol}})^s$ are quite surprising because in general polarization and taking powers are not very well compatible operations.

Acknowledgements. The authors would like to thank the referee for informing them about the references [2], [11], [8] and for suggesting comparing each of their upper bounds with our bound.

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