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AN INTERPOLATION OF OHNO'S RELATION TO COMPLEX FUNCTIONS

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Abstract

Ohno's relation is a well known formula among multiple zeta values. In this paper, we present its interpolation to complex functions.

1. Introduction

For complex numbers $s_1, \ldots, s_r \in \mathbb{C}$, we define the *multiple zeta function* (MZF) by

$$\zeta(s_1, \ldots, s_r) := \sum_{1 \le n_1 < \ldots < n_r} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}}.$$

Matsumoto [6] proved that this series is absolutely convergent in the domain

$$\{(s_1, \ldots, s_r) \in \mathbb{C}^r \mid \Re(s(\ell, r)) > r - \ell + 1 \ (1 \le \ell \le r)\},$$

where $s(\ell, r) := s_\ell + \cdots + s_r$. Akiyama, Egami and Tanigawa [1] and Zhao [13] independently proved that $\zeta(s_1, \ldots, s_r)$ is meromorphically continued to the whole space $\mathbb{C}^r$. The special values $\zeta(k_1, \ldots, k_r)$ ($k_i \in \mathbb{Z}_{\ge 1}$ for $i = 1, \ldots, r-1$, $k_r \in \mathbb{Z}_{\ge 2}$) of MZF are called the multiple zeta values (MZVs). The MZVs are real numbers and known to satisfy many kinds of algebraic relations over $\mathbb{Q}$. One of the most well-known formulas in this field is Ohno's relation. We say that an index $(k_1 \ldots, k_r) \in \mathbb{Z}_{\ge 1}^r$ is *admissible* if $k_r \ge 2$.

Definition 1.1. For an admissible index

$$\mathbf{k} := (\underbrace{1, \ldots, 1}_{a_1 - 1}, b_1 + 1, \ldots, \underbrace{1, \ldots, 1}_{a_\ell - 1}, b_\ell + 1) \quad (a_p, b_q \ge 1),$$

we define the *dual index* of $\mathbf{k}$ by

$$\mathbf{k}^\dagger := (\underbrace{1, \ldots, 1}_{b_\ell - 1}, a_\ell + 1, \ldots, \underbrace{1, \ldots, 1}_{b_1 - 1}, a_1 + 1).$$

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THEOREM 1.2 (Ohno's relation [11]). *For an admissible index $(k_1, \ldots, k_r) \in \mathbb{Z}_{\geq 1}^r$ and $m \in \mathbb{Z}_{\geq 0}$, we have*

$$\sum_{\substack{e_1+\cdots+e_r=m \\ e_i \geq 0\,(1\leq i\leq r)}} \zeta(k_1+e_1, \ldots, k_r+e_r) = \sum_{\substack{e'_1+\cdots+e'_{r'}=m \\ e'_i \geq 0\,(1\leq i\leq r')}} \zeta(k'_1+e'_1, \ldots, k'_{r'}+e'_{r'}),$$

where the index $(k'_1, \ldots, k'_{r'})$ is the dual index of $(k_1, \ldots, k_r)$.

From an analytic point of view, Matsumoto [7] raised the question whether the known relations among MZVs are valid only for positive integers or not. It is known that the harmonic relations, e.g., $\zeta(s_1)\zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1+s_2)$ are valid not only for positive integers but for complex numbers. Regarding this problem, there are no known such relations except for the harmonic relations above (for details, see Ikeda and Matsuoka [4]) or relations which contain infinite sums of MZF obtained by the authors (see Hirose, Murahara and Onozuka [3]).

As a weaker version of the question, we can consider the problem of whether the known relations among MZVs can be interpolated by complex variable functions. Related to this question, there are several studies using various multiple zeta functions (see Tsumura [12], Matsumoto and Tsumura [8], and Nakamura [10], for example). Based on such circumstances, we give a complex variable interpolation of Theorem 1.2.

For an admissible index $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{Z}_{\geq 1}^r$ and $s \in \mathbb{C}$, we define

$$I_{\mathbf{k}}(s) := \sum_{i=1}^{r} \sum_{0<n_1<\cdots<n_r} \frac{1}{n_1^{k_1}\cdots n_r^{k_r}} \cdot \frac{1}{n_i^s} \prod_{j\neq i} \frac{n_j}{n_j - n_i}.$$

By Zhao and Zhou [14, Proposition 2.1], we can easily check that this series converges absolutely when $\Re(s) > -1$. This is a sum of special cases of $\zeta_{\mathfrak{sl}(r+1)}(s)$ which is called the Witten MZF associated with $\mathfrak{sl}(r+1)$. This function was first introduced in Matsumoto and Tsumura [9], it is also called the zeta function associated with the root system of type A_r (for more details, see Komori, Matsumoto and Tsumura [5]), and continued meromorphically to the whole complex space $\mathbb{C}^{r(r+1)/2}$. Hence $I_{\mathbf{k}}(s)$ can be continued meromorphically to $\mathbb{C}$.

THEOREM 1.3. *For an admissible index* $\mathbf{k}$ *and* $s \in \mathbb{C}$, *we have*

$$I_{\mathbf{k}}(s) = I_{\mathbf{k}^\dagger}(s).$$

REMARK 1.4. As we shall see in the next section, Theorem 1.3 is a generalization of Theorem 1.2 (see Lemma 2.3).

2. Proof of Theorem 1.3

LEMMA 2.1. *For $m \in \mathbb{Z}_{\geq 0}$ and $a_1, \ldots, a_r \in \mathbb{R}$ with $a_i \neq a_j$ for $i \neq j$, we have*

$$\sum_{\substack{e_1+\cdots+e_r=m \\ e_i \geq 0\,(1\leq i\leq r)}} a_1^{e_1}\cdots a_r^{e_r} = \sum_{i=1}^{r} a_i^{m+r-1}\prod_{j\neq i}(a_i-a_j)^{-1}.$$

PROOF. By putting

$$A_i := a_i^{r-1}\prod_{j\neq i}(a_i-a_j)^{-1},$$

we have

$$\frac{1}{1-a_1x}\cdots\frac{1}{1-a_rx} = \frac{A_1}{1-a_1x}+\cdots+\frac{A_r}{1-a_rx}. \tag{1}$$

Then we find the desired result.

REMARK 2.2. In the above proof, the reason that each A_i is concretely obtained is by multiplying both sides of equation (1) by $1-a_ix$, and then substituting $1/a_i$ for x.

LEMMA 2.3. *For an admissible index $\mathbf{k}=(k_1,\ldots,k_r)\in\mathbb{Z}_{\geq 1}^r$ and $m\in\mathbb{Z}_{\geq 0}$, we have*

$$I_{\mathbf{k}}(m) = \sum_{\substack{e_1+\cdots+e_r=m \\ e_i \geq 0\,(1\leq i\leq r)}} \zeta(k_1+e_1,\ldots,k_r+e_r).$$

PROOF. By Lemma 2.1, we have

$$\begin{aligned}
I_{\mathbf{k}}(m) &= \sum_{0<n_1<\ldots<n_r}\frac{1}{n_1^{k_1}\cdots n_r^{k_r}}\sum_{i=1}^{r}\left(\frac{1}{n_i}\right)^{m+r-1}\prod_{j\neq i}\frac{n_in_j}{n_j-n_i} \\
&= \sum_{0<n_1<\ldots<n_r}\frac{1}{n_1^{k_1}\cdots n_r^{k_r}}\sum_{i=1}^{r}\left(\frac{1}{n_i}\right)^{m+r-1}\prod_{j\neq i}\left(\frac{1}{n_i}-\frac{1}{n_j}\right)^{-1} \\
&= \sum_{0<n_1<\ldots<n_r}\frac{1}{n_1^{k_1}\cdots n_r^{k_r}}\sum_{\substack{e_1+\cdots+e_r=m \\ e_i \geq 0\,(1\leq i\leq r)}}\left(\frac{1}{n_1}\right)^{e_1}\cdots\left(\frac{1}{n_r}\right)^{e_r} \\
&= \sum_{\substack{e_1+\cdots+e_r=m \\ e_i \geq 0\,(1\leq i\leq r)}}\zeta(k_1+e_1,\ldots,k_r+e_r).
\end{aligned}$$

Then we find the desired result.

In the proof of Theorem 1.3, we use the following lemma (for details, see e.g., Apostol [2]).

LEMMA 2.4. *Let two Dirichlet series*

$$F(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \quad \textit{and} \quad G(s) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s},$$

both absolutely convergent for $\Re(s) > \sigma_a$ *be given. If* $F(s) = G(s)$ *for each* s *in an infinite sequence* $\{s_k\}$ *such that* $\Re(s_k) \to \infty$*, then* $f(n) = g(n)$ *for every* n.

PROOF OF THEOREM 1.3. By Theorem 1.2 and Lemma 2.3, we have $I_{\mathbf{k}}(s) = I_{\mathbf{k}^\dagger}(s)$ for $s \in \mathbb{Z}_{\geq 0}$. Since

$$\sum_{n_i=1}^{\infty} \frac{1}{n_i^s} \left(\sum_{0<n_1<\ldots<n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}} \prod_{j \neq i} \frac{n_j}{n_j - n_i} \right)$$

is a Dirichlet series for each i, the function $I_{\mathbf{k}}(s)$ is also a Dirichlet series. By Lemma 2.4, we have $I_{\mathbf{k}}(s) = I_{\mathbf{k}^\dagger}(s)$ for $\Re(s) > -1$. Since

$$\begin{aligned} I_{\mathbf{k}}(s) &= \sum_{i=1}^{r} \sum_{0<n_1<\ldots<n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}} \cdot \frac{1}{n_i^s} \prod_{j \neq i} \frac{n_j}{n_j - n_i} \\ &= \sum_{i=1}^{r} (-1)^{i-1} \sum_{m_1,\ldots,m_r=1}^{\infty} \frac{1}{m_1^{k_1-1}(m_1+m_2)^{k_2-1} \cdots (m_1+\cdots+m_r)^{k_r-1}} \\ &\quad \times \frac{1}{(m_1+\cdots+m_i)^{s+1}} \prod_{j<i} \frac{1}{m_{j+1}+\cdots+m_i} \prod_{j>i} \frac{1}{m_{i+1}+\cdots+m_j}, \end{aligned}$$

$I_{\mathbf{k}}(s)$ can be regarded as the sum of the zeta functions associated with the root system of type A_r. Thus $I_{\mathbf{k}}(s)$ can be meromorphically continued to the whole space $\mathbb{C}$.

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