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SOME VITALI THEOREMS FOR LEBESGUE MEASURE

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1. Introduction and summary.

We shall work in N-dimensional euclidean space R^N. The distance between points or subsets of R^N is denoted by $d(\cdot,\cdot)$. The interior of a set is denoted by int $(\cdot)$. For $x \in \mathsf{R}^N$ and $\delta \in \mathsf{R}_+$ the closed ball with center x and radius δ is denoted by $B[x,\delta]$. If $B \subseteq \mathsf{R}^N$ is a ball, cen (B) denotes the center and rad (B) the radius of B.

We use λ for Lebesgue measure and λ^* for the outer Lebesgue measure. A family $\mathscr{B}$ of subsets of R^N is called a *packing* if $\mathscr{B}$ consists of pairwise disjoint sets. Let $\mathscr{B}$ be a family of balls in R^N. We say that $A \subseteq \mathsf{R}^N$ *can be packed with sets from* $\mathscr{B}$ if there exists a packing $\mathscr{B}^* \subseteq \mathscr{B}$ such that

$$\lambda^*(A \setminus \bigcup \{B \mid B \in \mathscr{B}^*\}) = 0 .$$

In this paper we shall use the special notation $\tilde{A}$ for the class of closed balls disjoint with the subset A of R^N:

$$\tilde{A} = \{B \text{ closed ball} \mid B \cap A = \emptyset\} .$$

In many cases below the class $\tilde{A}$ could have been enlarged by requiring only that $\lambda^*(B \cap A) = 0$ instead of $B \cap A = \emptyset$.

For a given family $\mathscr{B}$ of closed balls it is our main goal to establish theorems to the effect that "large" regions A of R^N can be packed with balls from $\mathscr{B}$. To achieve this we shall not use accidentally occurring large balls in $\mathscr{B}$ but rather the local geometrical positions of the small balls in $\mathscr{B}$. More precisely, we shall only try to pack subsets of the *local set* which is the set

$$A_{\text{loc}} = \{x \in \mathsf{R}^N \mid \forall \delta \in \mathsf{R}_+ \ \exists B \in \mathscr{B} : B \subseteq B[x,\delta]\} .$$

All our results may be expressed in terms of certain functions which tell us how large balls in $\mathscr{B}$ we can find close to a point x. First consider the function which to any set $V \subseteq \mathsf{R}^N$ assigns the value

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$$\varrho(V) = \sup\{\lambda(B) \mid B \in \mathscr{B},\ B \subseteq V\}$$

(with $\sup \emptyset = 0$). Note that even though this definition is expressed in a measure theoretical way, it is really purely geometrical, since there is a simple relationship between the Lebesgue measure of a ball and its radius.

We write $\varrho(x,\delta)$ in place of $\varrho(B[x,\delta])$ and call the function $(x,\delta) \curvearrowright \varrho(x,\delta)$ the *ϱ-function* (*associated with* $\mathscr{B}$). Clearly, the local set A_{loc} consists of all points x for which $\varrho(x,\delta) > 0$ for all $\delta \in \mathsf{R}_+$.

By the *relative ϱ-function* we understand the function

$$\varrho^*(x,\delta) = \sup\{\lambda(B)/\lambda(B[x,\delta]) \mid B \in \mathscr{B},\ B \subseteq B[x,\delta]\}; \qquad x \in \mathsf{R}^N,\ \delta \in \mathsf{R}_+\ ,$$

i.e.

$$\varrho^*(x,\delta) = \varrho(x,\delta)/\lambda(B[x,\delta])\ .$$

Clearly, the ϱ-function is monotone:

$$\text{(1)} \qquad t > s \Rightarrow \varrho(x,t) \geqq \varrho(x,s)\ .$$

For the relative ϱ-function we only have the weaker result:

$$\text{(2)} \qquad t > s \Rightarrow \varrho^*(x,t) \geqq \varrho^*(x,s)\cdot(s/t)^N\ .$$

Furthermore, $\varrho^*(x,t) \leqq 1$ always holds.

We introduce the class Φ of non-negative measurable functions φ defined on an interval of the form $]0,t[$ with $0 < t \leqq +\infty$ for which

$$\int_0^{t_0} \varphi(t)\frac{dt}{t} = +\infty$$

for all sufficiently small t_0. We shall in the following always assume that $\varphi \leqq 1$ so that the integral above is either divergent for all $0 < t_0 < 1$ or convergent for all $0 < t_0 < 1$. We require that $\delta \to \varphi(\delta)\delta$ is non-decreasing.

Then, with the given class $\mathscr{B}$ of balls we associate the following sets

$$\begin{aligned} A_0 &= \{x \in \mathsf{R}^N \mid \varrho^*(x,\cdot) \in \Phi\}\ , \\ A_{\log} &= \{x \in \mathsf{R}^N \mid \varrho^*(x,\cdot)/|\log \varrho(x,\cdot)| \in \Phi\}\ , \\ A_{\text{hom}} &= \{x \in \mathsf{R}^N \mid \limsup_{\delta\to 0} \varrho^*(x,\delta) > 0\}\ . \end{aligned}$$

If, in a given context, it is desirable to stress the dependence of the above defined objects on the family $\mathscr{B}$, we may use such notation as $A_{\text{loc}}(\mathscr{B})$, $A_0(\mathscr{B})$, $A_{\log}(\mathscr{B})$ and $A_{\text{hom}}(\mathscr{B})$, and also we may use the notation $\varrho(x,\delta \mid \mathscr{B})$ and $\varrho^*(x,\delta \mid \mathscr{B})$.

LEMMA 1. *For any class $\mathscr{B}$ of balls the associated sets satisfy the inclusions*

$$A_{\text{hom}} \cup A_{\log} \subseteqq A_0 \subseteqq A_{\text{loc}}\,.$$

PROOF. Only the inclusion $A_{\text{hom}} \subseteqq A_0$ is non-trivial. To establish this inclusion, assume that $x \in A_{\text{hom}}$. Then there exists a constant $c \in \mathsf{R}_+$ and a sequence $1 = \delta_0 > \delta_1 > \delta_2 > \dots$ converging to 0, such that $\varrho^*(x, \delta_n) \geqq c$ for all $n \in \mathsf{N}$. Then by (2),

$$\begin{aligned}\int_0^1 \varrho^*(x,t)\frac{dt}{t} &\geqq \sum_{n=0}^{+\infty} \int_{\delta_{n+1}}^{\delta_n} \varrho^*(x,\delta_{n+1})\cdot(\delta_{n+1}/t)^N \frac{dt}{t} \\ &\geqq \frac{c}{N}\sum_{n=0}^{+\infty}(1-[\delta_{n+1}/\delta_n]^N)\,.\end{aligned}$$

Now, the infinite product $\prod_{n=0}^{+\infty}[\delta_{n+1}/\delta_n]^N$ is 0-divergent, hence the infinite sum in the above inequalities diverges. This shows that $x \in A_0$.

In order to formulate one of the main results we need a variant of the ϱ-function. Let $c \in \mathsf{R}_+$ be a constant and define the *neighbouring ϱ-function* with parameter c by

$$\varrho_c(x,\delta) = \sup\{\min(\lambda(B_1),\lambda(B_2)) \mid B_1 \in \mathscr{B},\ B_2 \in \mathscr{B},\ B_1 \cup B_2 \subseteqq B[x,\delta]\,,\ d(\operatorname{cen} B_1, \operatorname{cen} B_2) \geqq c\delta\}\,.$$

The corresponding relative function is defined by

$$\varrho_c^*(x,\delta) = \varrho_c(x,\delta)/\lambda(B[x,\delta])\,.$$

We may then define the following sets

$$\begin{aligned} A_{\text{nbh}}^c &= \{x \in \mathsf{R}^N \mid \varrho_c^*(x,\cdot) \in \Phi\}, \qquad c \in \mathsf{R}_+\,, \\ A_{\text{nbh}} &= \bigcup\{A_{\text{nbh}}^c \mid c \in \mathsf{R}_+\}\,. \end{aligned}$$

The following two results constitute the core of our paper. The first result, due to M. Talagrand, is a negative one:

THEOREM 1. (M. Talagrand). *There exists a class $\mathscr{B}$ of closed balls such that it is impossible to pack the set $A_0 = A_0(\mathscr{B})$ by sets from $\mathscr{B}$. Indeed, one can achieve that*

$$\lambda(A_0 \setminus \bigcup\{B \mid B \in \mathscr{B}\}) > 0\,.$$

Actually, this gives a counterexample to a conjecture raised in [2] (cf. the discussion following Corollary 1 of [2]). Nevertheless, it is our opinion that it is "almost" possible to pack the set A_0. This is supported by the following result.

THEOREM 2. *Let $\mathscr{B}$ be a class of closed balls in R^N. Then it is possible to pack each of the sets A_{hom}, $A_{\log}$ and A_{nbh} by sets from $\mathscr{B}$.*

This implies (by Lemma 4 below) that there exists a packing $\mathscr{B}^* \subseteqq \mathscr{B}$, such that

$$\lambda((A_{hom} \cup A_{\log} \cup A_{nbh}) \setminus \bigcup\{B \mid B \in \mathscr{B}^*\}) = 0 .$$

Note that Theorem 2 as far as A_{hom} is concerned is a classical result, cf. the discussion of Theorem 1 in [2] (noting the regrettable mistake with lim inf instead of lim sup).

In the discussion above we have seen how we to a given class $\mathscr{B}$ of balls may associate various sets acting as candidates for sets to be packed by balls from $\mathscr{B}$. It is convenient to isolate this aspect in an abstract definition.

Let $\mathfrak{V}$ be a family of pairs $(A, \mathscr{B})$ where $A \subseteqq \mathsf{R}^N$ and where $\mathscr{B}$ is a class of closed balls in R^N. Following [2], we call $\mathfrak{V}$ a *Vitali system* if the following two axioms hold:

$$(A, \mathscr{B}) \in \mathfrak{V},\ D \subseteqq A \Rightarrow (D, \mathscr{B}) \in \mathfrak{V}$$

$$(A, \mathscr{B}) \in \mathfrak{V},\ F \text{ closed} \Rightarrow (A \setminus F, \mathscr{B} \cap \tilde{F}) \in \mathfrak{V} .$$

We say that the *packing theorem* holds for $\mathfrak{V}$ if, whenever $(A, \mathscr{B}) \in \mathfrak{V}$, it is possible to pack A with sets from $\mathscr{B}$.

Especially we shall discuss the concrete Vitali systems $\mathfrak{V}_{\mathrm{loc}}$, $\mathfrak{V}_0$, $\mathfrak{V}_{\mathrm{hom}}$, $\mathfrak{V}_{\log}$ and $\mathfrak{V}_{\mathrm{nbh}}$ which are defined as follows:

$$(A, \mathscr{B}) \in \mathfrak{V}_{\mathrm{loc}} \Leftrightarrow A \subseteqq A_{\mathrm{loc}}(\mathscr{B}) ,$$

$$(A, \mathscr{B}) \in \mathfrak{V}_0 \Leftrightarrow A \subseteqq A_0(\mathscr{B}) ,$$

$$(A, \mathscr{B}) \in \mathfrak{V}_{\mathrm{hom}} \Leftrightarrow A \subseteqq A_{\mathrm{hom}}(\mathscr{B}) ,$$

$$(A, \mathscr{B}) \in \mathfrak{V}_{\log} \Leftrightarrow A \subseteqq A_{\log}(\mathscr{B}) ,$$

$$(A, \mathscr{B}) \in \mathfrak{V}_{\mathrm{nbh}} \Leftrightarrow A \subseteqq A_{\mathrm{nbh}}(\mathscr{B}) .$$

The main results then say that the packing theorem fails for $\mathfrak{V}_0$ (and of course for $\mathfrak{V}_{\mathrm{loc}}$) whereas it holds for each of the Vitali systems $\mathfrak{V}_{\mathrm{hom}}$, $\mathfrak{V}_{\log}$ and $\mathfrak{V}_{\mathrm{nbh}}$.

All the concrete Vitali systems we have considered are of local type in the sense that $(A, \mathscr{B}) \in \mathfrak{V}$ if and only if $(\{x\}, \mathscr{B}) \in \mathfrak{V}$ for all $x \in A$. This fact may be formalised by defining a *local property* p as a property defined for every pair $(x, \mathscr{B})$, with $x \in \mathsf{R}^N$ and $\mathscr{B}$ a class of closed balls in R^N, such that

$$p(x, \mathscr{B}) \text{ is true} \Rightarrow p(x, \mathscr{B}_\delta) \text{ is true for all } \delta \in \mathsf{R}_+, \text{ where}$$

$$\mathscr{B}_\delta = \{B \in \mathscr{B} \mid B \subseteqq B[x, \delta]\} .$$

Then the *Vitali system* $\mathfrak{V}_p$ *induced by* p is defined by

$$(A, \mathscr{B}) \in \mathfrak{V}_p \Leftrightarrow p(x, \mathscr{B}) \text{ is true for all } x \in A .$$

The following trivial observation is sometimes useful:

LEMMA 2. *Assume that the packing theorem holds for the Vitali system* $\mathfrak{V}$. *If* A *is a subset of* R^N *and* $\mathscr{B}$ *a family of closed balls such that*

$$(A, \mathscr{B} \cup \tilde{A}) \in \mathfrak{V} ,$$

then A *can be packed with balls from* $\mathscr{B}$.

An interesting special case arises if $\mathscr{B} = \emptyset$. This leads to the following concept: Let $\mathfrak{V}$ be a Vitali system and define the class $\mathfrak{Z} = \mathfrak{Z}_{\mathfrak{V}}$ of $\mathfrak{V}$-*meagre* sets by

$$Z \in \mathfrak{Z} \Leftrightarrow (Z, \tilde{Z}) \in \mathfrak{V} .$$

By Lemma 2, every $\mathfrak{V}$-meagre set is a Lebesgue nullset provided the packing theorem holds for $\mathfrak{V}$. The concrete criteria for Lebesgue nullsets which can be obtained in this way from Theorem 2 are left for the reader to formulate.

To justify the terminology above it may be remarked that for the *local Vitali system* $\mathfrak{V}_{\text{loc}}$ the class of $\mathfrak{V}_{\text{loc}}$-meagre sets consists precisely of *all nowhere dense sets.*

Let $Z \subseteqq \mathsf{R}^N$. Then the *gap function* η_Z and the *relative gap function* η_Z^* are defined by

$$\eta_Z(x, \delta) = \varrho(x, \delta \mid \tilde{Z}) ,$$

$$\eta_Z^*(x, \delta) = \varrho^*(x, \delta \mid \tilde{Z}) .$$

We see that Z is a $\mathfrak{V}_0$-meagre set if and only if $\eta_Z^*(x, \cdot) \in \Phi$ for all $x \in Z$.

As other applications of packing theorems we mention differentiation and density theorems. We take as starting point a local property p. With p we associate a notion of contraction (cf. [2]) defined as follows: If $(B_\alpha)_{\alpha \in D}$ is a net of closed balls in R^N and if $x \in \mathsf{R}^N$, we write $B_\alpha \to x[p]$ if, for every $\delta \in \mathsf{R}_+$, B_α is eventually contained in $B[x, \delta]$ and if, for every $\alpha_0 \in D$, the property $p(x, \{B_\alpha \mid \alpha \geqq \alpha_0\})$ is true. From Proposition 2 of [2] we have

PROPOSITION 1. *Let* p *be a local property such that the packing theorem holds for the Vitali system* $\mathfrak{V}_p$.

If $f \colon \mathsf{R}^N \to \mathsf{R}$ *is locally integrable, then there exists a Lebesgue nullset* Δ *such that, for every* $x \in \mathsf{R}^N \setminus \Delta$ *and for every net* (B_α) *with* $B_\alpha \to x[p]$, $f(x)$ *is a point of accumulation of the net*

$$\left([\lambda(B_\alpha)]^{-1}\int_{B_\alpha} f\,d\lambda\right)_\alpha .$$

If f is an indicator function, we get a density theorem rather than a differentiation theorem. Of course, this density theorem may be viewed upon as a generalization of the previously mentioned criteria for Lebesgue nullsets since, if Z is a $\mathfrak{B}_p$-meagre set, then to any $x \in Z$ there exists $B_\alpha \to x[p]$ with $B_\alpha \cap Z = \varnothing$ for all α.

Just as in [2], a main technical tool in establishing packing theorems is the following

LEMMA 3. *Let $\mathfrak{B}$ be a Vitali system and assume that there exists a constant $c \in \mathsf{R}_+$ such that for every $(A,\mathscr{B}) \in \mathfrak{B}$ with A bounded, there is a packing $\mathscr{B}^* \subseteqq \mathscr{B}$ for which*

$$\lambda(\bigcup \{B \mid B \in \mathscr{B}^*\}) \geqq c\lambda^*(A) .$$

Then the packing theorem holds for $\mathfrak{B}$.

From this lemma we deduce

LEMMA 4. *Assume that the packing theorem holds for each of the Vitali systems $\mathfrak{B}_n$, $n \in \mathsf{N}$. Then the packing theorem also holds for the Vitali system $\mathfrak{B} = \oplus_{n=1}^{+\infty} \mathfrak{B}_n$ defined by*

$$(A,\mathscr{B}) \in \mathfrak{B} \Leftrightarrow \text{there exist } A_n,\ n \in \mathsf{N}, \text{ with } \bigcup A_n = A$$
$$\text{such that } (A_n,\mathscr{B}) \in \mathfrak{B}_n \text{ for all } n \in \mathsf{N} .$$

The proof is left to the reader (first establish the result for a finite collection $\mathfrak{B}_n$, $n \leqq n_0$, of Vitali systems, and then prove the general result; employ Lemma 3 in both parts of the proof).

It may be remarked that if $\mathfrak{B} = \mathfrak{B}_p$ is a Vitali system of local type, then $\mathfrak{B} \oplus \mathfrak{B} \oplus \ldots = \mathfrak{B}$.

We end our introduction by showing how the main result of [2] may be deduced from Theorem 2.

THEOREM 3. (cf. Theorem 2 of [2]). *Let $\mathscr{B}$ be a class of closed balls in R^N and let $\varphi \in \Phi$. Then it is possible to pack the set $A_\varphi = A_\varphi(\mathscr{B})$ defined by*

$$A_\varphi = \{x \in \mathsf{R}^N \mid \exists\, \delta_0 \in \mathsf{R}_+ \ \forall\, \delta \leqq \delta_0 \colon \varrho^*(x,\delta) \geqq \varphi(\delta)\}$$

with balls from $\mathscr{B}$.

PROOF. Clearly, $\mathfrak{V}_\varphi$ defined by

$$(A,\mathscr{B}) \in \mathfrak{V}_\varphi \Leftrightarrow A \subseteqq A_\varphi(\mathscr{B})$$

is a Vitali system (a screened Vitali system in the terminology of [2]). Without loss of generality, we may assume that $\varphi(\delta) \leqq 2^{1-N}$.

Choose, to the given family $\mathscr{B}$, a number $\delta_0 \in \mathbf{R}_+$ such that

$$\lambda^*(A_{\varphi,\delta_0}) \geqq \tfrac{1}{2}\lambda^*(A_\varphi)$$

where

$$A_{\varphi,\delta_0} = \{x \in \mathbf{R}^N \mid \forall \delta \leqq \delta_0 \colon \varrho^*(x,\delta) \geqq \varphi(\delta)\} .$$

By Lemma 3, it suffices to show that A_{φ,δ_0} can be packed with balls from $\mathscr{B}$. This we prove by showing that

$$A_{\varphi,\delta_0} \subseteqq A_{\mathrm{nbh}}(\mathscr{B} \cup \tilde{A}_{\varphi,\delta_0}) . \tag{3}$$

That this leads to the desired conclusion follows from Theorem 2 and Lemma 2. Put

$$\mathscr{B}^* = \mathscr{B} \cup \tilde{A}_{\varphi,\delta_0} .$$

Assume that $x \in A_{\varphi,\delta_0}$. Let $\delta \leqq \delta_0$. Choose closed balls B' and B'' such that

$$d(x, \mathrm{cen}\, B') = d(x, \mathrm{cen}\, B'') = \tfrac{5}{8}\delta ,$$

$$d(\mathrm{cen}\, B', \mathrm{cen}\, B'') = \tfrac{5}{4}\delta ,$$

$$\mathrm{rad}\,(B') = \mathrm{rad}\,(B'') = \tfrac{1}{8}\delta .$$

We shall use B' and B'' to define two balls B_1 and B_2 in $\mathscr{B}^*$. If $B' \in \mathscr{B}^*$, we put $B_1 = B'$. Then $B_1 \subseteqq B[x,\delta]$ and

$$\lambda(B_1)/\lambda(B[x,\delta]) = 8^{-N} \geqq \tfrac{1}{2}\cdot 4^{-N}\cdot \varphi(\tfrac{1}{4}\delta) .$$

If $B' \notin \mathscr{B}^*$, we choose $x' \in B' \cap A_{\varphi,\delta_0}$. Then we choose $B_1 \in \mathscr{B}$ such that

$$B_1 \subseteqq B[x', \tfrac{1}{4}\delta] \subseteqq B[x,\delta] ,$$

$$\lambda(B_1)/\lambda(B[x',\tfrac{1}{4}\delta]) \geqq \tfrac{1}{2}\varrho^*(x',\tfrac{1}{4}\delta) \geqq \tfrac{1}{2}\varphi(\tfrac{1}{4}\delta) .$$

Then

$$\lambda(B_1)/\lambda(B[x',\tfrac{1}{4}\delta]) \geqq \tfrac{1}{2}\varrho^*(x',\tfrac{1}{4}\delta) \geqq \tfrac{1}{2}\varphi(\tfrac{1}{4}\delta) .$$

The ball B_2 is constructed similarly from B''. By construction,

$$d(\mathrm{cen}\, B_1, \mathrm{cen}\, B_2) \geqq \tfrac{1}{2}\delta .$$

Then

$$\varrho^*_{\frac{1}{4}}(x,\delta \mid \mathscr{B}^*) \geqq \tfrac{1}{2}\cdot 4^{-N}\cdot \varphi(\tfrac{1}{4}\delta) .$$

As this holds for every $\delta \leqq \delta_0$, it follows that $\varrho_{\frac{1}{2}}^*(x, \cdot \mid \mathscr{B}^*) \in \Phi$, hence $x \in A_{\mathrm{nbh}}(\mathscr{B}^*)$.

Note that by Theorem 1 it is not always possible to pack the union $\bigcup\{A_\varphi \mid \varphi \in \Phi\}$ with balls from $\mathscr{B}$ as this union is identical with A_0. However, by Lemma 3, if $\Phi^* \subseteqq \Phi$ is countable, it is always possible to pack $\bigcup\{A_\varphi \mid \varphi \in \Phi^*\}$ with balls from $\mathscr{B}$.

All our efforts have been attempts to prove a unified and "natural" packing theorem. Especially, our criterion to such a result has been that it should contain the classical result for $\mathfrak{B}_{\mathrm{hom}}$ as well as the previously established Theorem 3. Actually, we have not reached this goal. We feel, that if there exists a Vitali system with the desirable properties, it must be a subsystem of $\mathfrak{B}_0$ which, as we recall, was defined via the function class Φ. That this class of functions plays a certain canonical role has been demonstrated with the present results; also, it should be recalled that the condition $\varphi \in \Phi$ in Theorem 3 is necessary (cf. Theorem 2 of [2]).

2. Talagrand's example.

In this section we shall prove Theorem 1. We start with a closer examination of the condition $\varrho^*(x, \cdot) \in \Phi$ appearing in the definition of the Vitali system $\mathfrak{B}_0$.

Let $\mathscr{B}$ be a class of closed balls in R^N. For each $n \in \mathsf{N}_0$ we define functions $\alpha_n \colon \mathsf{R}^N \to [0, +\infty]$ and $\tilde{\alpha}_n \colon \mathsf{R}^N \to [0, +\infty]$ by

$$\alpha_n(x) = \inf\{d(x, \operatorname{cen}(B)) \mid B \in \mathscr{B},\ 2^{-n-1} < \operatorname{rad}(B) \leqq 2^{-n}\}$$

$$\tilde{\alpha}_n(x) = \inf\{d(x, B) \mid B \in \mathscr{B},\ 2^{-n-1} < \operatorname{rad}(B) \leqq 2^{-n}\}\ .$$

Roughly speaking, these functions give the distance to the closest ball in $\mathscr{B}$ of radius approximately equal to 2^{-n}. If there is no ball in $\mathscr{B}$ with $2^{-n-1} < \operatorname{rad}(B) \leqq 2^{-n}$, we have $\alpha_n(x) = \tilde{\alpha}_n(x) = +\infty$ for all $x \in \mathsf{R}^N$. As usual, the dependence of α_n and $\tilde{\alpha}_n$ on $\mathscr{B}$ is notationally suppressed.

LEMMA 5. *Let $\mathscr{B}$ be a class of closed balls in R^N and let $x \in \mathsf{R}^N$. Then the following four conditions are equivalent:*

(i) $$\varrho^*(x, \cdot) \in \Phi\ ,$$

(ii) $$\sum_{n=0}^{+\infty} \varrho^*(x, 2^{-n}) = +\infty\ ,$$

(iii) $$\forall n_0 \in \mathbf{N}: \sum_{n=n_0}^{+\infty} (2^{-n}/\alpha_n(x))^N = +\infty ,$$

(iv) $$\forall n_0 \in \mathbf{N}: \sum_{n=n_0}^{+\infty} (2^{-n}/\tilde{\alpha}_n(x))^N = +\infty .$$

PROOF. To simplify the notation, assume that $N=1$. We may also assume that $x \in A_{\mathrm{loc}}$. We introduce the function

$$r(x,\delta) = \sup\{\mathrm{rad}\,(B) \mid B \in \mathscr{B},\, B \subseteqq B[x,\delta]\} .$$

Then (as $N=1$),

$$\varrho^*(x,\delta) = r(x,\delta)/\delta .$$

The equivalence of (i) and (ii) was noted in [2]. As $\tilde{\alpha}_n \leqq \alpha_n$, the implication (iii) $\Rightarrow$ (iv) is obvious. The implications (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (ii) then remain to be shown. We start with some general remarks.

For each $n \in \mathbf{N}_0$ determine an integer k_n such that

$$2^{-k_n-1} < r(x,2^{-n}) \leqq 2^{-k_n} .$$

Then $k_n \geqq n$ and $k_0 \leqq k_1 \leqq \dots$.

By a small argument it is seen that if $k_n \leqq 3+n$ for infinitely many n then (ii), (iii), and (iv) all hold. We may therefore assume that $k_n > 3+n$, eventually. In order to simplify the notation we assume that

$$k_n > 3+n \qquad \text{for all } n \in \mathbf{N}_0 .$$

Also we may and do assume that $\tilde{\alpha}_n(x) > 0$ for all $n \in \mathbf{N}_0$. Clearly,

$$\alpha_{k_n}(x) \leqq 2^{-n}, \qquad n \in \mathbf{N}_0 .$$

Determine numbers $s(0) < s(1) < \dots$ such that

$$k_{s(\nu-1)+1} = k_{s(\nu-1)+2} = \dots = k_{s(\nu)} < k_{s(\nu)+1}, \qquad \nu \in \mathbf{N}_0 ,$$

with $s(-1) = -1$. Put $k(\nu) = k_{s(\nu)}$. Then it is easily seen that

$$\sum_{n=0}^{+\infty} \varrho^*(x,2^{-n}) = +\infty \Leftrightarrow \sum_{\nu=0}^{+\infty} 2^{-(k(\nu)-s(\nu))} = +\infty .$$

Since

$$\sum_{n=0}^{+\infty} (2^{-n}/\alpha_n(x)) \geqq \sum_{\nu=0}^{+\infty} (2^{-k(\nu)}/\alpha_{k(\nu)}(x)) \geqq \sum_{\nu=0}^{+\infty} 2^{-(k(\nu)-s(\nu))} ,$$

it is clear that (ii) implies (iii).

To prove the remaining implication we remark that for $\nu \in \mathbf{N}_0$ there is no ball in $\mathscr{B}$ with a radius $> 2^{-k(\nu+1)}$ which is entirely contained in $B[x, 2^{-s(\nu)-1}]$. It follows that for any n with $k(\nu) \leqq n < k(\nu+1)$ we have

$$\tilde{\alpha}_n(x) \geqq 2^{-s(\nu)-1} - 2 \cdot 2^{-n} \geqq 2^{-s(\nu)-1} - 2 \cdot 2^{-k(\nu)}$$
$$\geqq 2^{-s(\nu)-1} - 2 \cdot 2^{-s(\nu)-3} = \tfrac{1}{4} \cdot 2^{-s(\nu)} .$$

Then

$$\sum_{n=k(0)}^{+\infty} (2^{-n}/\tilde{\alpha}_n(x)) = \sum_{\nu=0}^{+\infty} \sum_{n=k(\nu)}^{k(\nu+1)-1} (2^{-n}/\tilde{\alpha}_n(x)) \leqq 8 \sum_{\nu=0}^{+\infty} 2^{-(k(\nu)-s(\nu))} ,$$

and the implication (iv) $\Rightarrow$ (ii) follows.

REMARK. The term $2^{-n}/\alpha_n(x)$ is essentially equal to the reciprocal of the number of steps of length 2^{-n} we have to take in order to get from x to a center of a ball in $\mathscr{B}$ with a radius in the interval $]2^{-n-1}, 2^{-n}]$.

We shall establish Theorem 1 by showing the existence of a $\mathfrak{V}_0$-meagre set of positive Lebesgue measure. The set we shall find will be a subset of $[0, 1]$. For any $Z \subseteqq [0, 1]$ we write $\alpha_n(x \mid Z)$ for the α_n-function associated with $\tilde{Z}$.

LEMMA 6. *If to any $\varepsilon \in \mathbf{R}_+$ and any positive constant K there exists a set $Z \subseteqq [0, 1]$ such that*

(i) $$\lambda(Z) \geqq 1 - \varepsilon ,$$

(ii) $$\sum_{n=0}^{+\infty} (2^{-n}/\alpha_n(x \mid Z)) \geqq K \quad \text{a.e. on } Z ,$$

then there exists a $\mathfrak{V}_0$-meagre set of positive Lebesgue measure.

PROOF. Let $(\varepsilon_\nu)_{\nu \in \mathbf{N}}$ and $(K_\nu)_{\nu \in \mathbf{N}}$ be suitably chosen sequences of positive numbers with $K_\nu \uparrow +\infty$. Let $Z_\nu \subseteqq [0, 1]$ satisfy the stipulated conditions with respect to ε_ν and K_ν. We may even assume that the inequality (ii) for Z_ν holds for all $x \in Z_\nu$ since the sum in (ii) is increased when Z_ν is decreased.

Let $Z = \bigcap_{\nu=1}^{+\infty} Z_\nu$. Then $\lambda(Z) > 0$ if the ε_ν's are suitably chosen, and as $Z \subseteqq Z_\nu$ for all ν we have

$$\sum_{n=0}^{+\infty} (2^{-n}/\tilde{\alpha}_n(x \mid Z)) = +\infty \qquad \text{for all } x \in Z .$$

Thus, the set we obtain has positive measure and furthermore by Lemma 5, it is $\mathfrak{V}_0$-meagre.

We now embark upon the essential construction. Let $(n_p)_{p\in\mathbf{N}}$ be a sequence of positive integers, define

$$\varepsilon_p = n_p \cdot 2^{-2^p}, \quad p \in \mathbf{N},$$

and make sure that

$$\sum_{p=1}^{+\infty} \varepsilon_p \leqq 1, \quad \sum_{p=1}^{+\infty} \varepsilon_p \log \varepsilon_p^{-1} = +\infty . \tag{4}$$

Given constants $\varepsilon \in \mathbf{R}_+$ and $K \in \mathbf{R}_+$ (cf. Lemma 6), choose p_0 such that

$$\sum_{p=p_0}^{+\infty} \varepsilon_p \leqq \varepsilon . \tag{5}$$

We shall construct $Z \subseteqq [0,1]$ *fulfilling* (*i*) *and* (*ii*) *in Lemma* 6 *such that*

$$[0,1] \setminus Z = \bigcup_{p=p_0}^{+\infty} B_p ,$$

where the B_p's are disjoint, and each B_p is a union of precisely n_p dyadic intervals of length 2^{-2^p}. More will be required below, but first we consider any such sequence $(B_p)_{p\geqq p_0}$.

For each $p \geqq p_0$ define a function β_p: $[0,1] \to \mathbf{N}_0$ counting the number of dyadic intervals of length 2^{-2^p} you have to visit in order to come from a point x in $[0,1]$ to a B_p-interval (cf. Figure 1).

The β-functions are related to the $\tilde{\alpha}$-functions associated with Z by means of the obvious formula

$$2 \cdot 2^{-(2^p+1)}/\tilde{\alpha}_{2^p+1}(x \mid Z) \geqq 1/\beta_p(x) . \tag{6}$$

For $p \geqq p_0 - 1$ we define

$$H_p = \left\{ x \in [0,1] \;\middle|\; \sum_{\nu=p_0}^{p} 1/\beta_\nu(x) < K \right\}$$

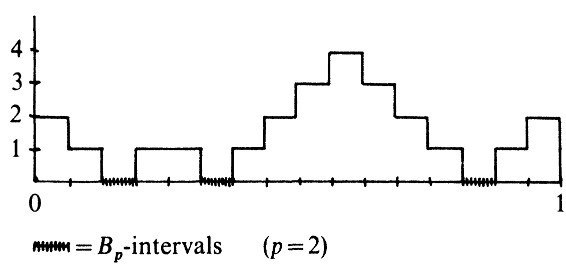


Figure 1.

(for $p=p_0-1$ we get $H_p=[0,1]$). We note that $(H_p)_{p\geqq p_0-1}$ is decreasing. The following observation will be important in the sequel:

$$\text{(7)}\qquad \sum_{p=p_0}^{+\infty} \beta_{p+1}(x)^{-1}\cdot 1_{H_p\setminus B_{p+1}}(x) \leqq K+1 \qquad \text{for all } x\in[0,1]\,.$$

The proof is left to the reader (consider the first ν, if any, such that $x \notin H_{\nu+1}$).

Clearly, H_p is disjoint with $B_{p_0}\cup\ldots\cup B_p$. We may describe H_p as a union over a certain set $\mathscr{I}_p$ of dyadic intervals of length 2^{-2^p}.

The actual (inductive) construction of the B_p-sets is so designed that the n_{p+1} dyadic intervals making up B_{p+1} are evenly spread out over H_p for each $p\geqq p_0-1$. In more detail, what we mean by this is the following:

If $\varepsilon_{p+1}>\lambda(H_p)$ we just make sure that $B_{p+1}\supseteq H_p$ [and $B_{p+1}\cap(B_{p_0}\cup\ldots\cup B_p)=\varnothing$, $\lambda(B_{p+1})=\varepsilon_{p+1}$]. In the case $\varepsilon_{p+1}\leqq\lambda(H_p)$ we put

$$\nu_{p+1} = [\varepsilon_{p+1}\cdot 2^{2^p}/\lambda(H_p)]$$

($[\cdot]=$"integer part") and for $I\in\mathscr{I}_p$ we select ν_I dyadic subintervals of I of length $2^{-2^{p+1}}$ such that

$$\nu_{p+1} \leqq \nu_I \leqq \nu_{p+1}+1, \qquad I\in\mathscr{I}_p\,,$$

$$\sum_{I\in\mathscr{I}_p} \nu_I = n_{p+1}\,,$$

and such that, for each $I\in\mathscr{I}_p$ the number s_{p+1} of dyadic intervals of length $2^{-2^{p+1}}$ lying between the first ν_{p+1} of the ν_I selected intervals is given by

$$\text{(8)}\qquad s_{p+1}=\text{largest integer such that } \nu_{p+1}+1+(\nu_{p+1}-1)s_{p+1}\leqq 2^{2^p}$$

(note that I consists of 2^{2^p} dyadic intervals of length $2^{-2^{p+1}}$), cf. Figure 2.

We have now given a complete description of the construction.

By (5), $\lambda(Z)\geqq 1-\varepsilon$. By Lemma 6 and (6), we realize that we only have to prove that

$$\text{(9)}\qquad \lambda(H_p)\downarrow 0\,.$$

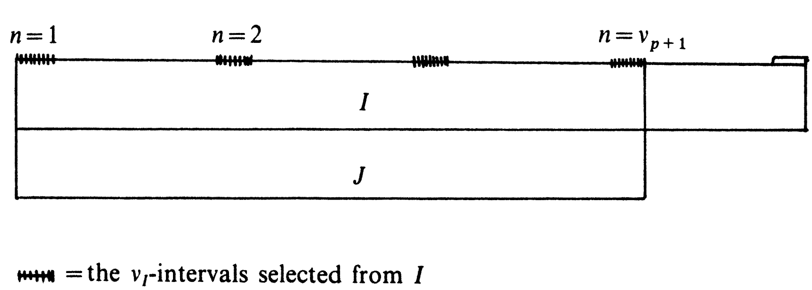


Figure 2.

This is clear if, for infinitely many p we have $\varepsilon_{p+1} \geqq \frac{1}{3}\lambda(H_p)$ [since then $H_p \downarrow \varnothing$]. Assume now that

$$\varepsilon_{p+1} < \tfrac{1}{3}\lambda(H_p) \quad \text{for } p \geqq p_1 .$$

By a tedious calculation it follows that

$$s_{p+1} \geqq \tfrac{1}{3}\lambda(H_p)/\varepsilon_{p+1} \quad \text{for } p \geqq p_1 .$$

For $p \geqq p_1$ we now evaluate the integral of the summand in (7). Let $I \in \mathscr{I}_p$ and define J as indicated on Figure 2. We have

$$\begin{aligned}\int \beta_{p+1}(x)^{-1} . 1_{H_p \setminus B_{p+1}}(x)\,dx &\geqq \lambda(H_p) \cdot 2^{2^p} \int_{J \setminus B_{p+1}} \beta_{p+1}(x)^{-1}\,dx \\ &\geqq \lambda(H_p) \cdot 2^{2^p} \cdot v_{p+1} \cdot \left(\frac{1}{1} + \ldots + \frac{1}{s_{p+1}}\right) \cdot 2^{-2^{p+1}} \\ &\geqq \lambda(H_p) \cdot 2^{2^p} \cdot (\varepsilon_{p+1} \cdot 2^{2^p} \cdot \lambda(H_p)^{-1} - 1) \log s_{p+1} \cdot 2^{-2^{p+1}} \\ &\geqq \varepsilon_{p+1} \log(\lambda(H_p)/3\varepsilon_{p+1}) - 2^{-2^p} \log 2^{2^p} .\end{aligned}$$

Put

$$\gamma = \sum_{p=1}^{+\infty} 2^{-2^p} \log 2^{2^p} .$$

Then $\gamma < +\infty$ and summing the inequality above we find by (7) that

$$\sum_{p=p_1}^{+\infty} \varepsilon_{p+1} \log\left(\tfrac{1}{3}\lambda(H_p)/\varepsilon_{p+1}\right) \leqq K + 1 + \gamma .$$

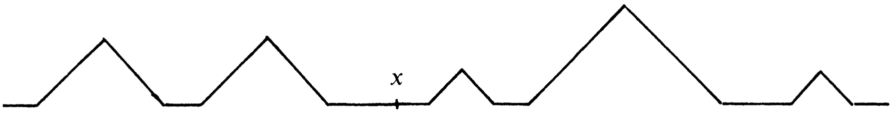
By (4) this implies (9).

Let us give a reformulation of Talagrand's result.

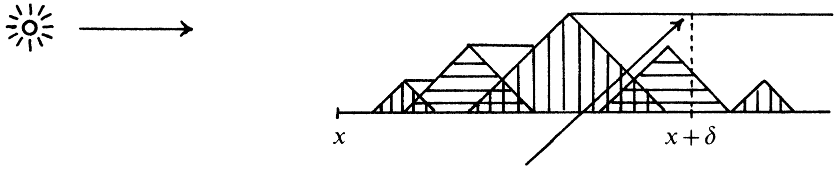
THEOREM 4. *There exists a subset $Z \subseteqq \mathsf{R}$ of positive measure such that, for every $x \in Z$, the sunshine function obtained by reflecting the function $t \to d(t, Z)$ in x belongs to the class Φ.*

The more precise meaning of this theorem follows from Figure 3.

This negative result should be compared with the result, which follows from Theorem 3, namely that if there exists a fixed function in the class Φ such that each sunshine function dominates this fixed function for sufficiently small values of the argument δ, then the set in question is a null-set.



the function $t \to d(t, Z)$



the sunshine function at δ

Figure 3.

3. Selection of a class $\{T_n\}$ of closed, disjoint balls from $\mathscr{B}$.

Let $(A, \mathscr{B})$ be a member of some Vitali system $\mathfrak{V}$. In the selection of a class $\{T_n\}$ of closed, disjoint balls from $\mathscr{B}$ we shall use the following procedure, disregarding the special geometrical conditions imposed on $\mathfrak{V}$. The procedure is similar to that given in [1], p. 20.

Without loss of generality we may assume that $\operatorname{rad}(B) \leqq \frac{1}{4}$ for $B \in \mathscr{B}$, that $A \subseteqq [0,1]^N$ and that $B \subseteqq [0,1]^N$ for $B \in \mathscr{B}$. (As R^N can be written as a countable union of closed cubes, it is enough to consider this special case.) From technical reasons we shall in the following use the maximum norm instead of the euclidean norm. Due to Lemma 3 this is no restriction. Thus, a ball from $\mathscr{B}$ is in fact a cube, and we may use both designations interchangeably.

We define $\mathscr{B}_0 = \mathscr{B}$. Choose a $T_1 \in \mathscr{B}_0$, such that

$$\lambda(T_1) \geqq \tfrac{1}{2} \sup \{\lambda(T) \mid T \in \mathscr{B}_0\} ,$$

and define

$$\mathscr{B}_1 = \mathscr{B}_0 \cap \tilde{T}_1 .$$

As $(A, \mathscr{B}_0)$ is a member of the Vitali system $\mathfrak{V}$ we conclude that $(A \setminus T_1, \mathscr{B}_1) \in \mathfrak{V}$, and thus the process may be repeated. By induction we choose a $T_n \in \mathscr{B}_{n-1}$, such that

$$\lambda(T_n) \geqq \tfrac{1}{2} \sup \{\lambda(T) \mid T \in \mathscr{B}_{n-1}\} , \tag{10}$$

and we define

$$\mathscr{B}_n = \mathscr{B}_{n-1} \cap \tilde{T}_n . \tag{11}$$

Then $(A \setminus \bigcup_{i=1}^{n} T_i, \mathscr{B}_n) \in \mathfrak{V}$.

In this way we define a system $\{T_n \mid n \in J\}$ of closed, disjoint balls from $\mathscr{B}$, where $J=\{1,\ldots,n\}$ or $J=\mathbf{N}$. Suppose that J is finite. Then $\mathscr{B}_n=\emptyset$ for a sufficiently large n. If $\mathfrak{V}$ is anyone of the systems $\mathfrak{V}_{\log}$, $\mathfrak{V}_{\mathrm{nbh}}$, $\mathfrak{V}_{\mathrm{hom}}$, one readily concludes that $\lambda^*(A \setminus \bigcup_{i=1}^{n} T_i)=0$. Thus, in the following we may assume that $J=\mathbf{N}$.

As $\{T_n \mid n \in \mathbf{N}\}$ is a disjoint sequence of closed sets contained in $[0,1]^N$, it follows that

$$\sum_{n=1}^{+\infty} \lambda(T_n) = \lambda\left(\bigcup_{n=1}^{+\infty} T_n\right) \leqq 1 ,$$

which shows that $\lambda(T_n) \to 0$ for $n \to +\infty$. We shall assume that $\{T_n\}$ has been enumerated, such that $\{\lambda(T_n)\}$ is a nonincreasing sequence of positive numbers.

We prove some lemmata concerning the chosen class $\{T_n\}$ of closed, disjoint balls from $\mathscr{B}$.

LEMMA 7. *To each $T \in \mathscr{B}$ there exists a $j \in \mathbf{N}$, such that $T \cap T_j \neq \emptyset$ and $\lambda(T) \leqq 2\lambda(T_j)$.*

PROOF. Suppose that $T \in \mathscr{B}$ is a closed ball for which $T \cap T_n=\emptyset$ for all $n \in \mathbf{N}$. According to (11) this means that T belongs to each system $\mathscr{B}_n$, $n \in \mathbf{N}_0$. As $\lambda(T_n) \to 0$ for $n \to +\infty$ there exists an integer $n \in \mathbf{N}$, such that $\lambda(T_n)<\frac{1}{2}\lambda(T)$. Using (10) and that $T \in \mathscr{B}_{n-1}$ we get the contradiction

$$\tfrac{1}{2}\lambda(T) > \lambda(T_n) \geqq \tfrac{1}{2}\sup\{\lambda(B) \mid B \in \mathscr{B}_{n-1}\} \geqq \tfrac{1}{2}\lambda(T) ,$$

and thus we conclude that $J=\{n \in \mathbf{N} \mid T \cap T_n \neq \emptyset\}$ is not empty. Let $j=\min J$. Then $T \in \mathscr{B}_{j-1}$ and thus

$$\lambda(T_j) \geqq \tfrac{1}{2}\sup\{\lambda(B) \mid B \in \mathscr{B}_{j-1}\} \geqq \tfrac{1}{2}\lambda(T) .$$

LEMMA 8. *Let $\mathscr{B}$ be a given system of closed balls, and let $A \subseteq A_{\mathrm{loc}}(\mathscr{B})$. Let $\{T_n\}$ be chosen according to the procedure above with respect to $(A,\mathscr{B})$. To any $x \in A$ and $r \in \mathbf{R}_+$ there exists an integer $n \in \mathbf{N}$, such that*

$$T_n \cap B[x,r] \neq \emptyset \quad \textit{and} \quad \varrho(x,r) \leqq 4\lambda(T_n) .$$

PROOF. As $A \subseteq A_{\mathrm{loc}}(\mathscr{B})$ there exists a $B \in \mathscr{B}$, such that $B \subseteq B[x,r]$ and $\lambda(B) \geqq \frac{1}{2}\varrho(x,r)$. By Lemma 7 one can find an element T_n of the sequence $\{T_n\}$, such that $\emptyset \neq B \cap T_n \subseteq T_n \cap B[x,r]$ and $4\lambda(T_n) \geqq 2\lambda(B) \geqq \varrho(x,r)$.

At last we prove a lemma concerning the neighbouring Vitali systems.

LEMMA 9. *Let* $\mathscr{B}$ *be a given system of closed balls, and let* $A \subseteqq A^1_{\mathrm{nbh}}(\mathscr{B})$. *Let* $\{T_n\}$ *be chosen according to the procedure above with respect to* $(A, \mathscr{B})$. *If* $x \in A$ *and* $r \in \mathbf{R}_+$, *then at least one of the following conditions is fulfilled*:

1) *There exists an integer* $n \in \mathbf{N}$, *such that* $T_n \cap B[x,r] \neq \emptyset$ *and* $\lambda(T_n) \geqq 12^{-N}\lambda(B[x,r])$.
2) *There exist two integers* $n_1, n_2 \in \mathbf{N}$, *such that* $T_{n_i} \cap B[x,r] \neq \emptyset$, $\lambda(T_{n_i}) \geqq \frac{1}{4}\varrho_1^*(x,r)$, $i=1, 2$, *and at least one of the sets* T_{n_1}, T_{n_2} *is disjoint from* $B[x, 2^{-3}r]$.

PROOF. Suppose for given $x \in A$ and $r \in \mathbf{R}_+$ that if $T_n \cap B[x,r] \neq \emptyset$, then $\lambda(T_n) < 12^{-N}\,\lambda(B[x,r])$ and $T_n \cap B[x, 2^{-3}r] \neq \emptyset$. As $x \in A^1_{\mathrm{nbh}}(\mathscr{B})$ there exist $B_1, B_2 \in \mathscr{B}$, such that $B_1 \cup B_2 \subseteqq B[x,r]$, $\lambda(B_i) \geqq \frac{1}{2}\varrho_1^*(x,r)$, $i=1,2$, and $d(\mathrm{cen}\,(B_1), \mathrm{cen}\,(B_2)) \geqq r$. By Lemma 7 there exist T_{n_1}, T_{n_2} from $\{T_n\}$ such that

$$\lambda(B_i) \leqq 2\lambda(T_{n_i}) < 6^{-N}\lambda(B[x,r]) ,$$

or $\mathrm{rad}\, B_i < r/6$ and $\mathrm{rad}\, T_{n_i} < r/12$. Then

$$\begin{aligned}
&d(\mathrm{cen}\,(B_1), \mathrm{cen}\,(B_2)) \\
&\quad \leqq d(\mathrm{cen}\,(B_1), \mathrm{cen}\,(T_{n_1})) + d(\mathrm{cen}\,(T_{n_1}), x) \\
&\qquad + d(x, \mathrm{cen}\,(T_{n_2})) + d(\mathrm{cen}\,(T_{n_2}), \mathrm{cen}\,(B_2)) \\
&\quad \leqq \mathrm{rad}\,(B_1) + \mathrm{rad}\,(T_{n_1}) + \mathrm{rad}\,(T_{n_1}) + 2^{-3}r + 2^{-3}r + \mathrm{rad}\,(T_{n_2}) \\
&\qquad + \mathrm{rad}\,(T_{n_2}) + \mathrm{rad}\,(B_2) \\
&\quad < r \cdot \{6^{-1} + 12^{-1} + 12^{-1} + 4^{-1} + 12^{-1} + 12^{-1} + 6^{-1}\} < r ,
\end{aligned}$$

contradicting $d(\mathrm{cen}\,(B_1), \mathrm{cen}\,(B_2)) \geqq r$. Thus, either $\lambda(T_n) \geqq 12^{-N}\lambda(B[x,r])$, or there exists at least one T_{n_i}, such that $T_{n_i} \cap B[x, 2^{-3}r] = \emptyset$. Finally, $\lambda(T_{n_i}) \geqq \frac{1}{2}\lambda(B_i) \geqq \frac{1}{4}\varrho_1^*(x,r)$.

LEMMA 10. *Let* $\mathscr{B}$ *be a given system of closed balls, and let* $A \subseteqq A^c_{\mathrm{nbh}}(\mathscr{B})$, *where* $c \in \,]0, 2[$. *Let* $\{T_n\}$ *be chosen according to the procedure above with respect to* $(A, \mathscr{B})$. *Let* $m \in \mathbf{N}$ *be a constant, such that* $2^{-m} < c/6$. *If* $x \in A$ *and* $r \in \mathbf{R}_+$, *then at least one of the following conditions is fulfilled*:

1) *There exists an integer* $n \in \mathbf{N}$, *such that* $T_n \cap B[x,r] \neq \emptyset$ *and* $\lambda(T_n) \geqq (c/12)^N \lambda(B[x,r])$.
2) *There exist two integers* $n_1, n_2 \in \mathbf{N}$, *such that* $T_{n_i} \cap B[x,r] \neq \emptyset$, $\lambda(T_{n_i}) \geqq \frac{1}{4}\varrho_c^*(x,r)$, $i=1,2$, *and at least one of the sets* T_{n_1}, T_{n_2} *is disjoint from* $B[x, 2^{-m}r]$.

The proof, which is a simple generalization of the proof of Lemma 9, is left to the reader.

4. A packing theorem for $\mathfrak{B}_{\log}$.

In this section we prove the following proposition, which is a part of Theorem 2.

PROPOSITION 2. *Let $\mathscr{B}$ be a class of closed balls in $\mathbf{R}^N$. If $A \subseteqq A_{\log}$, then one can pack A with balls from $\mathscr{B}$.*

PROOF. Without loss of generality we may assume that $A \subseteqq [0,1]^N$. It follows from the inequalities

$$2^{-(N+1)} \sum_{n=1}^{+\infty} \varrho^*(x,2^{-n})/|\log \varrho(x,2^{-n})| \leqq \int_0^1 \varrho^*(x,t)/|\log \varrho(x,t)| \frac{dt}{t}$$
$$\leqq 2^N \sum_{n=0}^{+\infty} \varrho^*(x,2^{-n})/|\log \varrho(x,2^{-n})|$$

that $\varrho^*(x,\cdot)/|\log \varrho(x,\cdot)| \in \Phi$, if and only if

$$\sum_{n=0}^{+\infty} \varrho^*(x,2^{-n})/|\log \varrho(x,2^{-n})| = +\infty \quad \text{for all } x \in A . \tag{12}$$

Thus, by Lemma 3 it is sufficient to prove that (12) implies the existence of a positive constant c and a class of disjoint sets $\{B_n\}$ from $\mathscr{B}$, such that

$$\lambda(\bigcup B_n) \geqq c\lambda^*(A) .$$

We assume that the sequence $\{T_n\}$ has been chosen by the procedure described in section 3, such that $\{\lambda(T_n)\}$ is a non-increasing sequence of positive numbers.

By Lemma 8 there exist integers $n \in \mathbf{N}$ to each $x \in A$ and $k \in \mathbf{N}$, such that

$$T_n \cap B[x,2^{-k}] \neq \emptyset \quad \text{and} \quad \varrho(x,2^{-k}) \leqq 4\lambda(T_n) . \tag{13}$$

Let $n(x,k)$ be the smallest integer n, for which (13) holds. As $\{\lambda(T_n)\}$ is non-increasing, we conclude that $\{n(x,k)\}_{k\in\mathbf{N}}$ is a non-decreasing sequence of positive integers for each fixed $x \in A$.

It follows immediately from (13) that

$$\tfrac{1}{4}\varrho(x,2^{-k}) \leqq \lambda(T_{n(x,k)}), \quad k \in \mathbf{N},\ x \in A , \tag{14}$$

and we conclude from (12) and (14) that

$$\sum_{k=1}^{+\infty} 2^{Nk}\lambda(T_{n(x,k)})/|\log \lambda(T_{n(x,k)})| = +\infty \quad \text{for all } x \in A . \tag{15}$$

Next, we use a property of the outer Lebesgue measure λ^*. To given constants $\varepsilon \in \,]0,1[$ and $K \in \mathsf{R}_+$ there exists a positive integer $m = m(\varepsilon, K)$, such that $\lambda^*(A') \geqq (1-\varepsilon)\lambda^*(A)$, where A' is defined by

$$(16) \qquad A' = \left\{ x \in A \;\middle|\; \sum_{k=1}^{m+1} 2^{Nk}\lambda(T_{n(x,k)})/|\log \lambda(T_{n(x,k)})| \geqq K \right\}.$$

Thus, by Lemma 3 it is enough to prove the existence of a constant $c \in \mathsf{R}_+$, such that

$$(17) \qquad \lambda(\bigcup \{T_{n(x,k)} \mid x \in A', k = 1, \ldots, m+1\}) \geqq c\lambda^*(A'),$$

and we may without loss of generality in the following assume that $A' = A$ and that $\lambda^*(A) > 0$. We shall prove that (17) is satisfied with a finite number of sets $T_{n(x,k)}$, and as the series (15) is divergent, the procedure of defining A' can be iterated. Thus, it is sufficient to prove that to a given constant $c \in \mathsf{R}_+$ [$c = (1+4^N)^{-1}$, say] one can select a finite subclass $\{T_{n_1}, T_{n_2}, \ldots, T_{n_p}\}$ of $\{T_n\}$, independent of $x \in A$, where this subclass in a sense made precise below is geometrically "equivalent" to the system $\{T_{n(x,k)} \mid x \in A, k = 1, \ldots, m+1\}$, such that

$$\sum_{i=1}^{p} \lambda(T_{n_i}) \geqq c\lambda^*(A).$$

We choose once for all $K = 2^{3N}/\log 2^N$ in (16). This means that there exists a positive integer $m \in \mathsf{N}$, such that

$$(18) \qquad \sum_{k=1}^{m+1} 2^{Nk}\lambda(T_{n(x,k)})/\log \lambda(T_{n(x,k)})| \geqq 2^{3N}/\log 2^N \qquad \text{for all } x \in A.$$

In our next step we describe the choice of the finite class $\{T_{n_1}, T_{n_2}, \ldots, T_{n_p}\}$, and we derive some properties of this class. For $n \in \mathsf{N}_0$, let $\mathfrak{N}_n$ denote the set of 2^{nN} (closed) meshes, each of sidelength 2^{-n}, obtained by dividing each side of $[0,1]^N$ into 2^n intervals of equal length. As $\{\lambda(T_n)\}$ is non-increasing, T_1 is the biggest ball in $\{T_n\}$. We choose $T_{n_1} = T_1$. Let B_1 be the union of all meshes in $\mathfrak{N}_1$, not intersecting T_{n_1}, thus $B_1 = \bigcup \{Q \mid Q \in \mathfrak{N}_1 \cap \tilde{T}_{n_1}\}$. If $B_1 \neq \emptyset$ we define $n_2 = \min\{n \in \mathsf{N} \mid T_n \cap B_1 \neq \emptyset\}$. This means that T_{n_2} is the biggest ball from $\{T_n\}$ that intersects the union of those meshes in $\mathfrak{N}_1$, which do not intersect T_{n_1}. Let B_2 be the union of all meshes in $\mathfrak{N}_1$, not intersecting $T_{n_1} \cup T_{n_2}$,

$$B_2 = \bigcup \{Q \mid Q \in \mathfrak{N}_1 \cap \tilde{T}_{n_1} \cap \tilde{T}_{n_2}\}.$$

If $B_2 \neq \emptyset$, we define $n_3 = \min\{n \in \mathsf{N} \mid T_n \cap B_2 \neq \emptyset\}$. In this way we continue.

By the passage from B_1 to B_2, at least one of the 2^N meshes of $\mathfrak{N}_1$ has disappeared, $B_2 \subset B_1$, $B_2 \neq B_1$. As the same is true for all the succeeding steps we conclude that there exist i cubes $\{T_{n_1}, \ldots, T_{n_i}\}$ from $\{T_n\}$, $i \leqq 2^N$, chosen by the procedure described above, such that either

$$Q \cap (T_{n_1} \cup T_{n_2} \cup \ldots \cup T_{n_i}) \neq \emptyset \quad \text{or} \quad Q \cap \bigcup T_n = \emptyset \quad \text{for all } Q \in \mathfrak{N}_1 .$$

Next, we consider $\mathfrak{N}_2$, defining B_i as the union of all meshes in $\mathfrak{N}_2$ not intersecting $T_{n_1} \cup T_{n_2} \cup \ldots \cup T_{n_i}$ (if any). Thus,

$$B_i = \bigcup \{Q \mid Q \in \mathfrak{N}_2 \cap \tilde{T}_{n_1} \cap \tilde{T}_{n_2} \cap \ldots \cap \tilde{T}_{n_i}\} ,$$

and the process is repeated on meshes from $\mathfrak{N}_2$, until every mesh from $\mathfrak{N}_2$ either intersects $T_{n_1} \cup T_{n_2} \cup \ldots \cup T_{n_i} \cup \ldots \cup T_{n_j}$ or does not meet $\bigcup T_n$. At this stage we change the scale to $\mathfrak{N}_3$. In this way we continue to level $m+1$, when the process is terminated. Thus, we have selected a finite system $\{T_{n_1}, T_{n_2}, \ldots, T_{n_p}\}$ from our infinite system $\{T_n\}$ of pairwise disjoint balls from $\mathscr{B}$, and $\{T_{n_1}, T_{n_2}, \ldots, T_{n_p}\}$ has the property that for each $i \in \{1, 2, \ldots, p\}$ one can find a $v \in \{1, \ldots, m+1\}$ and a mesh $Q \in \mathfrak{N}_v$, such that T_{n_i} is the biggest cube from $\{T_n\}$, for which $T_n \cap Q \neq \emptyset$.

Let $x \in A$ and $k \in \{1, 2, \ldots, m+1\}$ be given. Then the closed ball $B[x, 2^{-k}]$ intersects some of the sets $\{T_{n_i}\}$ associated with the meshes in $\mathfrak{N}_v$, $v = 0, 1, 2, \ldots, k$. Let T_{n_j} be the biggest cube associated with the meshes in $\mathfrak{N}_v$, $v = 0, 1, \ldots, k$, for which $T_{n_j} \cap B[x, 2 \cdot 2^{-k}] \neq \emptyset$. By definition, $T_{n(x,k)}$ is the biggest ball from $\{T_n\}$, for which $T_n \cap B[x, 2^{-k}] \neq \emptyset$, so $T_{n(x,k)}$ is one of the candidates for T_{n_j}. Thus, $\lambda(T_{n_j}) \geqq \lambda(T_{n(x,k)})$, and the class $\{T_{n(x,k)} \mid x \in A;\ k = 1, \ldots, m+1\}$ can in this sense be compared with the chosen finite subclass $\{T_{n_1}, T_{n_2}, \ldots, T_{n_p}\}$ of $\{T_n\}$. More explicitly, we have proved that

$$\forall x \in A, \forall k \leqq m+1\ \exists i \leqq p : B[x, 2^{1-k}] \cap T_{n_i} \neq \emptyset ,\qquad \lambda(T_{n_i}) \geqq \lambda(T_{n(x,k)}) . \tag{19}$$

Since the subclass $\{T_{n_1}, T_{n_2}, \ldots, T_{n_p}\}$ is finite, we can find a constant $M \in \mathbf{N}$, such that $2\,\mathrm{rad}\,(T_{n_i}) \geqq 2^{-M}$ for all $i = 1, \ldots, p$. To each T_{n_i} we can choose 2^N meshes from $\mathfrak{N}_{v_i}$, which make up a cube of sidelength 2^{-v_i+1} (not necessarily a mesh in $\mathfrak{N}_{v_i-1}$), such that T_{n_i} is contained in this new cube (cf. [2]). Here v_i is defined by the condition

$$2^{-v_i-1} \leqq 2\,\mathrm{rad}\,(T_{n_i}) < 2^{-v_i} . \tag{20}$$

In the terminology of [2] this cube consisting of 2^N meshes from $\mathfrak{N}_{v_i}$ is called the γ-cube associated with T_{n_i}. We denote this γ-cube by γ_i. Hence, $T_{n_i} \subseteqq \gamma_i$, and from (20) we derive that $\lambda(T_{n_i}) \geqq 4^{-N} \lambda(\gamma_i)$. If $\Gamma = \bigcup_{i=1}^p \gamma_i$, we get

$$\lambda\left(\bigcup_{i=1}^{p} T_{n_i}\right) = \sum_{i=1}^{p} \lambda(T_{n_i}) \geqq 4^{-N} \sum_{i=1}^{p} \lambda(\gamma_i) \geqq 4^{-N} \lambda\left(\bigcup_{i=1}^{p} \gamma_i\right) = 4^{-N} \lambda(\Gamma) ,$$

and $\bigcup_{i=1}^p T_{n_i}$ covers at least the fraction 4^{-N} of Γ. If we can prove that

$$\lambda^*(A \setminus \Gamma) \leqq \lambda\left(\bigcup_{i=1}^{p} T_{n_i}\right) , \tag{21}$$

then

$$\lambda^*(A) \leqq \lambda(\Gamma)+\lambda^*(A\setminus\Gamma) \leqq (1+4^N)\lambda\left(\bigcup_{i=1}^{p} T_{n_i}\right),$$

or $\lambda(\bigcup_{i=1}^{p} T_{n_i}) \geqq (1+4^N)^{-1}\lambda^*(A) = c\cdot\lambda^*(A)$, and the theorem is proved according to our previous remarks.

Let us turn to the proof of (21). The set Γ is composed of meshes from $\mathfrak{N}_{M+1}$. Hence the closure of $[0,1]^N\setminus\Gamma$ is again composed of meshes from $\mathfrak{N}_{M+1}$:

$$\mathrm{cl}\,([0,1]^N\setminus\Gamma) = \bigcup_{j=1}^{q'} Q_j, \quad Q_j \in \mathfrak{N}_{M+1}.$$

There exists a $q \leqq q'$, such that after a rearrangement of the Q_j

$$(22) \qquad A\setminus\Gamma \subseteqq \bigcup_{j=1}^{q} Q_j; \quad Q_j \in \mathfrak{N}_{M+1}, \quad A \cap \mathrm{int}\, Q_j \neq \emptyset, \quad j=1,\ldots,q.$$

[If $q=0$, the set $A\setminus\Gamma$ is empty, and (21) is trivial.] Suppose that $q \in \mathbf{N}$ is given. We shall prove that the measure of $\bigcup_{j=1}^{q} Q_j$ is smaller than or equal to the measure of $\bigcup_{i=1}^{p} T_{n_i}$. It should be remarked that the two sets $\bigcup_{j=1}^{q} Q_j$ and $\bigcup_{i=1}^{p} T_{n_i}$ of course are disjoint.

To each T_{n_i} there exists a unique integer v_i, such that (20) is fulfilled. As $2\,\mathrm{rad}\,(T_{n_i})$ is the sidelength of T_{n_i}, an easy calculation gives that

$$(23) \qquad v_i < |\log\lambda(T_{n_i})|/\log 2^N \leqq v_i+1, \quad i=1,\ldots,p.$$

If $v_i \leqq 1$ for some i, then by (20),

$$\lambda(T_{n_i}) \geqq 2^{-2N} \geqq 4^{-N}\lambda^*(A) \geqq c\cdot\lambda^*(A),$$

where $c=(1+4^N)^{-1}$, and the estimate $\lambda(\bigcup_{i=1}^{p} T_{n_i}) \geqq c\cdot\lambda^*(A)$ is trivial. Hence, we may assume that $v_i \geqq 2$ for $i=1,\ldots,p$.

For each $v \in \{1,\ldots,v_i\}$ there exist at most 2^N meshes from $\mathfrak{N}_v$ intersecting T_{n_i}, and thus at most 2^{2N} meshes from $\mathfrak{N}_v$ containing points with a distance $\leqq 2^{-v}$ from T_{n_i}. This means that we from all the levels $v=1,\ldots,v_i$ have meshes $Q_{i,1},\ldots,Q_{i,m_i}$, $m_i \leqq v_i 2^{2N}$, for which we shall use T_{n_i} as a scale. We divide T_{n_i} into $v_i 2^{2N}$ "gauges" of equal size, and each of the meshes $Q_{i,k}$, $k=1,\ldots,m_i$, is given the gauge $v_i^{-1}2^{-2N}\lambda(T_{n_i})$.

If $Q_{i,k} \in \mathfrak{N}_v$, then $Q_{i,k}$ is composed of at most $2^{N(M+1-v)}$ of the sets Q_j, defined in (22). We divide the gauge of size $v_i^{-1}2^{-2N}\lambda(T_{n_i})$ into $2^{N(M+1-v)}$ subgauges of equal size. If a Q_j from (22) is contained in a $Q_{i,k}$ from level v, then Q_j is given the subgauge of size

$$v_i^{-1}2^{-2N}\lambda(T_{n_i})2^{-N(M+1-v)} = v_i^{-1}2^{-N(M+3-v)}\lambda(T_{n_i})$$

from $Q_{i,k} \in \mathfrak{N}_v$ associated with T_{n_i}. We note that $\lambda(Q_j)=2^{-N(M+1)}$ for

$Q_j \in \mathfrak{N}_{M+1}$ so the use of T_{n_i} as a scale may be summarized in the following way:

Every T_{n_i} is used to measure some meshes from the levels $v=1,\dots,v_i$. Each level is from T_{n_i} given the gauge $v_i^{-1}\lambda(T_{n_i})$. From each level v we pick up those meshes, which contain points with a distance $\leqq 2^{-v}$ from T_{n_i}. Equivalently, these meshes $Q_{i,k}$ from $\mathfrak{N}_v$ are described by the condition

$$Q_{i,k} \cap B[\mathrm{cen}\,(T_{n_i}), \mathrm{rad}\,(T_{n_i})+2^{-v}] \neq \varnothing .$$

At a given level $v \leqq v_i$ there are at most 2^{2N} such meshes, each of which is given the subgauge $v_i^{-1}2^{-2N}\lambda(T_{n_i})$. Next, each of these meshes from *level* v is again divided into meshes from $\mathfrak{N}_{M+1}$, and we pick up those meshes Q_j that also occur in (22). In particular $A \cap \mathrm{int}\, Q_j \neq \varnothing$, so we may use our ϱ-function for some $x \in A \cap \mathrm{int}\, Q_j$. The number of the Q_j associated with a mesh $Q_{i,k}$ from level v is at most $2^{N(M+1-v)}$, so each $Q_j \subseteqq Q_{i,k}$ is given the subgauge

$$v_i^{-1}2^{-N(M+3-v)}\lambda(T_{n_i}) = (v_i^{-1}2^{-2N}2^{vN}\lambda(T_{n_i}))\lambda(Q_j) \tag{24}$$

from one mesh $Q_{i,k} \in \mathfrak{N}_v$ associated with a particular T_{n_i}. By (24) we conclude that Q_j through $Q_{i,k}$ is given a subgauge that may cover *the fraction* $v_i^{-1}2^{-2N}2^{vN}\lambda(T_{n_i})$ of Q_j. We note that it follows from (23) that

$$v_i^{-1}2^{-2N}2^{vN}\lambda(T_{n_i}) > (\log 2^N)2^{-2N}2^{vN}\lambda(T_{n_i})/|\log \lambda(T_{n_i})| . \tag{25}$$

In the last step of the proof we show that each Q_j, $j=1,\dots,q$, in (22) is totally "covered" by such gauges from all the T_{n_i}. Let Q_j be any of these meshes. We shall prove that

$$\sum_{(i,v)}{}' \; v_i^{-1}2^{-2N}2^{vN}\lambda(T_{n_i}) \geqq 1 , \tag{26}$$

where the summation is performed over all pairs (i,v), $i=1,\dots,p$; $v=1,\dots,v_i$, for which there exists a $Q_{i,k} \in \mathfrak{N}_v$ associated with T_{n_i}, such that $Q_j \subseteqq Q_{i,k}$. More precisely, the summation is performed over all such $Q_{i,k}$.

Let $x \in A \cap \mathrm{int}\, Q_j$. By (19), to each $k \leqq m+1$ there exists a $T_{n_i} = T_{p(x,k)}$, such that

$$B[x, 2^{1-k}] \cap T_{p(x,k)} \neq \varnothing \quad \text{and} \quad \lambda(T_{p(x,k)}) \geqq \lambda(T_{n(x,k)}) .$$

The first condition implies that Q_j is given a gauge from some T_{n_i}, $n_i = p(x,k)$, stemming from level $k-1$, and as the same $x \in A \cap \mathrm{int}\, Q_j$ may be used for all $k \in \{1,\dots,m+1\}$, we get the estimates [by using (18) and (25)]

$$\begin{aligned} 1 &\leqq (\log 2^N)2^{-3N}\sum_{k=1}^{m+1} 2^{Nk}\lambda(T_{n(x,k)})/|\log \lambda(T_{n(x,k)})| \\ &\leqq (\log 2^N)2^{-3N}\sum_{k=1}^{m+1} 2^{Nk}\lambda(T_{p(x,k)})/|\log \lambda(T_{p(x,k)})| \end{aligned}$$

$$\leqq (\log 2^N)2^{-3N}(\log 2^N)^{-1}2^{3N} \sum_{(i,\nu)}' \nu_i^{-1}2^{-2N}2^{\nu N}\lambda(T_{n_i}) ,$$

from which (26) follows. As this is true for all Q_j in (22), we conclude that

$$\lambda^*(A \setminus \Gamma) \leqq \lambda\left(\bigcup_{j=1}^{q} Q_j\right) \leqq \lambda\left(\bigcup_{i=1}^{p} T_{n_i}\right),$$

and the proof is complete.

5. A packing theorem for $\mathfrak{V}_{\mathrm{nbh}}$.

In this section we prove another part of Theorem 2.

PROPOSITION 3. *Let $\mathscr{B}$ be a class of closed balls in $\mathbf{R}^N$. If $A \subseteqq A_{\mathrm{nbh}}$, then one can pack A with balls from $\mathscr{B}$.*

PROOF. It is obvious that $A^s_{\mathrm{nbh}} \subseteqq A^t_{\mathrm{nbh}}$ for $t<s$ and that

$$A_{\mathrm{nbh}} = \bigcup \{A^s_{\mathrm{nbh}} \mid s \in \mathbf{R}_+\} = \bigcup \{A^{1/n}_{\mathrm{nbh}} \mid n \in \mathbf{N}\} .$$

According to Lemma 4 it is enough to prove the proposition for A contained in $A^{1/n}_{\mathrm{nbh}}$. In the following we shall prove the assertion for $n=1$ using Lemma 9. The general case is with trivial modifications proved in the same way using Lemma 10 instead. As the calculations for $n \neq 1$ are fairly long and tedious compared with the case $n=1$, we shall omit the general proof.

It is easy to see that $\varrho_1^*(x,\cdot) \in \Phi$ if and only if

$$\sum_{n=0}^{+\infty} \varrho_1^*(x,2^{-n}) = +\infty \qquad \text{for all } x \in A , \tag{27}$$

[compare the derivation of the equivalence of $\varrho^*(x,\cdot)/|\log \varrho(x,\cdot)| \in \Phi$ and (12)]. We assume that the sets $\{T_n\}$ have been chosen according to the procedure described in section 3 with respect to $(A,\mathscr{B})$.

Using a property of the outer Lebesgue measure λ^* we see that to each $\varepsilon \in]0,1[$ one can find a constant $m=m(\varepsilon) \in \mathbf{N}$, such that $\lambda^*(A') \geqq (1-\varepsilon)\lambda^*(A)$, where

$$A' = \left\{x \in A \;\middle|\; \sum_{k=1}^{m} \varrho_1^*(x,2^{-k}) \geqq 12\cdot 2^N\right\} .$$

Because of Lemma 3 it is sufficient to prove that $\lambda^*(A')$ is smaller than some constant times the measure of the union of a specially selected finite subclass of $\{T_n\}$. We may therefore assume that $A'=A$, so we are given a constant $m \in \mathbf{N}$, such that

$$\sum_{k=1}^{m} \varrho_1^*(x, 2^{-k}) \geqq 12 \cdot 2^N \quad \text{for all } x \in A . \tag{28}$$

By an open 2^N-tant in R^N defined by a cube Q we shall understand a connected component of the set $\{x \in \mathsf{R}^N \mid x_k \neq \operatorname{cen}(Q)_k,\ k=1,\ldots,N\}$ where $x = (x_1,\ldots,x_N)$ and $\operatorname{cen}(Q) = (\operatorname{cen}(Q)_1,\ldots,\operatorname{cen}(Q)_N)$. By a 2^N-tant we shall understand the closure of an open 2^N-tant.

We shall prove the existence of a finite subclass $\{T_{n_i}\}$ of $\{T_n\}$, such that (cf. Lemma 3)

$$\lambda(\bigcup T_{n_i}) \geqq c\lambda^*(A) , \tag{29}$$

where $c = (1+768^N)^{-1}$, say, and where $A \subseteq [0,1]^N$.

Let $\mathfrak{N}_p$, $p \in \mathsf{N}$, be the set of meshes introduced in section 4, here used on the whole space R^N. We put

$$\{Q \in \mathfrak{N}_{m+4} \mid A \cap \operatorname{int} Q \neq \emptyset\} = \{R_l \mid l = 1,\ldots,M\} , \tag{30}$$

and in each R_l we choose a point $x_l \in A \cap \operatorname{int} R_l$. Let $P = \{x_l \mid l=1,\ldots,M\}$. By Lemma 9, to each $x_l \in P$ and each $k \in \{1,2,\ldots,m+4\}$ we have two possibilities: Either

1) There exists an integer $n \in \mathsf{N}$, such that

$$T_n \cap B[x_l, 2^{-k}] \neq \emptyset \quad \text{and} \quad \lambda(T_n) \geqq 12^{-N}\lambda(B[x,2^{-k}]) = 6^{-N}2^{-kN} ,$$

or

2) There exist two integers $n_1, n_2 \in \mathsf{N}$, such that

$$T_{n_i} \cap B[x_l, 2^{-k}] \neq \emptyset \quad \text{and} \quad \lambda(T_{n_i}) \geqq \tfrac{1}{4}\varrho_i^*(x, 2^{-k}), \qquad i=1,2 , \tag{31}$$

and at least one of the sets T_{n_1}, T_{n_2} does not intersect $B[x, 2^{-k-3}]$.

For later use, the chosen sets T_n for a given k are called I_{l1}^k and I_{l2}^k. In case 1) we have $I_{l1}^k = I_{l2}^k = T_n$, whereas $I_{l1}^k \cap I_{l2}^k = \emptyset$ in case 2). Thus, to each $x_l \in P$ we are given a finite set of pairs (I_{l1}^k, I_{l2}^k), $k=1,\ldots,m+4$, of cubes from our sequence $\{T_n\}$. We choose our finite subclass of $\{T_n\}$ as the set $\{I_{li}^k \mid k=1,\ldots,m+4;\ l=1,\ldots,M;\ i=1,2\}$. As each I_{li}^k belongs to $\{T_n\}$, this finite subclass can be rewritten as $\{T_{n_1}, T_{n_2},\ldots,T_{n_p}\}$, where $n_1 < n_2 < \ldots < n_p$. Note that $\{T_n\}$ in section 3 has been chosen according to size, so $\lambda(T_{n_1}) \geqq \lambda(T_{n_2}) \geqq \ldots \geqq \lambda(T_{n_p})$. It is convenient in the following to use both notations for different purposes.

If $\lambda(T_{n_1}) \geqq c = (1+768^N)^{-1}$, there is nothing to prove, as (29) becomes trivial because $A \subseteq [0,1]^N$. Otherwise, to each $i=1,\ldots,p$ there is a uniquely determined integer k_i, such that

$$12^{-N}2^{-k_iN} \leqq \lambda(T_{n_i}) < 6^{-N}2^{-k_iN} . \tag{32}$$

Hence, we can choose 2^N meshes from $\mathfrak{N}_{k_i}$ which make up a cube $S^i_{k_i}$ containing T_{n_i}. It follows from (32) that

$$\lambda(T_{n_i}) \geqq 24^{-N}\lambda(S^i_{k_i}) . \tag{33}$$

The cube $S^i_{k_i}$ is contained in another cube $S^i_{k_i-1}$ consisting of 2^N meshes from $\mathfrak{N}_{k_i-1}$. In this way we continue defining a finite sequence $S^i_1 \supset \ldots \supset S^i_{k_i}$ of cubes associated with T_{n_i}, where in case of more than one possible S^i_j the cube is chosen such that $d(\text{cen}\,(T_{n_i}), \text{cen}\,(S^i_j))$ is minimized, i.e.

$$d(\text{cen}\,(T_{n_i}), \text{cen}\,(S^i_j)) \leqq 2^{-j-1} .$$

Note that the S^i_j need not be contained in $[0,1]^N$. We define R^i_j as the cube consisting of 4^N meshes from $\mathfrak{N}_j$ with S^i_j in the center.

Let the meshes $J^i_{j,k}$, $k=1,\ldots,4^N$, of R^i_j be enumerated in a natural way according to position in the cube (one might as well have used multi-indices instead, but this would overburden the notation). Then each $J^i_{1,k}$ defines a uniquely determined string

$$\mathscr{J}^i_k = \{J^i_{j,k} \mid j=1,\ldots,m+4\} .$$

As each S^i_j has been chosen such that T_{n_i} is as close to the center of S^i_j as possible, we easily see that all the sets of $\mathscr{J}^i_k$ belong to the same 2^N-tant defined by R^i_1, though they may not at all be ordered by inclusion.

We define $\gamma_i = R^i_{k_i-3}$ and $\Gamma = [0,1]^N \cap \bigcup_{i=1}^p \gamma_i$. Then by (33), $\lambda(T_{n_i}) \geqq 8^{-N}\cdot 4^{-N}\cdot 24^{-N}\lambda(\gamma_i) = 768^{-N}\lambda(\gamma_i)$, and thus

$$\lambda\left(\bigcup_{i=1}^{p} T_{n_i}\right) = \sum_{i=1}^{p} \lambda(T_{n_i}) \geqq 768^{-N} \sum_{i=1}^{p} \lambda(\gamma_i) \geqq 768^{-N}\lambda(\Gamma) . \tag{34}$$

If we can prove that

$$\sum_{i=1}^{p} \lambda(T_{n_i}) \geqq \lambda^*(A \setminus \Gamma) , \tag{35}$$

we also have proved (29) and hence the theorem. In fact, it follows from (34) and (35) that

$$\lambda^*(A) \leqq \lambda^*(A \setminus \Gamma) + \lambda(\Gamma) \leqq (1+768^N)\lambda\left(\bigcup_{i=1}^{p} T_{n_i}\right) .$$

As the series in (27) is divergent, the process may be iterated infinitely many times proving the theorem. Thus, we shall only prove (35). As $\bigcup_{i=1}^p T_{n_i}$ and $A \setminus \Gamma$ are disjoint, we "cut" the sets T_{n_i} into suitable gauges instead. These gauges are given to selected meshes, the union of which contains $A \setminus \Gamma$. The distribution of gauges is performed in such a way that each of these meshes is totally covered by the corresponding gauges.

Let $K=\max\{k_1,\ldots,k_p,m+4\}$. Then the closure of $[0,1]^N\setminus\Gamma$ is composed of meshes from $\mathfrak{N}_K$:

$$\mathrm{cl}\,([0,1]^N\setminus\Gamma) = \bigcup_{j=1}^{q'} Q_j, \quad Q_j\in\mathfrak{N}_K\,.$$

Suppose that the Q_j have been enumerated such that we for the first q of these have

$$A\setminus\Gamma \subseteqq \bigcup_{j=1}^{q} Q_j \quad \text{and} \quad A\cap \mathrm{int}\,Q_j \neq \emptyset \quad \text{for } j=1,\ldots,q\,. \tag{36}$$

For each Q_j in (36) there exists an R_l defined by (30), such that $Q_j\subseteqq R_l$. For the point $x_l\in R_l$ we have

$$\max\{d(x_l,y)\mid y\in Q_j\} \leqq 2^{-m-4}\,,$$

so $x_l\in P$ can be chosen close to the center of a cube made up by 2^N meshes from $\mathfrak{N}_{m+1}$.

Let us look at one particular point $x_l\in P$. According to the construction of the T_{n_i} we are given an ordered set of pairs (I_{l1}^k,I_{l2}^k), $k=1,\ldots,m+4$, associated with x_l. We shall prove that if there exists a pair, for which $I_{l1}^k=I_{l2}^k(=T_{n_i})$, then $x_l\in\gamma_i$. But if $I_{l1}^k=I_{l2}^k=T_{n_i}$, then T_{n_i} has been chosen according to rule 1), so $T_{n_i}\cap B[x_l,2^{-k}]\neq\emptyset$ and [using (32)]

$$6^{-N}2^{-kN} \leqq \lambda(T_{n_i}) < 6^{-N}2^{-k_iN}\,,$$

from which we conclude that $k_i<k\leqq m+4$, and thus $1-k\leqq -k_i$ as k and k_i are integers. We shall prove that even $B[x_l,2^{-k}]\subseteqq\gamma_i=R^i_{k_i-3}$. By definition $S^i_{k_i}$ is chosen, such that $d(\mathrm{cen}\,(T_{n_i}),\mathrm{cen}\,(S^i_{k_i}))\leqq 2^{-k_i-1}$. Furthermore, it follows from (32) that $\mathrm{rad}\,(T_{n_i})<12^{-1}\cdot 2^{-k_i}$, and as $T_{n_i}\cap B[x_i,2^{-k}]\neq\emptyset$ it follows for any $y\in B[x_l,2^{-k}]$ that

$$\begin{aligned} d(\mathrm{cen}\,(S^i_{k_i}),y) &\leqq d(\mathrm{cen}\,(S^i_{k_i}),\mathrm{cen}\,(T_{n_i}))+\mathrm{rad}\,(T_{n_i})+2^{1-k} \\ &\leqq 2^{-1}2^{-k_i}+12^{-1}\cdot 2^{-k_i}+2^{-k_i} < 2^{-(k_i-1)}\,, \end{aligned}$$

or $y\in R^i_{k_i}\subset R^i_{k_i-3}=\gamma_i$.

If we consider a particular Q_j in (36) and the corresponding $x_l\in P$, then $\Gamma\cap\mathrm{int}\,Q_j=\emptyset$, so according to the discussion above none of the pairs (I_{l1}^k,I_{l2}^k) has been chosen by rule 1). As at least one of the sets I_{l1}^k, I_{l2}^k does not intersect $B[x,2^{-k-3}]$ by the alternative rule 2), we conclude that $(I_{l1}^k,I_{l2}^k)\neq(I_{l1}^{k+3},I_{l2}^{k+3})$, so at least one member of the pair (I_{l1}^k,I_{l2}^k) has been replaced by a necessarily smaller cube within three steps.

We divide each T_{n_i} into $3\cdot 4^N$ gauges of equal size, namely $3^{-1}4^{-N}\lambda(T_{n_i})$, and we are left with the problem of how to distribute these gauges among the chosen meshes Q_j, $j=1,\ldots,q$.

First, each string $\mathscr{J}_k^i$, $k = 1, \ldots, 4^N$, associated with T_{n_i}, is given three gauges. Then we use that the T_{n_i} have been arranged according to size. We start with the biggest cube T_{n_1}. The three gauges given to each string $\mathscr{J}_k^1$ are attached to the first three members of the string, $J_{1,k}^1, J_{2,k}^1, J_{3,k}^1$. Then we turn to T_{n_2}. In this case we attach the three gauges given to $\mathscr{J}_k^2$ to the first three *vacant* members of the string, i.e. beginning with the first mesh $J_{j,k}^2$ in the string, which has not got a gauge from T_{n_1}. In this way we continue. In general, the three gauges of the string $\mathscr{J}_k^i$ are given to the first three members of the string, which have not got gauges from the sets $T_{n_1}, \ldots, T_{n_{i-1}}$. In this way all the gauges are distributed to meshes of different sizes. (Occasionally, some of the gauges are given to meshes that do not contain points from $A \setminus \Gamma$, but this will only improve the final estimate.)

As we always give the gauges associated with a given string to three levels (note that these need not be succeeding; if they are not attached to succeeding levels the final estimate will also be improved) and as $(I_{l1}^k, I_{l2}^k) \neq (I_{l1}^{k+3}, I_{l2}^{k+3})$ by rule 2), it follows that each mesh Q from $\mathfrak{N}_k$ containing some Q_j from (36) associated with a point $x_l \in P$ is given a gauge of at least the size [cf. (31)]

$$3^{-1} \cdot 4^{-N} \cdot 4^{-1} \cdot \varrho_1(x_l, 2^{-k}) = 12^{-1} \cdot 4^{-N} \varrho_1(x_l, 2^{-k}), \tag{37}$$

which at least covers *the fraction* $12^{-1} \cdot 2^{-N} \varrho_1^*(x_l, 2^{-k})$ of Q. In fact, if I_{l2}^k has just replaced a bigger cube, then I_{12}^k still satisfies (31), but I_{12}^k *may be too small to be used at this very step as a gauge*, because we may still have larger gauges at our disposal. (The replacement of the sets I_{li}^k may take place more often than just once per three steps, which implies that they are used as gauges for much smaller meshes than originally intended. This will of course give a better estimate.) Starting once more from T_{n_1} we easily conclude that each mesh Q from $\mathfrak{N}_k$, for which $Q_j \subset Q$ and $x_l \in P$ are given as above, has in fact been given a gauge of at least the size (37).

At last, each gauge given to a mesh $Q \in \mathfrak{N}_k$ is cut into $2^{N(K-k)}$ subgauges of equal size, and each mesh $Q' \in \mathfrak{N}_K$ contained in Q is given one subgauge. We note that the subgauge of course covers a *fraction* $\geqq 12^{-1} 2^{-N} \varrho_1^*(x_l, 2^{-k})$ of Q_j, if Q_j and x_l are given as above. Adding all these subgauges given to one particular Q_j associated with $x_l \in P$ we find that Q_j has been totally covered by these subgauges, because

$$12^{-1} 2^{-N} \sum_{k=1}^{m} \varrho_1^*(x, 2^{-k}) \geqq 1$$

according to (28). As this is true for all Q_j in (36) and as the sum of all gauges is equal to $\lambda(\bigcup_{i=1}^{p} T_{n_i})$ we conclude that

$$\lambda\left(\bigcup_{i=1}^{p} T_{n_i}\right) \geqq \lambda\left(\bigcup_{j=1}^{q} Q_j\right) \geqq \lambda^*(A \setminus \Gamma),$$

and the proof is complete.

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