SOME REMARKS ON THE C*-ALGEBRAS
ASSOCIATED WITH CERTAIN
TOPOLOGICAL MARKOV CHAINS

JOACHIM CUNTZ and DAVID E. EVANS

In this note we show how certain algebras $\mathcal{O}_A$, of the type constructed in [5], and associated to certain topological Markov chains, can be constructed as crossed products and fixed point algebras of certain natural group actions on $\mathcal{O}_n$, the algebra of the full shift. This helps to clarify their structure somewhat, including their $K$-theory. For simplicity, we work in detail with actions of $\mathbb{Z}_2$ only, and merely indicate what happens with more general group actions.

Let $\mathcal{O}_n$ be generated by $n$ isometries $S_1, \ldots, S_n$ on a Hilbert space $H$, with $\sum S_iS_i^* = 1$. If $r_0, r_1 \in \mathbb{N}$, with $r_0 + r_1 = n$, then by [2], there is an action $\alpha = \alpha(r_0, r_1)$ of $\mathbb{Z}_2$ on $\mathcal{O}_n$ given by $S_i \rightarrow S_i, i \leq r_0$, and $S_i \rightarrow -S_i$ if $i > r_0$.

**Theorem**

$\mathbb{Z}_2 \times \mathcal{O}_n \cong \mathcal{O}_A$, where $A = \begin{pmatrix} r_0 & r_1 \\ r_1 & r_0 \end{pmatrix}$.

**Proof.** The crossed product acts on $H \oplus H$, and is generated by

$$\left\{ \pi(X) = \begin{pmatrix} X & 0 \\ 0 & \alpha(X) \end{pmatrix} : X \in \mathcal{O}_n \right\},$$

and the unitary $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

where $\pi(\alpha(X)) = U \pi(X) U^*$, $X \in \mathcal{O}_n$. If $U = R_0 - R_1$ is the spectral decomposition of $U$, then

$$\mathbb{Z}_2 \times \mathcal{O}_n = C^*(\pi(S_i), R_j : 1 \leq i \leq n, 0 \leq j \leq 1)$$

$$= C^*(\pi(S_i)R_j : 1 \leq i \leq n, 0 \leq j \leq 1)$$

where $\pi(S_i)R_j \neq 0, \forall i, j$. Now $\alpha(S_iS_i^*) = S_iS_i^*, \forall i$, hence if $P_i$ denotes the range projection $S_iS_i^*$, then $\pi(P_i)$ commutes with each spectral projection $R_j$. Hence we can decompose $H \oplus H$ into $2n$ orthogonal subspaces by

$$1 = \sum_{j=0}^{1} \sum_{i=1}^{n} \pi(P_i)R_j .$$

Work supported by the Deutsche Forschungsgemeinschaft.

Received March 14, 1980.
Then $\mathbb{Z}_2 \times O_n$ is generated by certain transitions, given by the operators $\pi(S_i)R_j$, relative to this decomposition. This enables $\mathbb{Z}_2 \times O_n$ to be expressed as a $C^*$-algebra of the form $O_B$, where $B$ is a $2n \times 2n$ matrix, with entries consisting entirely of zeros and ones. The admissible transitions can be obtained from

$$(R_0 - R_1)\pi(S_i)(R_0 - R_1) = \begin{cases} S_i, & i \leq r_0, \\ -S_i, & i > r_0, \end{cases}$$

so that certainly

$$R_0 \pi(S_i)R_1 = R_1 \pi(S_i)R_0 = R_0 \pi(S_j)R_0 = R_1 \pi(S_j)R_1 = 0$$

for $i \leq r_0 < j$. Moreover, $R_0 \pi(S_i)R_0$, $R_1 \pi(S_i)R_1$, $R_0 \pi(S_j)R_1$, $R_1 \pi(S_j)R_0$ are all non-zero for $i \leq r_0 < j$. Thus we see

$$\mathbb{Z}_2 \times O_n = O_B,$$

where $B = [b_{\alpha\beta}]_{\alpha, \beta \in A}$, $A = \{1, \ldots, n\} \times \{0, 1\}$ is given by

- $b_{\alpha\beta} = 0$ if $\alpha = (j, 1)$, $\beta = (i, 0)$, $i \leq r_0$
- $\alpha = (j, 0)$, $\beta = (i, 1)$, $i \leq r_0$
- $\alpha = (i, 0)$, $\beta = (j, 0)$, $j > r_0$
- $\alpha = (i, 1)$, $\beta = (j, 1)$, $j > r_0$

and $b_{\alpha\beta} = 1$ otherwise. This matrix can be reduced to an equivalent $2 \times 2$ matrix $A$ as follows: First define

$$A = \begin{pmatrix} r_0 & r_1 \\ r_1 & r_0 \end{pmatrix} = [a_{ij}]$$

and set

$$A' = \{(i, \xi, j) : 1 \leq i, j \leq 2, \xi = a_{ij}\}.$$

Then $A'$ can be matched with $A$ by

- $(1, i, 1) \leftrightarrow (i, 0)$, $1 \leq i \leq r_0$
- $(1, j, 2) \leftrightarrow (j + r_0, 0)$, $1 \leq j \leq r_1$
- $(2, i, 1) \leftrightarrow (i, 1)$, $1 \leq i \leq r_0$
- $(2, j, 2) \leftrightarrow (j + r_0, 2)$, $1 \leq j \leq r_1$

Then in the notation of [5, Remark 2.16], it is easily checked that $A' = B$, so that $O_A = O_B$. Hence $\mathbb{Z}_2 \times O_n = O_A$.

**Remark 1.** Note that the fixed point algebra
\[ \mathcal{O}_n^* \cong R_0(\mathbb{Z}_2 \times \mathcal{O}_n)R_0 \cong \mathcal{O}_C, \]

where
\[ C = \begin{pmatrix} r_0 & r_1 \\ r_1 & r_0 \end{pmatrix} \text{ if } r_1 \neq 0, \quad \text{and} \quad C = n \text{ if } r_1 = 0. \]

In fact this can be seen directly as follows. If \( r_0 = 0 \), then
\[ \mathcal{O}_n^* = C^*(S_i S_j : 1 \leq i, j \leq n) \cong \mathcal{O}_n^2, \]
and if \( r_0 \neq 0 \), then \( \mathcal{O}_n^* \) is generated by \( \mathcal{F}_n^* S_i^*, (S_i^*)^r \mathcal{F}_n^*, r = 0, 1, \ldots \), where \( \mathcal{F}_n^* \) is the fixed point algebra of the UHF algebra \( \mathcal{F}_n \) under the restricted action of \( \alpha \). This is an AF algebra with homogeneous embeddings
\[ \begin{pmatrix} r_0 & r_1 \\ r_1 & r_0 \end{pmatrix} \]
by [7]. Hence
\[ \mathcal{O}_n^* \cong \mathcal{O}_C, \quad \text{where} \quad C = \begin{pmatrix} r_0 & r_1 \\ r_1 & r_0 \end{pmatrix}. \]

**Corollary.** If \( r_0, r_1, s_0, s_1 \in \mathbb{Z} \) with \( r_0 + r_1 = n = s_0 + s_1 \), then \( \alpha(r_0, r_1) \) is not conjugate to \( \alpha(s_0, s_1) \) if \( r_0 \neq 1 + s_1, s_0 \).

**Proof.** If \( \theta = \alpha(r_0, r_1) \) is conjugate to \( \varphi = \alpha(s_0, s_1) \), then \( \mathbb{Z}_2 \not\cong \mathcal{O}_n \) is isomorphic to \( \mathbb{Z}_2 \times \mathcal{O}_n \), that is, \( \mathcal{O}_{A_1} \) is isomorphic to \( \mathcal{O}_{A_2} \) where
\[ A_1 = \begin{pmatrix} r_0 & r_1 \\ r_1 & r_0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} s_0 & s_1 \\ s_1 & s_0 \end{pmatrix}. \]

Hence by considering weak extensions [5],
\[ |\det (1 - A_1')| = |\det (1 - A_2')|. \]

Straightforward computation shows that
\[ |\det (1 - A_1')| = |\det (1 - A_1)| = |(1 - r_0 + r_1)(n - 1)|. \]

Thus \( |1 - r_0 + r_1| = |1 - s_0 + s_1|. \)

Let \( \varphi_n \) denote the endomorphism \( X \rightarrow \sum S_i X S_i^* \) on \( \mathcal{O}_n \), which commutes with the action \( \alpha \) of \( \mathbb{Z}_2 \), and hence extends to an endomorphism \( \tilde{\varphi}_n \) of \( \mathbb{Z}_2 \times \mathcal{O}_n \), by
\[ \tilde{\varphi}_n(X_0 + X_1 U) = \varphi_n(X_0) + \varphi_n(X_1)U, \quad X_i \in \mathcal{O}_n. \]
Proposition. \( \hat{\varphi}_n \) is homotopic to the identity on \( \mathbb{Z}_2 \times \mathcal{O}_n \).

Proof. In the notation of [3], \( \varphi_n = \hat{\lambda}_{V} \), where \( V = \sum_{i,j} S_i S_j S_i^* S_j^* \) is a self adjoint unitary with spectral decomposition \( V = E_1 - E_2 \) say. Then \( \alpha(E_i) = E_i \), since \( \alpha(V) = V \). Let \( V_t \) denote the unitary \( E_1 - tE_2 \), for \( t \in \mathbb{T} \), so that \( \hat{\lambda}_{V_t} \) gives a homotopy from the identity to \( \varphi_n \). Now \( \lambda = \hat{\lambda}_D \), where

\[
D = \sum_{i \leq r_0} S_i S_i^* - \sum_{j > r_0} S_j S_j^*,
\]

and so \( \lambda_X(D) = X^*DX \) for all unitaries \( X \) in \( \mathcal{O}_n \). Then

\[
\hat{\lambda}_D \hat{\lambda}_{V_t} = \hat{\lambda}_{D \hat{\lambda}_D(V_t)} = \hat{\lambda}_{DV_t} \quad \text{as} \quad \alpha(V_t) = V_t
\]

\[
= \hat{\lambda}_{V_t} \hat{\lambda}_{V_t}(D) \quad \text{as} \quad \hat{\lambda}_{V_t}(D) = V_t^*DV_t
\]

\[
= \hat{\lambda}_{V_t} \hat{\lambda}_D,
\]

i.e. \( \alpha \) commutes with \( \hat{\lambda}_{V_t} \), and so \( \hat{\lambda}_{V_t} \) extends to an endomorphism \( \hat{\chi}_{V_t} \) on \( \mathbb{Z}_2 \times \mathcal{O}_n \), which gives a homotopy from the identity to \( \hat{\varphi}_n \) on the crossed product.

Corollary. If \( \hat{\alpha} \) denotes the dual action of \( \hat{\mathbb{Z}}_2 \) on \( \mathbb{Z}_2 \times \mathcal{O}_n \), then \( \chi = r_0 \chi + r_1 \hat{\alpha}_*(\chi) \), for all \( \chi \) in \( K_i(\mathbb{Z}_2 \times \mathcal{O}_n) \) \((i = 0, 1)\).

Proof. Consider first the case \( i = 0 \). By [4] it is enough to consider classes \( \chi = [F]_0 \), where \( F \) is a projection majorized by \( R_0 \), thus \( F = ER_0 \), with \( E \in \mathcal{O}_n \). Then \( F = \frac{1}{2}(E + EU) \), and so

\[
\hat{\varphi}_n(F) = \frac{1}{2}(\varphi_n(E) + \varphi_n(E)U) = \varphi_n(E)R_0
\]

\[
= \sum_{i = 1}^n S_i ES_i^* R_0 = \sum_{i \leq r_0} S_i ER_0 S_i^* + \sum_{i > r_0} S_i ER_1 S_i^*
\]

\[
= \sum_{i \leq r_0} S_i FS_i^* + \sum_{i > r_0} S_i \hat{\alpha}(F) S_i^*
\]

since \( S_i R_0 = R_0 S_i, \ i \leq r_0 \), \( S_i R_0 = R_1 S_i, \ i > r_0 \), and \( \hat{\alpha}(ER_0) = ER_1 \). Thus by the proposition,

\[
[F]_0 = r_0[F]_0 + r_1[\hat{\alpha}(F)]_0.
\]

To get the result for \( K_1(\mathbb{Z}_2 \times \mathcal{O}_n) \) it suffices, again by [4], to consider equivalence classes of unitaries \( U \) in \( \mathbb{Z}_2 \times \mathcal{O}_n \) of the form \( U = U' + R_1 \), where \( U' \) is a unitary in \( R_0(\mathbb{Z}_2 \times \mathcal{O}_n) R_0 \). The argument then works just as for the case of \( K_0 \).

Remark 2. This global condition is enough in a large number of interesting cases to deduce that \( K_i(\mathcal{O}_A) \) are torsion groups. In this case one can proceed in
the spirit of [4] to compute \( K_0(\mathcal{O}_A) \). Consider the action \( \alpha(r_0 + 1, r_1) \) of \( Z_2 \) on \( \mathcal{O}_{n+1} \) which leaves

\[ \mathcal{O}_n = C^*(S_1, \ldots, S_{r_0}, S_{r_0+2}, \ldots S_{n+1}) \]

globally invariant. If \( \mathcal{I} \) is the ideal of \( \mathcal{O}_n \) generated by \( S_{r_0+1}, S^*_{r_0+1} \), which is also invariant and isomorphic to the compacts \( \mathcal{K} \), then

\[ Z_2 \times \mathcal{O}_n \big/ Z_2 \times \mathcal{I} \cong Z_2 \times \mathcal{O}_n, \quad \text{and} \quad Z_2 \times \mathcal{I} \cong \mathcal{K} \oplus \mathcal{K}. \]

Then one can show using exact sequences as in [4], that

\[ K_0(\mathcal{O}_A) = Z^2/(1-A)Z^2 = Z_{n-1} \oplus Z_{r_1-r_0-1}, \]

where the components are generated by \([1]_0, [R_1]_0 - [R_0]_0\), at least if \( Z^2/(1-A)Z^2 \) is 2-torsion free.

Alternatively, one can use the decomposition of \( \mathcal{K} \otimes \mathcal{O}_A \) as a crossed product of an AF-algebra by a single automorphism and apply the exact sequence [9, 2.4] to show that

\[ K_0(\mathcal{O}_A) = Z^n/(1-A)Z^n, \quad K_1(\mathcal{O}_A) = \text{Ker} (1-A) \]
on \( Z^n \) for every \( n \times n \)-matrix \( A \) that satisfies condition (I) of [5].

**Remark 3.** A similar analysis works for the actions of finite abelian groups, e.g. if \( \omega = e^{2\pi i/3} \), and \( r_0, r_1, r_2 \in Z \) with \( r_0 + r_1 + r_2 = n \), let \( \alpha \) denote the action of \( Z_3 \) on \( \mathcal{O}_n \) by

\[ S_i \rightarrow \omega^k S_i \quad \text{if} \quad \sum_{0}^{r-1} r_j \leq i < \sum_{0}^{r} r_j. \]

Then \( Z_3 \times \mathcal{O}_n \cong \mathcal{O}_A \), where

\[ A = \begin{bmatrix} r_0 & r_1 & r_2 \\ r_2 & r_0 & r_1 \\ r_1 & r_2 & r_0 \end{bmatrix}. \]

In particular the matrix

\[ A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \]

for which \( \text{Ext} (\mathcal{O}_A) = Z_2 \oplus Z_2 \) arises in this way.

**Remark 4.** For a compact abelian group \( G \), let \( \gamma_1, \ldots, \gamma_n \in \hat{G} \) and let \( \alpha \) be the corresponding action of \( G \) on \( \mathcal{O}_n \) given by \( \alpha(g)(S_i) = \gamma_i(g)S_i, \ g \in G, \ i = 1, \ldots, n. \)
For example, if $G = T$, and $\gamma_i(t) = t$, $\forall i$, then $T \times \mathcal{O}_n$ is stably isomorphic to a Glimm algebra and so simple [8, Remark]. However, if $G = T^2$, $n = 2$, and if $\gamma_i(t_1, t_2) = t_i$, then $T^2 \times \mathcal{O}_2$ is not simple (see [6]), and cf. [1] for actions on UHF algebras. Some insight into the distinction may be found by considering the related objects

$$C^\ast(R_n \pi(S_i) : 1 \leq i \leq n, \quad \eta \in \hat{G})$$

where $(\pi, U)$ is the natural covariant representation of $(\mathcal{O}_n, G, \alpha)$ on $L^2(G, H)$ and where $R_n$ are the projections with $U(g) = \sum_{\eta \in \hat{G}} \eta(g) R_n$. Then $C^\ast(R_n \pi(S_i))$ can be expressed, modulo the fact that $B$ does not satisfy condition (I) in [5], as $\mathcal{O}_B$, where $B = [b_{\alpha \beta}]_{\alpha, \beta \in A}$ is the matrix with $A = \{1, \ldots, n\} \times \hat{G}$ and $\{0, 1\}$ entries of admissible transitions $(i, \eta) \rightarrow (j, \eta - \gamma_j), 1 \leq i, j \leq n, \eta \in \hat{G}$. In both examples cited above, the matrices obtained are not irreducible in the strict sense. However, in the first case, if $\alpha, \beta \in A$, then there exists a path of transitions from either $\alpha$ to $\beta$ or $\beta$ to $\alpha$, but not both if $\alpha \neq \beta$. This is not the case in the second example. Further investigation of the ideal structure of these algebras, might shed more light on the question of simplicity of the crossed products themselves.

Acknowledgements. The second named author would like to thank J. Cuntz and his colleagues at Heidelberg for their warm hospitality during the visit when this work was pursued.

References