MATH. SCAND. 126 (2020), 276–292

# COMPACT GROUP ACTIONS ON OPERATOR ALGEBRAS AND THEIR SPECTRA

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## Abstract

We consider a class of dynamical systems with compact non-abelian groups that include C*-, W*- and multiplier dynamical systems. We prove results that relate the algebraic properties such as simplicity or primeness of the fixed point algebras to the spectral properties of the action, including the Connes and strong Connes spectra.


## 1. Introduction

In [2], Connes introduced the invariant $\Gamma(U)$ known as the Connes spectrum of the action $U$ of a locally compact abelian group on a von Neumann algebra and used it in his seminal classification of type III von Neumann factors. Soon after, Olesen [10] defined the Connes spectrum of an action of a locally compact abelian group on a C*-algebra. The paper [11] proves, using the definition of the Connes spectrum in [10], an analog of a result of Connes and Takesaki [3, Chapter III, Corollary 3.4] regarding the significance of the Connes spectrum of a locally compact abelian group action on a C*-algebra for the ideal structure of the crossed product. In particular, [11] discusses a spectral characterization for the crossed product to be a prime C*-algebra. This definition of the Connes spectrum in [10] cannot be used to prove similar results for the simplicity of the crossed product, unless the group is discrete [11]. Kishimoto [8] defined the strong Connes spectrum for C*-dynamical systems with locally compact abelian groups in a way that coincides with the Connes spectrum for the W*-dynamical system and with the Connes spectrum defined by Olesen for discrete abelian group actions on C*-algebras and he proved the Connes-Takesaki result for simple crossed products. In [2] Connes obtained results that relate the spectral properties of the action of abelian compact groups with the algebraic properties of the fixed point algebra. These results were extended in [12] to C*-algebras and compact abelian groups. In [6], [14] we considered the problems of simplicity and primeness of the crossed product by compact, non-abelian group actions. In particular, in [6] we defined the Connes and strong Connes

Received 13 September 2018, in final form 21 January 2019. Accepted 24 April 2019.
DOI: https://doi.org/10.7146/math.scand.a-119079

spectra for such actions in a way that coincide with Connes spectra [2], [10], respectively with the strong Connes spectra [8] for compact abelian groups. Further, in [15] we considered the case of one-parameter $\mathscr{F}$-dynamical systems that include the C*- the W*- and the multiplier one-parameter dynamical systems. In particular, we obtained extensions of some results in [2], [12] for W*-, respectively C*-, dynamical systems to the case of $\mathscr{F}$-dynamical systems with compact abelian groups [15, Theorems 3.2 and 3.4].

In this paper we will prove results for $\mathscr{F}$-dynamical systems with compact non-abelian groups. Our results contain and extend to the case of compact non-abelian groups the following: [2, Proposition 2.2.2b) and Theorem 2.4.1], [12, Theorem 2], [13, Theorem 8.10.4] and [15, Theorems 3.2 and 3.4]. In §2 we will set up the framework and state some results that will be used in the rest of the paper. In §3 we discuss the connection between the strong Connes spectrum, $\widetilde{\Gamma}_{\mathscr{F}}(\alpha)$, of the action and the $\mathscr{F}$-simplicity of the fixed point algebras $(X \otimes B(H_\pi))^{\alpha\otimes \operatorname{ad}\pi}$. In §4 we will get similar results about the connection between the $\mathscr{F}$-primeness of the fixed point algebras and the Connes spectrum, $\Gamma_{\mathscr{F}}(\alpha)$, of the action.

## 2. Notation and preliminary results

This section contains the definitions of the basic concepts used in the rest of the paper, the notation and some preliminary results.

Definition 2.1 ([1], [16]). A *dual pair of Banach spaces* is, by definition, a pair $(X, \mathscr{F})$ of Banach spaces with the following properties:

(a) $\mathscr{F}$ is a Banach subspace of the dual $X^*$ of $X$;

(b) $\|x\| = \sup\{|\varphi(x)| : \varphi \in \mathscr{F}, \|\varphi\| \le 1\}$, for every $x \in X$;

(c) $\|\varphi\| = \sup\{|\varphi(x)| : x \in X, \|x\| \le 1\}$, for every $\varphi \in \mathscr{F}$;

(d) the convex hull of every relatively $\sigma(X, \mathscr{F})$-compact subset of $X$ is relatively $\sigma(X, \mathscr{F})$-compact;

(e) the convex hull of every relatively $\sigma(\mathscr{F}, X)$-compact subset of $\mathscr{F}$ is relatively $\sigma(\mathscr{F}, X)$-compact.

In the rest of the paper, $X$ will be assumed to be a C*-algebra with the additional property:

(f) the involution of $X$ is $\sigma(X, \mathscr{F})$-continuous and the multiplication in $X$ is separately $\sigma(X, \mathscr{F})$-continuous.

From now on, we will use the terms $\mathscr{F}$-closed, $\mathscr{F}$-continuous, etc., instead of $\sigma(X, \mathscr{F})$-closed, $\sigma(X, \mathscr{F})$-continuous, etc.

Property (d) implies the existence of weak integrals of bounded continuous functions defined on a locally compact measure space, $(S, \mu)$, with values in $X$, endowed with the $\mathscr{F}$-topology (i.e. the $\sigma(X, \mathscr{F})$-topology): if $f$ is such a function, we will denote by

$$y = \int f(s)\, d\mu$$

the unique element $y$ of $X$ such that

$$\varphi(y) = \int_S \varphi(f(s))\, d\mu \tag{2.1}$$

for every $\varphi \in \mathscr{F}$ [1, Proposition 1.2]. Property (e) was used by Arveson [1, Proposition 1.4] to prove continuity in the $\mathscr{F}$-topology of some linear mappings on $X$ (in particular, the mappings $P_\alpha(\pi)$ and $(P_\alpha)_{ij}(\pi)$ defined below).

EXAMPLES 2.2. (a) [1] If $X$ is a C*-algebra and $\mathscr{F} = X^*$, then conditions (a)–(f) are satisfied.

(b) [1] If $X$ is a W*-algebra and $\mathscr{F} = X_*$ is its predual, then conditions (a)–(f) are satisfied.

(c) [4] If $X = M(Y)$ is the multiplier algebra of $Y$ and $\mathscr{F} = Y^*$, then conditions (a)–(f) are satisfied. In addition, in this case, the $\mathscr{F}$-topology on $X$ is compatible with the strict topology on $X = M(Y)$.

Let $(X, \mathscr{F})$ be a dual pair of Banach spaces, $G$ a compact group and $\alpha\colon G \to \mathrm{Aut}(X)$ a homomorphism from $G$ into the group of $*$-automorphisms of $X$. We say that $(X, G, \alpha)$ is an *$\mathscr{F}$-dynamical system* if the mapping

$$g \mapsto \varphi(\alpha_g(x))$$

is continuous for every $x \in X$ and $\varphi \in \mathscr{F}$.

EXAMPLES 2.3. (a) If $\mathscr{F} = X^*$, the dual of $X$, then by [7, p. 306] the above condition is equivalent to the continuity of the mapping $g \mapsto \alpha_g(x)$ from $G$ to $X$ endowed with the norm topology for every $x \in X$, so in this case $(X, G, \alpha)$ is a C*-dynamical system.

(b) If $X$ is a von Neumann algebra and $\mathscr{F} = X_*$, the predual of $X$ then $(X, G, \alpha)$ is a W*-dynamical system.

(c) If $X = M(Y)$ is the multiplier algebra of $Y$ and $\mathscr{F} = Y^*$, then $(X, G, \alpha)$ is said to be a multiplier dynamical system.

Let $(X, G, \alpha)$ be an $\mathscr{F}$-dynamical system with $G$ compact. Denote by $\widehat{G}$ the set of unitary equivalence classes of irreducible representations of $G$. For each $\pi \in \widehat{G}$ denote also by $\pi$ a fixed representative of that class on the Hilbert space $H_\pi$ of dimension $d_\pi$ and by $[\pi_{ij}(g)]$ the matrix of $\pi_g$ in a fixed orthonormal basis of $H_\pi$. If $\chi_\pi(g) = d_\pi^{-1} \sum_{i=1}^{d_\pi} \pi_{ii}(g^{-1}) = d_\pi^{-1} \sum \overline{\pi_{ii}(g)}$ is the character of $\pi$, write

$$P_\alpha(\pi)(x) = \int_G \chi_\pi(g)\alpha_g(x)\,dg,$$

where the integral is taken in the weak sense defined in (2.1) above. Then $P_\alpha(\pi)$ is a projection of $X$ onto the spectral subspace

$$X_1(\pi) = \{x \in X : P_\alpha(\pi)(x) = x\}.$$

As in [14], one can also define for every $1 \leq i, j \leq d_\pi$,

$$(P_\alpha)_{ij}(\pi)(x) = \int_G \overline{\pi_{ji}(g)}\alpha_g(x)\,dg$$

and show that

$$(P_\alpha)_{ij}(\pi)(X) \subset X_1(\pi). \tag{2.2}$$

Using [1, Proposition 1.4] it follows that $P_\alpha(\pi)$, $(P_\alpha)_{ij}(\pi)$ are $\mathscr{F}$-continuous. If $\pi$ is the identity one-dimensional representation $\iota$ of $G$, we will write

$$P_\alpha(\iota) = P_\alpha$$

and

$$X_1(\iota) = X^\alpha$$

is the fixed point algebra of the action.

Lemma 2.4.

$$\overline{\sum_{\pi\in\widehat{G}} X_1(\pi)}^{\sigma} = X,$$

*where* $\overline{\cdot}^{\sigma}$ *denotes closure in the* $\mathscr{F}$*-topology of* $X$.

Proof. Indeed, suppose that there exists $\varphi \in \mathscr{F}$ such that $\varphi(X_1(\pi)) = \{0\}$ for every $\pi \in \widehat{G}$. Since, as noticed above, $(P_\alpha)_{ij}(\pi)(X) \subset X_1(\pi)$, it follows that

$$\int_G \overline{\pi_{ij}(g)}\varphi(\alpha_g(x))\,dg = 0,$$

for every $x \in X$ and every $\pi \in \widehat{G}$. Since $\{\pi_{ij}(g) : \pi \in \widehat{G},\ 1 \leq i, j \leq d_\pi\}$ is an orthogonal basis of $L^2(G)$, and $\varphi(\alpha_g(x))$ is a continuous function of $g$, for every $x \in X$, it follows that $\varphi(x) = 0$ for every $x \in X$, so $\varphi = 0$ and we are done.

In [9], [14], [6] it is pointed out that the spectral subspaces

$$X_2(\pi) := \{a \in X \otimes B(H_\pi) : (\alpha_g \otimes \iota)(a) = a(1 \otimes \pi_g)\},$$

where $\iota$ is the identity automorphism of $B(H_\pi)$, are in some respects more useful. In [14] it is shown that

$$X_2(\pi) = \{[(P_\alpha)_{ij}(\pi)(x)] : x \in X, 1 \leqslant i, j \leqslant d_\pi\},$$

under the canonical identification of $X \otimes B(H_\pi)$ with $M_{d_\pi}(X)$. It is straightforward to prove that, if $a \in X_2(\pi)$ and $x = \sum_i a_{ii}$, then $a_{ij} = (P_\alpha)_{ij}(\pi)(x)$. In what follows, if $b \in X \otimes B(H_\pi)$, we will write

$$\mathrm{tr}(b) = \sum b_{ii},$$

which is a continuous linear mapping from $X \otimes B(H_\pi)$, with the $\sigma(X \otimes B(H_\pi), \mathscr{F} \otimes B(H_\pi)^*)$-topology, to $X$, with the $\sigma(X, \mathscr{F})$-topology. The following lemma is proven for compact non-abelian group actions on C*-algebras in [6, Lemma 2.3] and for compact abelian $\mathscr{F}$-dynamical systems in [15]. Since the proof is very similar to the proof of [6, Lemma 2.3], we will state it without proof.

LEMMA 2.5. *Let $(X, G, \alpha)$ be an $\mathscr{F}$-dynamical system with G compact and let $J$ be a two-sided ideal of $X^\alpha$. Then*

$$(\overline{XJX}^{\sigma})^{\alpha} = \mathscr{F}\text{-closed linear span of } \{\mathrm{tr}(X_2(\pi)JX_2(\pi)^*) : \pi \in \widehat{G}\},$$

*where, the meaning of $XJX$, $X_2(\pi)JX_2(\pi)^*$ is as in parts* (c) *and* (g) *of Notation 2.6 below.*

We will use the following notation:

NOTATION 2.6. Let $(X, \mathscr{F})$ be a dual pair of Banach spaces with $X$ a C*-algebra satisfying conditions (a)–(f) in Definition 2.1. If $Y$, $Z$ are subsets of $X$, then

(a) $\mathrm{lin}\{Y\}$ is the linear span of $Y$,

(b) $Y^* = \{y^* : y \in Y\}$,

(c) $YZ = \mathrm{lin}\{yz : y \in Y, z \in Z\}$,

(d) $\overline{Y}^{\sigma}$ is the $\mathscr{F}$-closure of $Y$ in $X$,

(e) $\overline{Y}^{\|\|}$ is the norm-closure of $Y$,

(f) $\overline{Y}^{w}$ is the $\sigma(\mathscr{F}^*, \mathscr{F})$-closure of $Y$ in $\mathscr{F}^*$,

(g) if $x \in X$, $H$ is a finite-dimensional Hilbert space and $a = [a_{ij}] \in X \otimes B(H)$, then $xa$ denotes the matrix $[xa_{ij}]$,

(h) if $H$ is a finite-dimensional Hilbert space and $Y \subset X \otimes B(H)$, we denote by $\overline{Y}^{\sigma}$ the $\sigma(X \otimes B(H), \mathscr{F} \otimes B(H)^*)$-closure of $Y$ in $X \otimes B(H)$.

If $(X, G, \alpha)$ is an $\mathscr{F}$-dynamical system, write

(i) $H^{\alpha}_{\sigma}(X)$ for the set of all non-zero globally $\alpha$-invariant $F$-closed hereditary C*-subalgebras of $X$.

Notice that if $(X, G, \alpha)$ is an $\mathscr{F}$-dynamical system and if $X_2(\pi)$ is the spectral subspace defined above, then $X_2(\pi)X_2(\pi)^*$ is a two-sided ideal of $X^{\alpha} \otimes B(H_{\pi})$ and $X_2(\pi)^*X_2(\pi)$ is a two-sided ideal of $(X \otimes B(H_{\pi}))^{\alpha \otimes \operatorname{ad} \pi}$, where $\alpha \otimes \operatorname{ad} \pi$ is the action

$$(\alpha_g \otimes \operatorname{ad} \pi_g)(a) = (1 \otimes \pi_g)[\alpha_g(a_{ij})](1 \otimes \pi_{g^{-1}})$$

on $X \otimes B(H_{\pi})$.

DEFINITION 2.7. (a) $\operatorname{sp}(\alpha) = \{\pi \in \widehat{G} : X_1(\pi) \neq \{0\}\}$.
(b) $\operatorname{sp}_{\mathscr{F}}(\alpha) = \{\pi \in \widehat{G} : X_2(\pi)^*X_2(\pi) \text{ is essential in } (X \otimes B(H_{\pi}))^{\alpha \otimes \operatorname{ad} \pi}\}$.
(c) $\widetilde{\operatorname{sp}}_{\mathscr{F}}(\alpha) = \left\{\pi \in \widehat{G} : \overline{X_2(\pi)^*X_2(\pi)}^{\sigma} = (X \otimes B(H_{\pi}))^{\alpha \otimes \operatorname{ad} \pi}\right\}$.
Corresponding to the above Arveson-type spectra (b) and (c) we define two Connes-type spectra:
(d) $\Gamma_{\mathscr{F}}(\alpha) = \bigcap\{\operatorname{sp}_{\mathscr{F}}(\alpha|_Y) : Y \in \mathscr{H}^{\alpha}_{\sigma}(X)\}$,
(e) $\widetilde{\Gamma}_{\mathscr{F}}(\alpha) = \bigcap\{\widetilde{\operatorname{sp}}_{\mathscr{F}}(\alpha|_Y) : Y \in \mathscr{H}^{\alpha}_{\sigma}(X)\}$.

Clearly, $\widetilde{\operatorname{sp}}_{\mathscr{F}}(\alpha) \subset \operatorname{sp}_{\mathscr{F}}(\alpha) \subset \operatorname{sp}(\alpha)$, so $\widetilde{\Gamma}_{\mathscr{F}}(\alpha) \subset \Gamma_{\mathscr{F}}(\alpha)$. The definition of $\widetilde{\Gamma}_{\mathscr{F}}(\alpha)$ is a direct generalization of the strong Connes spectrum of Kishimoto to compact non-abelian groups. Our motivation for the definition of $\Gamma_{\mathscr{F}}(\alpha)$ above (and $\Gamma(\alpha)$ for C*-dynamical systems in [6]) is the following observation.

LEMMA 2.8. (i) *If $(X, G, \alpha)$ is an $\mathscr{F}$-dynamical system with $G$ compact abelian, then*

$$\bigcap\{\operatorname{sp}(\alpha|_Y) : Y \in \mathscr{H}^{\alpha}_{\sigma}(X)\} = \bigcap\{\operatorname{sp}_{\mathscr{F}}(\alpha|_Y) : Y \in \mathscr{H}^{\alpha}_{\sigma}(X)\}$$

*and the left-hand side of the above equality is the Connes spectrum for W*- or C*-dynamical systems.*
(ii) *If $G$ is not abelian, the equality in part* (i) *is not true.*

PROOF. (i) We have to prove only one inclusion, the opposite one being obvious. Let $\gamma \in \bigcap\{\operatorname{sp}(\alpha|_Y) : Y \in \mathscr{H}^{\alpha}_{\sigma}(X)\}$ and $Y \in \mathscr{H}^{\alpha}_{\sigma}(X)$. Suppose that $aY^*_{\gamma}Y_{\gamma} = \{0\}$ for some $a \in Y^{\alpha}$, $a \neq 0$, where $Y_{\gamma} = \{y \in Y : \alpha_g(y) = \langle g, \gamma \rangle y,\ g \in G\}$ is the spectral subspace of $Y$ corresponding to $\gamma \in \widehat{G}$. Then $aY^*_{\gamma} = \{0\}$. Therefore, writing $Z = \overline{aYa^*}^{\sigma}$, it follows that $Z \in \mathscr{H}^{\alpha}_{\sigma}(X)$ and $Z^*_{\gamma} = \{0\}$, which is in contradiction with the hypothesis that $\gamma \in \bigcap\{\operatorname{sp}(\alpha|_Y) : Y \in \mathscr{H}^{\alpha}_{\sigma}(X)\} \subset \operatorname{sp}(\alpha|_Z)$.

(ii) In [14, Example 3.9] we provided an example of an action of $G = S_3$, the permutation group on three elements, on the algebra $X$ of $2 \times 2$ matrices, such that $\mathrm{sp}(\alpha) = \widehat{G}$, $H_\sigma^\alpha = \{X\}$, so

$$\bigcap\{\mathrm{sp}(\alpha|_Y) : Y \in \mathscr{H}_\sigma^\alpha(X)\} = \mathrm{sp}(\alpha)$$

and we have shown that there exists $\pi \in \widehat{G}$ such that $(X \otimes B(H_\pi))^{\alpha\otimes \mathrm{ad}\,\pi}$ has non-trivial center and, therefore, it is not a prime C*-algebra. By [6, Thm. 2.2], it follows that $\bigcap\{\mathrm{sp}(\alpha|_Y) : Y \in \mathscr{H}_\sigma^\alpha(X)\} \neq \bigcap\{\mathrm{sp}_{\mathscr{F}}(\alpha|_Y) : Y \in \mathscr{H}_\sigma^\alpha(X)\} = \Gamma(\alpha)$.

## 3. $\mathscr{F}$-simple fixed point algebras

Let $(X, G, \alpha)$ be an $\mathscr{F}$-dynamical system with $G$ compact. In the rest of this paper we will study how the $\mathscr{F}$-simplicity (respectively, $\mathscr{F}$-primeness) as defined below, of the fixed point algebras $(X \otimes B(H_\pi))^{\alpha\otimes \mathrm{ad}\,\pi}$ is reflected in the spectral properties of the action.

Definition 3.1. Let $(B, \mathscr{F})$ be a dual pair of Banach spaces with $B$ a C*-algebra.

(a) $B$ is called *$\mathscr{F}$-simple* if every non-zero two-sided ideal of $B$ is $\mathscr{F}$-dense in $B$.

(b) $B$ is called *$\mathscr{F}$-prime* if it is prime as a C*-algebra, i.e. if the annihilator of every non-zero two-sided ideal of $B$ is trivial, or, equivalently, every non-zero two-sided ideal of $B$ is an essential ideal.

Let $(X, G, \alpha)$ be an $\mathscr{F}$-dynamical system.

(c) $X$ is called *$\alpha$-simple* if every non-zero $\alpha$-invariant two-sided ideal of $X$ is $\mathscr{F}$-dense in $X$.

(d) $X$ is called *$\alpha$-prime* if every non-zero $\alpha$-invariant two-sided ideal of $X$ is an essential ideal.

Even though $X$ is $\mathscr{F}$-prime if and only if $X$ is prime as a C*-algebra, we will continue to use the name $\mathscr{F}$-prime since, traditionally, a von Neumann algebra which is prime as a C*-algebra is called a factor. Also in the case when $X = M(Y)$ and $\mathscr{F} = Y^*$, an $\mathscr{F}$-prime and $\mathscr{F}$-closed subalgebra of $X$ is a prime C*-algebra that is closed in the strict topology of $X$.

In the particular case when $B$ is a C*-algebra and $\mathscr{F} = B^*$ is its dual, then, clearly, the concepts of $\mathscr{F}$-simple and $\mathscr{F}$-prime in parts (a) and (b) of the above Definition 3.1 coincide with the usual concepts of simple and prime C*-algebras. Similarly, if $(X, G, \alpha)$ is a C*-dynamical system, that is if $X$ is a C*-algebra and $\mathscr{F} = X^*$ is its dual, then the notions of $\alpha$-simple and $\alpha$-prime coincide with the usual ones for C*-dynamical systems.

If $B$ is a von Neumann algebra and $\mathscr{F} = B_*$ is its predual, then, since the weak closure of every essential ideal equals $B$, it follows that $B$ is $\mathscr{F}$-simple if and only if $B$ is $\mathscr{F}$-prime, so, if and only if $B$ is a factor. It is also obvious that if $(X, G, \alpha)$ is a W*-dynamical system, that is if $X$ is a von Neumann algebra and $\mathscr{F} = X_*$ is its predual, then $X$ is $\alpha$-simple if and only if it is $\alpha$-prime, and this holds if and only if $\alpha$ acts ergodically on the center of $X$ (i.e. every fixed element in the center of $X$ is a scalar).

The above observations and the next remark show that for W*-dynamical systems, $(X, G, \alpha)$ with $G$ compact, the results in the current section and §4 are equivalent.

Remark 3.2. Let $(X, G, \alpha)$ be a W*-dynamical system, that is, an $\mathscr{F}$-dynamical system with $X$ a von Neumann algebra and $\mathscr{F} = X_*$ its predual. Then $\widetilde{\Gamma}_{\mathscr{F}}(\alpha) = \Gamma_{\mathscr{F}}(\alpha)$.

This follows from the fact that if $X$ is a von Neumann algebra, $p \in X^\alpha$ an $\alpha$-invariant projection and $\overline{pX_2(\pi)^* pX_2(\pi) p}^\sigma$ is essential in $(pXp \otimes B(H_\pi))^{\alpha \otimes \mathrm{ad}\,\pi}$, then $\overline{pX_2(\pi)^* pX_2(\pi) p}^\sigma = (pXp \otimes B(H_\pi))^{\alpha \otimes \mathrm{ad}\,\pi}$ since an essential ideal of a W*-algebra is $\sigma$-dense.

The next lemma will be used in the proofs of the main results of the current and next sections.

Lemma 3.3. *Let $(B, G, \alpha)$ be an $\mathscr{F}$-dynamical system with $G$ compact. Then*

(i) *If $\{e_\lambda\}$ is an approximate identity of $B^\alpha$ in the norm topology, then*

$$(\mathrm{norm})\lim_\lambda e_\lambda x = (\mathrm{norm})\lim_\lambda x e_\lambda = (\mathrm{norm})\lim_\lambda e_\lambda x e_\lambda = x,$$

*for every $x \in \overline{\sum_{\pi \in \widehat{G}} B_1(\pi)}^{\|\|}$.*

(ii) *If $b \in B$ is such that $B^\alpha b B^\alpha = \{0\}$ then $b = 0$,*

(iii) $\overline{B^\alpha B B^\alpha}^\sigma = \overline{B^\alpha B}^\sigma = \overline{B B^\alpha}^\sigma = B$,

(iv) $\overline{B^\alpha B_1(\pi)}^\sigma = \overline{B_1(\pi) B^\alpha}^\sigma = B_1(\pi)$, $\pi \in \widehat{G}$,

(v) $\overline{B^\alpha B_2(\pi)}^\sigma = \overline{B_2(\pi) B^\alpha}^\sigma = B_2(\pi)$, $\pi \in \widehat{G}$.

Proof. (i) This follows from the proof of [5, Lemma 2.7] in the more general case of compact quantum group actions.

(ii) If $\{e_\lambda\}$ is an approximate identity of $B^\alpha$, then $e_\lambda b e_\lambda = 0$ implies

$$e_\lambda (P_\alpha)_{ij}(\pi)(b) e_\lambda = (P_\alpha)_{ij}(\pi)(e_\lambda b e_\lambda) = 0,$$

for every $\pi \in \widehat{G}$, $1 \le i, j \le d_\pi$, so, by (i), $(P_\alpha)_{ij}(\pi)(b) = 0$. Therefore,

$$\varphi\big((P_\alpha)_{ij}(\pi)(b)\big) = \int_G \pi_{ji}(g) \varphi(\alpha_g(b))\, dg = 0,$$

for every $\varphi \in \mathscr{F}$, $\pi \in \widehat{G}$, $1 \le i, j \le d_\pi$. Since $\{\pi_{ij}(g) : \pi \in \widehat{G}, 1 \le i, j \le d_\pi\}$ form an orthogonal basis of $L^2(G)$, and $\varphi(\alpha_g(b))$ is continuous on $G$, it follows that $\varphi(\alpha_g(b)) = 0$ for every $g \in G$, $\varphi \in \mathscr{F}$, so $b = 0$.

(iii) We will prove only that $\overline{B^\alpha B B^\alpha}^\sigma = B$, the proofs of the other equalities being similar. Let $\{e_\lambda\}$ be an approximate identity of $B^\alpha$. By (i),

$$(\text{norm})\lim_\lambda e_\lambda x e_\lambda = x, \quad \text{for every } x \in \overline{\sum_{\pi \in \widehat{G}} B_1(\pi)}^{\|\,\|}.$$

Therefore

$$\overline{\sum_{\pi \in \widehat{G}} B_1(\pi)}^{\|\,\|} \subset \overline{B^\alpha B B^\alpha}^{\|\,\|} \subset \overline{B^\alpha B B^\alpha}^\sigma.$$

Since, by Lemma 2.4, the $\mathscr{F}$-closure of $\overline{\sum_{\pi \in \widehat{G}} B_1(\pi)}^{\|\,\|}$ equals $B$ it follows that

$$B = \overline{\sum_{\pi \in \widehat{G}} B_1(\pi)}^\sigma \subset \overline{B^\alpha B B^\alpha}^\sigma,$$

so $\overline{B^\alpha B B^\alpha}^\sigma = B$.

(iv) The proof is similar to the proof of part (iii).

(v) follows from (i), (iv) and the formula (2.2).

The following lemma will be used in the proofs of Theorems 3.5 and 4.1.

LEMMA 3.4. *Let $(X, G, \alpha)$ be an $\mathscr{F}$-dynamical system with G a compact group and $Y \in \mathscr{H}_\sigma^\alpha(X)$. Consider the ideal $J = \overline{X^\alpha Y^\alpha X^\alpha}^\sigma$ of $X^\alpha$ and let $Z \in \mathscr{H}_\sigma^\alpha(X)$ be defined as $Z = \overline{JXJ}^\sigma$. Then*

(i) $Z = \overline{Z^\alpha X Z^\alpha}^\sigma = \overline{X^\alpha Y X^\alpha}^\sigma$,

(ii) $Z_2(\pi) = \overline{X^\alpha Y_2(\pi) X^\alpha}^\sigma$ *and so* $\overline{Z_2(\pi)^* Z_2(\pi)}^\sigma = \overline{X^\alpha Y_2(\pi)^* Y_2(\pi) X^\alpha}^\sigma$,

(iii) $(Z \otimes B(H_\pi))^{\alpha \otimes \mathrm{ad}\,\pi} = \overline{X^\alpha (Y \otimes B(H_\pi))^{\alpha \otimes \mathrm{ad}\,\pi} X^\alpha}^\sigma$.

PROOF. (i) Since $Z$ is an $\mathscr{F}$-closed hereditary C*-subalgebra, we have $\overline{Z^\alpha X Z^\alpha}^\sigma \subset \overline{ZXZ}^\sigma = Z$. Using Lemma 3.3(iii) for $B = Z$, it follows that $Z = \overline{Z^\alpha Z Z^\alpha}^\sigma$, so $\overline{ZXZ}^\sigma \subset \overline{Z^\alpha X Z^\alpha}^\sigma$ and thus $Z = \overline{Z^\alpha X Z^\alpha}^\sigma$. The second equality in part (i) follows since, clearly, $Z^\alpha = \overline{X^\alpha Y^\alpha X^\alpha}^\sigma$. Parts (ii) and (iii) follow from (i) and from definitions.

Theorem 3.5 below is an extension of [2, Proposition 2.2.2b)] to the case of $\mathscr{F}$-dynamical systems with compact groups, not necessarily abelian, for the strong Connes spectrum, $\widetilde{\Gamma}_{\mathscr{F}}(\alpha)$.

THEOREM 3.5. *Let* $(X, G, \alpha)$ *be an* $\mathcal{F}$*-dynamical system with* $G$ *compact. Then*

$$\widetilde{\Gamma}_{\mathcal{F}}(\alpha) = \bigcap \big\{ \widetilde{\mathrm{sp}}_{\mathcal{F}}(\alpha|_{\overline{JXJ}^{\sigma}}) : J \subset X^{\alpha},\ \textit{non-zero } \mathcal{F}\textit{-closed two-sided ideal} \big\}. \tag{3.1}$$

PROOF. Denote the right-hand side of (3.1) by $\tilde{\gamma}_{\mathcal{F}}(\alpha)$. Clearly, since $\overline{JXJ}^{\sigma} \in \mathcal{H}_{\sigma}^{\alpha}(X)$,

$$\widetilde{\Gamma}_{\mathcal{F}}(\alpha) \subset \tilde{\gamma}_{\mathcal{F}}(\alpha).$$

Let $\pi \in \tilde{\gamma}_{\mathcal{F}}(\alpha)$ and $Y \in \mathcal{H}_{\sigma}^{\alpha}(X)$, so $Y^{\alpha} \in \mathcal{H}_{\sigma}(X^{\alpha})$. We will prove that $\overline{Y_2(\pi)^* Y_2(\pi)}^{\sigma} = (Y \otimes B(H_{\pi}))^{\alpha \otimes \mathrm{ad}\,\pi}$ and thus $\pi \in \widetilde{\mathrm{sp}}_{\mathcal{F}}(\alpha|_Y)$. Since $Y \in \mathcal{H}_{\sigma}^{\alpha}(X)$ is arbitrary, it will follow that $\pi \in \widetilde{\Gamma}_{\mathcal{F}}(\alpha)$. Denote by $J$ the following ideal of $X^{\alpha}$:

$$J = \overline{X^{\alpha} Y^{\alpha} X^{\alpha}}^{\sigma}.$$

It is clear that $J = \overline{JX^{\alpha}J}^{\sigma}$ (actually it is quite easy to show that this equality holds without the closure, but we do not need this fact). Also

$$\begin{aligned}\overline{Y^{\alpha} J Y^{\alpha}}^{\sigma} &= \overline{Y^{\alpha} X^{\alpha} Y^{\alpha} X^{\alpha} Y^{\alpha}}^{\sigma} = \overline{(Y^{\alpha} X^{\alpha} Y^{\alpha})(Y^{\alpha} X^{\alpha} Y^{\alpha})}^{\sigma} \\ &= \overline{Y^{\alpha} Y^{\alpha}}^{\sigma} = Y^{\alpha}.\end{aligned}$$

Denote $Z = \overline{JXJ}^{\sigma}$. From Lemma 3.4(i) it follows that

$$Z = \overline{X^{\alpha} Y X^{\alpha}}^{\sigma}. \tag{3.2}$$

Since $\pi \in \tilde{\gamma}_{\mathcal{F}}(\alpha)$, it follows that $\pi \in \widetilde{\mathrm{sp}}_{\mathcal{F}}(\alpha|_Z)$, so

$$\overline{Z_2(\pi)^* Z_2(\pi)}^{\sigma} = (Z \otimes B(H_{\pi}))^{\alpha \otimes \mathrm{ad}\,\pi}. \tag{3.3}$$

Using (3.2) above, and Lemma 3.4(ii), the relation (3.3) becomes

$$\overline{X^{\alpha} Y_2(\pi)^* Y_2(\pi) X^{\alpha}}^{\sigma} = \overline{X^{\alpha} (Y \otimes B(H_{\pi}))^{\alpha \otimes \mathrm{ad}\,\pi} X^{\alpha}}^{\sigma}.$$

Therefore, by applying Lemma 3.3(v) to $B = Y$, we get

$$\overline{X^{\alpha} Y^{\alpha} Y_2(\pi)^* Y_2(\pi) Y^{\alpha} X^{\alpha}}^{\sigma} = \overline{X^{\alpha} Y^{\alpha} (Y \otimes B(H_{\pi}))^{\alpha \otimes \mathrm{ad}\,\pi} Y^{\alpha} X^{\alpha}}^{\sigma}. \tag{3.4}$$

By multiplying (3.4) on the right and on the left by $Y^{\alpha}$ and taking into account that, by Lemma 3.3(iii) applied to $B = Y$, $\overline{Y^{\alpha} Y Y^{\alpha}}^{\sigma} = Y$ and, consequently, $\overline{Y^{\alpha} X^{\alpha} Y^{\alpha}}^{\sigma} = Y^{\alpha}$, it follows that

$$\overline{Y_2(\pi)^* Y_2(\pi)}^{\sigma} = (Y \otimes B(H_{\pi}))^{\alpha \otimes \mathrm{ad}\,\pi}.$$

Therefore, $\pi \in \widetilde{\mathrm{sp}}_{\mathcal{F}}(\alpha|_Y)$ and the proof is complete.

In the next Lemma and the rest of the paper, a subalgebra of $X \otimes B(H_\pi)$ will be called $\mathscr{F}$-simple (respectively, $\mathscr{F}$-prime) if it is $\mathscr{F} \otimes B(H_\pi)^*$-simple (respectively, $\mathscr{F} \otimes B(H_\pi)^*$-prime), where $B(H_\pi)^*$ denotes the dual of $B(H_\pi)$. Clearly, a subalgebra of $X \otimes B(H_\pi)$ is $\mathscr{F}$-prime if and only if it is a prime C*-algebra. The similar statement for the $\mathscr{F}$-simple case is not true.

LEMMA 3.6. *Let $(X, G, \alpha)$ be an $\mathscr{F}$-dynamical system with G compact. Then, if $X^\alpha$ is $\mathscr{F}$-simple, it follows that $(X \otimes B(H_\pi))^{\alpha\otimes \mathrm{ad}\,\pi}$ is $\mathscr{F}$-simple for every $\pi \in \widetilde{\mathrm{sp}}_{\mathscr{F}}(\alpha)$.*

PROOF. Let $\pi \in \widetilde{\mathrm{sp}}_{\mathscr{F}}(\alpha) \subset \mathrm{sp}(\alpha)$. Since $X^\alpha$ is $\mathscr{F}$-simple, so $X^\alpha \otimes B(H_\pi)$ is also $\mathscr{F}$-simple, and $X_2(\pi)X_2(\pi)^*$ is an ideal of $X^\alpha \otimes B(H_\pi)$, it follows that $\overline{X_2(\pi)X_2(\pi)^*}^\sigma = X^\alpha \otimes B(H_\pi)$. To prove that $(X \otimes B(H_\pi))^{\alpha\otimes \mathrm{ad}\,\pi}$ is $\mathscr{F}$-simple, let $I \subset (X \otimes B(H_\pi))^{\alpha\otimes \mathrm{ad}\,\pi}$ be an $\mathscr{F}$-closed non-zero ideal. Then it can be easily verified that

$$J = \overline{\mathrm{lin}}^\sigma\{yy^* : y \in X_2(\pi)I\} = \overline{X_2(\pi)IX_2(\pi)^*}^\sigma$$

is an ideal of $X^\alpha \otimes B(H_\pi)$ and, since the latter algebra is $\mathscr{F}$-simple, it follows that $J = X^\alpha \otimes B(H_\pi)$. Therefore, since $\pi \in \widetilde{\mathrm{sp}}_{\mathscr{F}}(\alpha)$, we have $\overline{X_2(\pi)^*X_2(\pi)}^\sigma = (X \otimes B(H_\pi))^{\alpha\otimes \mathrm{ad}\,\pi}$ and consequently, since, by Lemma 3.3(v) $\overline{X^\alpha X_2(\pi)}^\sigma = X_2(\pi)$, we have

$$(X \otimes B(H_\pi))^{\alpha\otimes \mathrm{ad}\,\pi} = \overline{X_2(\pi)^*JX_2(\pi)}^\sigma \subset \overline{X_2(\pi)^*X_2(\pi)IX_2(\pi)^*X_2(\pi)}^\sigma \subset I.$$

Thus $I = (X \otimes B(H_\pi))^{\alpha\otimes \mathrm{ad}\,\pi}$ and we are done.

The following result extends [2, Théorème 2.4.1], [12, Theorem 2:i) ⇔ ii)] and [15, Theorem 3.4] to the more general case of $\mathscr{F}$-dynamical systems and non-abelian compact groups $G$.

THEOREM 3.7. *Let $(X, G, \alpha)$ be an $\mathscr{F}$-dynamical system with G compact. The following conditions are equivalent:*

(i) *$(X \otimes B(H_\pi))^{\alpha\otimes \mathrm{ad}\,\pi}$ is $\mathscr{F}$-simple for all $\pi \in \mathrm{sp}(\alpha)$;*

(ii) *$X$ is $\alpha$-simple and $\mathrm{sp}(\alpha) = \widetilde{\Gamma}_{\mathscr{F}}(\alpha)$.*

PROOF. (i) ⇒ (ii) Suppose that $(X \otimes B(H_\pi))^{\alpha\otimes \mathrm{ad}\,\pi}$ is $\mathscr{F}$-simple for all $\pi \in \mathrm{sp}(\alpha)$. Then, it follows immediately from the definitions that $\mathrm{sp}(\alpha) = \widetilde{\mathrm{sp}}_{\mathscr{F}}(\alpha)$. Let $\pi \in \mathrm{sp}(\alpha)$ be arbitrary. Since, in particular, $X^\alpha$ is $\mathscr{F}$-simple, so it has no non-trivial $\mathscr{F}$-closed ideals, from Theorem 3.5 it follows that $\pi \in \widetilde{\Gamma}_{\mathscr{F}}(\alpha)$, so $\mathrm{sp}(\alpha) = \widetilde{\Gamma}_{\mathscr{F}}(\alpha)$. We will prove next that $X$ is $\alpha$-simple. If $I$ is an $\mathscr{F}$-closed non-zero $\alpha$-invariant ideal of $X$, then $I^\alpha$ is an $\mathscr{F}$-closed ideal of $X^\alpha$, so

$I^\alpha = X^\alpha$. By Lemma 3.3(iii) applied to $B = I$, and to $B = X$ it follows that $\overline{I^\alpha I I^\alpha}^\sigma = I$ and $\overline{X^\alpha X X^\alpha}^\sigma = X$, so

$$X = \overline{X^\alpha X X^\alpha}^\sigma = \overline{I^\alpha X I^\alpha}^\sigma \subset I.$$

Therefore, $I = X$, hence $X$ is $\alpha$-simple.

(ii) $\Rightarrow$ (i) Suppose that $X$ is $\alpha$-simple and $\mathrm{sp}(\alpha) = \widetilde{\Gamma}_{\mathscr{F}}(\alpha)$. We will prove first that $X^\alpha$ is $\mathscr{F}$-simple. Let $J \subset X^\alpha$ be a non-zero ideal and $\pi \in \widetilde{\Gamma}_{\mathscr{F}}(\alpha)$. Since $\overline{JXJ}^\sigma \in \mathscr{H}^\alpha_\sigma(X)$, and $\pi \in \widetilde{\Gamma}_{\mathscr{F}}(\alpha)$, it follows that

$$\overline{JX_2(\pi)^* JX_2(\pi) J}^\sigma = \overline{J(X \otimes B(H_\pi))^{\alpha\otimes \mathrm{ad}\,\pi} J}^\sigma. \tag{3.5}$$

By multiplying the above relation on the left by $X_2(\pi)$ and on the right by $X_2(\pi)^*$, we get

$$\overline{X_2(\pi) J X_2(\pi)^* J X_2(\pi) J X_2(\pi)^*}^\sigma = \overline{X_2(\pi) J (X \otimes B(H_\pi))^{\alpha\otimes \mathrm{ad}\,\pi} J X_2(\pi)^*}^\sigma. \tag{3.6}$$

From the above relations (3.5) and (3.6) it follows that

$$\begin{aligned} X_2(\pi) J X_2(\pi)^* &\subseteq \overline{X_2(\pi) J X_2(\pi)^*}^\sigma \\ &\subset \overline{X_2(\pi) J (X \otimes B(H_\pi))^{\alpha\otimes \mathrm{ad}\,\pi} J X_2(\pi)^*}^\sigma \\ &= \overline{X_2(\pi) J X_2(\pi)^* J X_2(\pi) J X_2(\pi)^*}^\sigma \\ &\subset \overline{(X^\alpha \otimes B(H_\pi)) J (X^\alpha \otimes B(H_\pi))}^\sigma \subset J \otimes B(H_\pi). \end{aligned}$$

It follows that $\mathrm{tr}(X_2(\pi) J X_2(\pi)^*) \subset J$. From Lemma 2.5 it follows that $(\overline{XJX}^\sigma)^\alpha \subset J$. Since $X$ is $\alpha$-simple, we have $\overline{XJX}^\sigma = X$, so $J = X^\alpha$ and therefore, $X^\alpha$ is $\mathscr{F}$-simple. Applying Lemma 3.6 it follows that $(X \otimes B(H_\pi))^{\alpha\otimes \mathrm{ad}\,\pi}$ is $\mathscr{F}$-simple for all $\pi \in \mathrm{sp}(\alpha) = \widetilde{\Gamma}_{\mathscr{F}}(\alpha)$.

## 4. $\mathscr{F}$-prime fixed point algebras

This section is concerned with the relationship between the $\mathscr{F}$-primeness of the fixed point algebras and the spectral properties, involving the Connes spectrum $\Gamma_{\mathscr{F}}(\alpha)$ of the $\mathscr{F}$-dynamical system $(X, G, \alpha)$.

Theorem 4.1 below is an extension of [2, Proposition 2.2.2b)] to the case of $\mathscr{F}$-dynamical systems with compact groups, not necessarily abelian, for the Connes spectrum, $\Gamma_{\mathscr{F}}(\alpha)$. By Remark 3.2 and the discussion preceding it, if $(X, G, \alpha)$ is a W*-dynamical system (that is $X$ is a von Neumann algebra and $\mathscr{F} = X_*$ its predual), then the next Theorem 4.1 is equivalent with Theorem 3.5.

THEOREM 4.1. *Let $(X, G, \alpha)$ be an $\mathscr{F}$-dynamical system. Then*

$$\Gamma_{\mathscr{F}}(\alpha) = \bigcap\{\mathrm{sp}_{\mathscr{F}}(\alpha|_{\overline{JXJ}^{\sigma}}) : J \subset X^{\alpha},\ \textit{non-zero } \mathscr{F}\textit{-closed two-sided ideal}\}. \tag{4.1}$$

PROOF. Denote the right-hand side of (4.1) by $\gamma_{\mathscr{F}}(\alpha)$. Since $\overline{JXJ}^{\sigma} \in \mathscr{H}_{\sigma}^{\alpha}(X)$ for every non-zero $\mathscr{F}$-closed two-sided ideal $J \subset X^{\alpha}$, we have,

$$\Gamma_{\mathscr{F}}(\alpha) \subset \gamma_{\mathscr{F}}(\alpha).$$

Now let $\pi \in \gamma_{\mathscr{F}}(\alpha)$ and $Y \in \mathscr{H}_{\sigma}^{\alpha}(X)$, so $Y^{\alpha} \in \mathscr{H}_{\sigma}(X^{\alpha})$. We will prove that $\overline{Y_2(\pi)^*Y_2(\pi)}^{\sigma}$ is essential in $(Y \otimes B(H_{\pi}))^{\alpha\otimes\mathrm{ad}\,\pi}$. As in the proof of Theorem 3.5, let $J = \overline{X^{\alpha}Y^{\alpha}X^{\alpha}}^{\sigma}$ and $Z = \overline{JXJ}^{\sigma} \in H_{\sigma}^{\alpha}(X)$. Since $\pi \in \gamma_{\mathscr{F}}(\alpha)$, we have that $\pi \in \mathrm{sp}_{\mathscr{F}}(\alpha|_Z)$. Therefore, $\overline{Z_2(\pi)^*Z_2(\pi)}^{\sigma}$ is essential in $(Z \otimes B(H_{\pi}))^{\alpha\otimes\mathrm{ad}\,\pi}$. By Lemma 3.4(ii) and (iii), we have

$$\overline{Z_2(\pi)^*Z_2(\pi)}^{\sigma} = \overline{X^{\alpha}Y_2(\pi)^*Y_2(\pi)X^{\alpha}}^{\sigma}.$$

and

$$(Z \otimes B(H_{\pi}))^{\alpha\otimes\mathrm{ad}\,\pi} = \overline{X^{\alpha}(Y \otimes B(H_{\pi}))^{\alpha\otimes\mathrm{ad}\,\pi}X^{\alpha}}^{\sigma}.$$

Let $a \in (Y \otimes B(H_{\pi}))^{\alpha\otimes\mathrm{ad}\,\pi}$ be such that

$$\overline{aY_2(\pi)^*Y_2(\pi)}^{\sigma} = \{0\}.$$

Then, by Lemma 3.3(iii) $Y = \overline{Y^{\alpha}Y}^{\sigma} = \overline{Y^{\alpha}X^{\alpha}Y^{\alpha}Y}^{\sigma}$ and, by Lemma 3.3(v), $Y_2(\pi)^* = \overline{Y^{\alpha}Y_2(\pi)^*}^{\sigma}$. Then, it follows that

$$\overline{aY_2(\pi)^*Y_2(\pi)}^{\sigma} = \overline{aY^{\alpha}Y_2(\pi)^*Y_2(\pi)}^{\sigma} = \overline{aY^{\alpha}X^{\alpha}Y^{\alpha}Y_2(\pi)^*Y_2(\pi)}^{\sigma} = \{0\}.$$

Therefore

$$\overline{Y^{\alpha}aY^{\alpha}X^{\alpha}Y^{\alpha}Y_2(\pi)^*Y_2(\pi)X^{\alpha}}^{\sigma} = \{0\}.$$

So, since $\overline{Z_2(\pi)^*Z_2(\pi)}^{\sigma} = \overline{X^{\alpha}Y_2(\pi)^*Y_2(\pi)X^{\alpha}}^{\sigma}$ is essential in $\overline{X^{\alpha}(Y \otimes B(H_{\pi}))^{\alpha\otimes\mathrm{ad}\,\pi}X^{\alpha}}^{\sigma}$, we have $Y^{\alpha}aY^{\alpha} = 0$ and therefore, by Lemma 3.3(ii) applied to $B = Y$, it follows that $a = 0$.

LEMMA 4.2. *Let $(X, G, \alpha)$ be an $\mathscr{F}$-dynamical system with G compact. Then, if $X^{\alpha}$ is $\mathscr{F}$-prime, it follows that $(X \otimes B(H_{\pi}))^{\alpha\otimes\mathrm{ad}\,\pi}$ is $\mathscr{F}$-prime for every $\pi \in \mathrm{sp}_{\mathscr{F}}(\alpha)$.*

PROOF. Since $X^{\alpha}$ is $\mathscr{F}$-prime, it follows that $X^{\alpha} \otimes B(H_{\pi})$ is $\mathscr{F}$-prime for every $\pi \in \widehat{G}$. Since $X_2(\pi)X_2(\pi)^*$ is a non-zero ideal of $X^{\alpha} \otimes B(H_{\pi})$, it follows that $X_2(\pi)X_2(\pi)^*$ is an essential ideal. To prove that $(X \otimes B(H_{\pi}))^{\alpha\otimes\mathrm{ad}\,\pi}$ is

$\mathscr{F}$-prime, let $I \subset (X \otimes B(H_\pi))^{\alpha\otimes \mathrm{ad}\,\pi}$ be a non-zero ideal. Then, as in the proof of Lemma 3.6, consider the following ideal of $X^\alpha \otimes B(H_\pi)$:

$$\begin{aligned} J &= \overline{\mathrm{lin}}^\sigma\{yy^* : y \in X_2(\pi)I\} \\ &= \overline{X_2(\pi)IX_2(\pi)^*}^\sigma. \end{aligned}$$

Since $X^\alpha \otimes B(H_\pi)$ is $\mathscr{F}$-prime, it follows that $J$ is essential in $X^\alpha \otimes B(H_\pi)$. Therefore, if $a \in (X \otimes B(H_\pi))^{\alpha\otimes \mathrm{ad}\,\pi}$ and $aI = \{0\}$, we have

$$aX_2(\pi)^*JX_2(\pi) \subset \overline{aX_2(\pi)^*X_2(\pi)IX_2(\pi)^*X_2(\pi)}^\sigma \subset \overline{aI}^\sigma = \{0\}.$$

So

$$(X_2(\pi)aX_2(\pi)^*J)X_2(\pi)X_2(\pi)^* = \{0\}.$$

Thus, since $X_2(\pi)X_2(\pi)^*$ is essential in $X^\alpha \otimes B(H_\pi)$, it follows that $X_2(\pi)aX_2(\pi)^*J = \{0\}$. Since $X_2(\pi)aX_2(\pi)^* \subset X^\alpha \otimes B(H_\pi)$, $J$ is essential in $X^\alpha \otimes B(H_\pi)$ and $\pi \in \mathrm{sp}_{\mathscr{F}}(\alpha)$, it follows that $a = 0$.

The next result extends [2, Théorème 2.4.1], and [13, Theorem 8.10.4] to the case of $\mathscr{F}$-dynamical systems with compact non-abelian groups.

THEOREM 4.3. *Let $(X, G, \alpha)$ be an $\mathscr{F}$-dynamical system with G compact. The following conditions are equivalent:*

(i) $(X \otimes B(H_\pi))^{\alpha\otimes \mathrm{ad}\,\pi}$ *is $\mathscr{F}$-prime for all* $\pi \in \mathrm{sp}(\alpha)$*;*

(ii) *$X$ is $\alpha$-prime and* $\mathrm{sp}(\alpha) = \Gamma_{\mathscr{F}}(\alpha)$.

PROOF. (i) $\Rightarrow$ (ii) Suppose that $(X \otimes B(H_\pi))^{\alpha\otimes \mathrm{ad}\,\pi}$ is $\mathscr{F}$-prime for all $\pi \in \mathrm{sp}(\alpha)$. Then, it follows immediately from (i) and the definitions that $\mathrm{sp}(\alpha) = \mathrm{sp}_{\mathscr{F}}(\alpha)$. Let $\pi \in \mathrm{sp}(\alpha)$ be arbitrary. We will use Theorem 4.1 to show that $\pi \in \Gamma_{\mathscr{F}}(\alpha)$. Indeed, let $J$ be a non-trivial ideal of $X^\alpha$ and $Z = \overline{JXJ}^\sigma \in H^\alpha_\sigma(X)$. We will show that $\pi \in \mathrm{sp}(\alpha|_Z)$, that is $Z_2(\pi)^*Z_2(\pi)$ is essential in $(Z \otimes B(H_\pi))^{\alpha\otimes \mathrm{ad}\,\pi}$. Notice that

$$Z_2(\pi) = JX_2(\pi)J,$$

so

$$Z_2(\pi)^*Z_2(\pi) = JX_2(\pi)^*JX_2(\pi)J.$$

Since, in particular, $X^\alpha$ is prime, and $J$ is an essential ideal of $X^\alpha$, we have $Z_2(\pi) \neq \{0\}$. Indeed as observed after 2.6, $X_2(\pi)X_2(\pi)^*$ is an ideal of $X^\alpha \otimes B(H_\pi)$, so, as $X^\alpha$ is $\mathscr{F}$-prime, it follows that $X^\alpha \otimes B(H_\pi)$ is a prime C*-algebra and therefore $JX_2(\pi)X_2(\pi)^* \neq \{0\}$, hence $JX_2(\pi) \neq \{0\}$ and $X_2(\pi)^*J \neq \{0\}$. So, $X_2(\pi)^*JX_2(\pi) \neq \{0\}$. Using the hypothesis that $(X \otimes B(H_\pi))^{\alpha\otimes \mathrm{ad}\,\pi}$ is $\mathscr{F}$-prime and the fact that $J$ is a non-trivial ideal of $X^\alpha$, it follows that

$X_2(\pi)^*JX_2(\pi)J \neq \{0\}$, so, $Z_2(\pi) \neq \{0\}$. As noticed above, $X_2(\pi)^*JX_2(\pi)$ is a non-trivial ideal of $(X \otimes B(H_\pi))^{\alpha\otimes\mathrm{ad}\,\pi}$. If $a \in (Z \otimes B(H_\pi))^{\alpha\otimes\mathrm{ad}\,\pi}$, $a \geq 0$, is such that $aZ_2(\pi)^*Z_2(\pi) = \{0\}$, then

$$aJX_2(\pi)^*JX_2(\pi)J = \{0\};$$

which implies

$$JaJX_2(\pi)^*JX_2(\pi)JaJ = \{0\}.$$

Hence

$$JaJX_2(\pi)^*JX_2(\pi) = \{0\}.$$

Since, as noticed above, $X_2(\pi)^*JX_2(\pi)$ is non-trivial and $(X \otimes B(H_\pi))^{\alpha\otimes\mathrm{ad}\,\pi}$ is $\mathscr{F}$-prime, it follows that

$$JaJ = \{0\},$$

so $Ja = \{0\}$. Hence $J\,\mathrm{tr}(a) = \{0\}$. Since $X^\alpha$ is $\mathscr{F}$-prime, we deduce that $\mathrm{tr}(a) = 0$, so $a = 0$ because $a$ was assumed to be non-negative. Therefore, $\pi \in \Gamma_{\mathscr{F}}(\alpha)$, so $\mathrm{sp}(\alpha) = \Gamma_{\mathscr{F}}(\alpha)$.

It remains to prove that $X$ is $\alpha$-prime. Let $I \subset X$ be an $\alpha$-invariant non-trivial ideal and $x \in X$, $x \geq 0$, be such that $xI = \{0\}$. Then, in particular, $xI^\alpha = \{0\}$, so $P_\alpha(x)I^\alpha = \{0\}$. Since $X^\alpha$ is $\mathscr{F}$-prime and $I^\alpha$ is a non-trivial ideal of $X^\alpha$, we have $P_\alpha(x) = 0$. So, since $P_\alpha(\pi)$ is faithful, it follows that $x = 0$.

(ii) $\Rightarrow$ (i) Suppose that $X$ is $\alpha$-prime and $\mathrm{sp}(\alpha) = \Gamma_{\mathscr{F}}(\alpha)$. We will prove first that $X^\alpha$ is $\mathscr{F}$-prime. Let $J \subset X^\alpha$ be a non-zero ideal and $a \in X^\alpha$, $a \geq 0$, $a \neq 0$, such that $Ja = \{0\}$. Since $X$ is $\alpha$-prime, and $XJX$ is a non-zero $\alpha$-invariant ideal of $X$, it follows that $XJXa \neq \{0\}$, so $JXa \neq \{0\}$. Therefore, since by Lemma 2.4

$$X = \overline{\sum_{\pi\in\mathrm{sp}(\alpha)} X_1(\pi)}^{\sigma},$$

there exists $\pi \in \mathrm{sp}(\alpha)$ such that

$$JX_1(\pi)a \neq \{0\}. \tag{4.2}$$

Write $Z = \overline{aXa}^{\sigma} \in H_\sigma^\alpha(X)$. Then, since $\pi \in \mathrm{sp}(\alpha) = \Gamma_{\mathscr{F}}(\alpha)$, $Z_2(\pi)^*Z_2(\pi)$ is essential in $(Z \otimes B(H_\pi))^{\alpha\otimes\mathrm{ad}\,\pi}$. But

$$\overline{Z_2(\pi)^*Z_2(\pi)}^{\sigma} = \overline{aX_2(\pi)^*a^2X_2(\pi)a}^{\sigma}$$

and

$$(Z \otimes B(H_\pi))^{\alpha\otimes\mathrm{ad}\,\pi} = \overline{a((X \otimes B(H_\pi))^{\alpha\otimes\mathrm{ad}\,\pi})a}^{\sigma}.$$

Taking into account that $X_2(\pi)a^2X_2(\pi)^* \subset X^\alpha \otimes B(H_\pi)$, we immediately get that

$$JX_2(\pi)a^2X_2(\pi)^* \subset J \otimes B(H_\pi).$$

Hence, since $Ja = \{0\}$, it follows that

$$\overline{(aX_2(\pi)^*J)X_2(\pi)a^2X_2(\pi)^*(a^2X_2(\pi)a)}^\sigma = \{0\}.$$

Therefore

$$\overline{(aX_2(\pi)^*JX_2(\pi)a)(aX_2(\pi)^*a^2X_2(\pi)a)}^\sigma = \{0\}.$$

It follows that

$$(aX_2(\pi)^*JX_2(\pi)a)(Z_2(\pi)^*Z_2(\pi)) = \{0\}.$$

Since $Z_2(\pi)^*Z_2(\pi)$ is essential in $(Z \otimes B(H_\pi))^{\alpha\otimes \mathrm{ad}\,\pi}$ and obviously $aX_2(\pi)^*JX_2(\pi)^*a \subset (Z \otimes B(H_\pi))^{\alpha\otimes \mathrm{ad}\,\pi}$, it follows that $JX_2(\pi)a = \{0\}$ and hence $JX_1(\pi)a = \{0\}$, but this contradicts (4.2), so $X^\alpha$ is $\mathscr{F}$-prime. From Lemma 4.2, it follows that $(X \otimes B(H_\pi))^{\alpha\otimes \mathrm{ad}\,\pi}$ is $\mathscr{F}$-prime for all $\pi \in \Gamma_{\mathscr{F}}(\alpha) = \mathrm{sp}_{\mathscr{F}}(\alpha) = \mathrm{sp}(\alpha)$ and we are done.

ACKNOWLEDGEMENTS. I am very grateful to the referee who made several valuable comments that helped improve the presentation of this paper.

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