SPREADING BASIC SEQUENCES 
AND SUBSPACES OF 
JAMES' QUASI-REFLEXIVE SPACE 

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Abstract.

We prove that every sequence from J having no nonzero weak cluster point has a subsequence equivalent to either the unit vector basis of $l_2$ or to a spreading basis for J. This implies that J embeds isomorphically in each of its non-reflexive subspaces, and that the only spreading models of J with basic fundamental sequence are J and $l_2$.

1. Introduction.

We discuss the existence and number of spreading basic sequences in James' quasi-reflexive Banach space J, and make applications of this study to spreading models of J and subspaces of J.

In section 2 we show that in one sense, J has many spreading basic sequences, yet only two, up to equivalence. Namely, any seminormalized sequence with no weak cluster point has a subsequence which is a spreading basic sequence. Moreover, the subsequence can be chosen to have complemented span and to be equivalent either to the unit vector basis of $l_2$ or to a certain basis for J. This latter result implies that J has, up to equivalence, precisely two spreading basic sequences. Since J is primary [4], it possesses exactly two spreading basic sequences in an essential manner, in that it is not the direct sum of two spaces each having a unique spreading basic sequence.

In section 3 we use the results of section 2 to study subspaces of J, and show that every non-reflexive subspace of J contains a complemented isomorph of J. This extends results of Casazza [4].

In section 4 we show that J has precisely two spreading models, itself and $l_2$. Here we assume that the fundamental sequence defined in [2] is a Schauder basis.

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We use the standard notation of Banach space theory. If \((z_n) \subset Z\), we denote the closed linear span of \((z_n)\) by \([z_n]\), and say \((z_n)\) is seminormalized if there exists a constant \(M\) such that \(M^{-1} \leq \|z_n\| \leq M\) for all \(n\).

Recall that sequences \((y_n)\) and \((z_n)\) in Banach spaces \(Y\) and \(Z\) are equivalent if there exists a constant \(K\) such that for any scalar sequence \((a_n)\),

\[
K^{-1}\|\sum a_n y_n\| \leq \|\sum a_n z_n\| \leq K\|\sum a_n y_n\|.
\]

A basic sequence \((y_n)\) in a Banach space \(Y\) is said to be spreading if \((y_n)\) is equivalent to each of its subsequences. If, in addition, \((y_n)\) is unconditional, it is said to be subsymmetric.

The notion of spreading model is due to Brunel and Sucheston [3]. They showed that if \((y_n)\) is a bounded sequence from a Banach space \(Y\), then there exists a subsequence \((y'_n)\) such that for every scalar sequence \((a_i)_{i=1}^n\), and every choice of integers \(k_1 < \ldots < k_n\), the limit

\[
(1) \quad L((a_i)) = \lim_{k_i \to \infty} \left\| \sum_{i=1}^n a_i y'_{k_i} \right\|
\]

exists. In the event that \((y_n)\) has no Cauchy subsequences, formula (1) defines a norm \(|\cdot|\) on the space \(S\) of all finite real sequences by

\[
(2) \quad \left| \sum_{i=1}^n a_i f_i \right| = L((a_i)_{i=1}^n).
\]

The sequence of unit vectors \((f_i)\) is clearly spreading. The completion of \(S\) under this norm is called the spreading model of \(Y\) with fundamental sequence \((f_n)\) based on \((y'_n)\).

James' space \(J\) [8, 9, 10], is the Banach space of all null sequences of scalars for which the squared-variation norm

\[
(3) \quad \left\| \sum_{i=1}^\infty a_i e_i \right\| = \sup_{p_0 < \ldots < p_n} \left[ \sum_{i=1}^n |a_{p_i} - a_{p_{i-1}}|^2 \right]^{\frac{1}{2}}
\]

is finite, and the second conjugate \(J^{**}\) is the space of all sequences for which the squared-variation norm is finite [7]. Notice that \(\|\sum a_i e_i\| < \infty\) implies \(\lim_{i \to \infty} a_i\) exists. We shall reserve the notation \((e_i)\) for the unit vector basis in James' space, \((e'_i)\) for the biorthogonal sequence, and \(P_n\) for the natural projections associated with \((e_i)\). We will regard \(e'_i\) and \(P_n\) as defined on \(J\) and also on \(J^{**}\).

The sequence \((x_{n})_{n=1}^\infty \subset J\) defined by \(x_n = \sum_{i=1}^n e_i\) is known to be a basis for \(J\), and the norm may be computed as

\[
(4) \quad \left\|\sum b_n x_n\right\| = \sup_{p_0 < \ldots < p_n} \left[ \sum_{i=1}^n \left| \sum_{j=p_{i-1}}^{p_i-1} b_j \right|^2 \right]^{\frac{1}{2}}.
\]
From either (3) or (4) it follows that \((x_n)\) is a spreading sequence. We shall reserve the notation \((x_n)\) for this spreading basis of \(J\).

The following lemma is obtained easily from (3) and (4). Proofs may be found in [4], [5].

**Lemma 1.1.** a) For any scalar sequence \((b_n)\),

\[
\|\sum b_n x_n\| \geq \left(\sum |b_n|^2\right)^{\frac{1}{4}}.
\]

(b) If \(w_n = \sum_{q_n}^{r_n} a_i e_i\), \(\|w_n\| = 1\) for all \(n\), and \(r_{n-1} + 1 < q_n \leq r_n\) for all \(n\), then for any scalar sequence \((b_n)\),

\[
\left[\sum b_n^2\right]^\frac{1}{4} \leq \|\sum b_n w_n\| \leq \sqrt{2} \left[\sum b_n^2\right]^\frac{1}{4}.
\]

2. **Spreading basic sequences in \(J\).**

In [5], Casazza, Lin, and Lohman proved that every subsymmetric basic sequence in \(J\) is equivalent to the unit vector basis of \(l_2\). One consequence of the main result of this section is that any spreading basic sequence in \(J\) is equivalent either to the unit vectors in \(l_2\) or to the spreading basis \((x_n)\) of \(J\).

The main result is

**Theorem 2.1.** Let \((z_k)\) be a seminormalized sequence from \(J\) having no non-zero weak cluster point. Then

a. if \((z_k)\) has a weakly null subsequence, then there is a subsequence \((z_{n_k})\) equivalent to the unit vectors in \(l_2\) with \([(z_{n_k})]\) complemented in \(J\) or

b. if \((z_n)\) has no weak cluster point, then there is a subsequence \((z_{n_k})\) equivalent to \((x_j)\) with \([(z_{n_k})]\) complemented in \(J\).

We present the proof in a sequence of propositions, and will use the following standard perturbation argument [11].

**Proposition 2.2.** Let \((y_n)\) be a basic sequence in a Banach space \(Y\) having basis constant \(M\), and assume \([(y_n)]\) is complemented by a projection \(P\). If \((z_n) \subset Y\) satisfies

\[
\sum_{n=1}^{\infty} \|y_n - z_n\| < \frac{1}{8M\|P\|},
\]

then \((z_n)\) is a basic sequence equivalent to \((y_n)\) and \([(z_n)]\) is complemented in \(Y\).

In the case that \((z_n)\) has a weakly null subsequence, the result follows from the argument of [5].

**Proposition 2.3.** If \((z_n) \subset J\) is a seminormalized sequence having a weakly null
subsequence, then there is a subsequence \((z_{n_k})\) having span complemented in \(J\) and equivalent to the unit vector basis of \(l_2\).

**Proof.** Assume \(\|z_n\| = 1\) for all \(n\) and \(z_n \to 0\) weakly. Let \((\varepsilon_k)\) be a sequence of positive real numbers such that \(\sum \varepsilon_k < 2^{-7}\) and choose an increasing sequence of integers \((n_k)\) such that with \(z'_{n_k} = (P_{n_k} - P_{n_k-1})z_{n_k}\), we have

\[
\frac{1}{2} \leq \|z'_{n_k}\| \leq 2
\]

and

\[
\|z_{n_k} - z'_{n_k}\| < \varepsilon_k.
\]

Then by Lemma 1.1b, \((z'_{n_k})\) is \(4/\sqrt{2}\)-equivalent to the unit vectors in \(l_2\), and is hence a spreading basic sequence with basis constant at most \(4/\sqrt{2}\). By Theorem 10 of [5], \([z'_{n_k}]\) is complemented in \(J\) by a projection \(P\) of norm at most \(2/\sqrt{2}\). Since

\[
\sum \|z'_{n_k} - z_{n_k}\| < \sum \varepsilon_k < 2^{-7} < \frac{1}{8M\|P\|},
\]

it follows from Proposition 2.2 that \((z_{n_k})\) is equivalent to the unit vectors in \(l_2\) and has complemented span.

**Remark 2.4.** Suppose now that \((z_n)\) has no weakly null subsequence. By regarding \((z_n)\) as a sequence in \(J^{**}\) and passing to a subsequence, \((z_n)\) has a weak* limit \(z\) in \(J^{**}\). Since \((z_n)\subset J\) has no weak limit, \(z \notin J\). The proof of Theorem 2.1b. is based on regarding the sequence \((z_n)\) as arising from its weak* limit in \(J^{**}\).

We begin with a simple yet important case.

**Proposition 2.5.** Suppose \(y \in J^{**} - J\) and \((n_k)\subset \mathbb{N}\) is an increasing sequence. Let \(y_k = P_{n_k}y\), and suppose \((n_k)\) is such that \(y_k \neq y_{k+1}\) for any \(k\). Then

a. \((y_k)\) is a spreading basic sequence equivalent to \((x_k)\). The constant of this equivalence has bound depending on \(y\) but not on \(n_k\).

b. \([y_k]\) is complemented in \(J\) by a projection \(P\), and \(\|P\| < B\), where \(B\) depends on \(y\) but not on \(n_k\).

**Proof.** The proof is simplest in the case where \(e_j^*(y) \neq 0\) for all \(j\), and we present this case first. Let \(T: J \to J\) be the operator defined by

\[
T(\sum a_i e_i) = \sum e_j^*(y) a_i e_i.
\]

That is, \(T\) is coordinatewise multiplication by \(y \in J^{**}\). Since \(J^{**}\) is the multiplier algebra of \(J\) \([1]\), \(T\) is bounded.
Moreover, $Tx_n = y_k$, so that for any scalar sequence $(a_n)$,
\[
\| \sum a_n y_n \| \leq \| T \| \| \sum a_n x_n \|
= \| T \| \| \sum a_n x_k \|
\leq 2 \| y \| \| \sum a_n x_k \|.
\]
Since $y \in J^{**} - J$ and $e_j^*(y) \neq 0$ for any $j$, we have that
\[
\lim_j e_j^*(y) \neq 0 \quad \text{and} \quad \max_j [\| e_j^*(y) \|^{-1}] < \infty.
\]
It follows that the sequence
\[
y' = (e_j^*(y)^{-1}) \in J^{**} \quad \text{and} \quad \| y' \| \leq \max_j [\| e_j^*(y) \|^{-1}]^2 \| y \|.
\]
Thus multiplication by $y'$, which is $T^{-1}$, is a bounded operator on $J$, and hence
\[
\| \sum a_n x_n \| \leq \| T^{-1} \| \| \sum a_n y_n \|
\]
for all scalar sequences $(a_n)$. Thus $(x_k)$ is equivalent to $(y_k)$ and the constant of the equivalence is
\[
\| T \| \| T^{-1} \| \leq 4 \| y \|^2 \max_j (|e_j^*(y)|^{-2}).
\]
Casazza [4] has proved that for any sequence $(n_k)$, $(x_{n_k})$ is complemented by a contractive projection $P$. But then $TPT^{-1}$ is a projection from $J$ onto $[(y_k)]$ with norm independent of $(n_k)$.

In the case that there do exist $j$ with $e_j^*(y) = 0$, since $y \in J^{**} - J$ it follows that \{ $j : e_j^*(y) = 0$ \} is finite, say \{ $j_1 < j_2 < \ldots < j_i$ \}. The permutation $\tau$ of $\mathbb{N}$ defined by $\tau(j) = i$ if $j = j_i$, $\tau(j) = j$ for $j > j_i$, and the requirement that $\tau$ preserves order in the remaining cases induces an automorphism $U$ of $J$ [1]. Denoting by $S$ the operator on $J$ defined by $Se_n = e_{n+p}$, the sequence $(y_k)$ is equivalent to the sequence $Uy_k$ which has no nonzero coordinates when regarded as a sequence in the space $SJ$, which is isometric to $J$. By the preceding arguments, $(Uy_k)$ is equivalent to $(x_k)$ and complemented in $SJ$. It follows that $(y_k)$ is equivalent to $(x_k)$ and is complemented in $J$. The constant of the equivalence and norm of the projection now depend on $U$, but $U$ is determined by $y$ and is independent of $(n_k)$.

We shall use the following lemma.

**Lemma 2.6.** Let $(q_i)$, $(r_i)$ be sequences of natural numbers such that for all $i$, $q_i < r_i < r_i + 1 < q_{i+1}$, and define a projection $Q$ on $J^{**}$ by

\[
e_j^*(Qy) = \begin{cases} e_{r_i+1}^*(y) & q_i \leq j \leq r_i \\ e_j^*(y) & \text{otherwise}. \end{cases}
\]

Then $\| Q \| = 1$ and $Qy \in J$ if and only if $y \in J$. 
Proof. Any estimate by (3) of \( ||Qy|| \) is also an estimate of \( ||y|| \). Hence \( ||Qy|| \leq ||y|| \) for any \( y \in J \). For \( y \in J^{**} \), \( ||y|| = \lim_k ||P_k y||_J \), so \( Q \) has norm one on \( J^{**} \) also.

Since \( \lim_j e_j^*(Qy) = \lim_j e_j^*(y) \), it follows that \( Qy \in J \) if and only if \( y \in J \). The proof of Theorem 2.1 is now completed by

**Proposition 2.7.** Let \( (z_n) \) be a seminormalized sequence from \( J \) having no weak cluster point. Then \( (z_n) \) has a subsequence \( (z_{n_k}) \) equivalent to \( (x_k) \) and such that \( [(z_{n_k})] \) is complemented in \( J \).

Proof. Assume \( M^{-1} \leq ||z_n|| \leq M \) for all \( n \). Regarding \( (z_n) \) as a sequence from \( J^{**} \), and passing to a subsequence we may assume, by Remark 2.4, that \( (z_n) \) has a weak* limit \( y \in J^{**} - J \). Let \( (\varepsilon_k) \) be a sequence of reals decreasing to zero, and choose a subsequence, also denoted by \( (z_k) \), and an increasing sequence of integers \( (n_k) \) such that

\[
||P_{n_k}z_k - z_k|| < \varepsilon_k
\]

and

\[
\frac{1}{2}M^{-1} \leq ||P_{n_k}z_k|| \leq M .
\]

Let \( z'_k = P_{n_k}z_k \). The sequence \( (z'_k) \subset J^{**} \) converges in the weak* topology to \( y \), so we may select an increasing sequence of natural numbers \( (m_k) \) such that

\[
||P_{n_{m-1} + 2}(z'_m - y)|| < \varepsilon_k .
\]

Define

\[
z''_k = (P_{n_{m-1} + 2})y + (I - P_{n_{m-1} + 2})(z'_m)
\]

\[
= y_k + w_k .
\]

We may write

\[
w_k = \sum_{j=q_k}^{r_k} b_j e_j , \quad \text{with}
\]

\[
q_k = n_{m_{k-1}} + 3, \quad \text{and}
\]

\[
r_k = n_{m_k} ,
\]

so that the sequences \( (q_k) \) and \( (r_k) \) satisfy the hypotheses of Lemmas 1.1.b and 2.6. Since \( ||w_k|| \leq 2M \) for all \( k \), it follows from Lemma 1.1 that

\[
||\sum a_kw_k|| \leq 2M[\sum |a_k|^2]^\dagger
\]

for all scalar sequences \((a_k)\). Hence for any sequence \((a_k)\)
\[ \| \sum a_k z_k' \| \leq \| \sum a_k y_k \| + \| \sum a_k w_k \| \]
\[ \leq 2 \| y \| \| \sum a_k x_k \| + 2M[\sum |a_k|^2]^\frac{1}{2} \]
\[ \leq 2(\| y \| + M)\| \sum a_k x_k \| , \]

by Proposition 2.5 and Lemma 1.1.

Let \( Q \) be the projection defined in Lemma 2.6 using sequences \((r_i)\) and \((q_i)\) defined in (5). Let \( y' = Qy \in J^{**} - J \), and let \( y'_k = (P_{n_{m_k}+2})y \). Since
\[ e_{r_{i+1}}^*(w_k) = 0 \]
for all \( i \) and \( k \), it follows that
\[ Q z_k'' = (P_{n_{m_k}+2})y' = y'_k . \]

Thus for any scalar sequence \((a_k)\),
\[ \| \sum a_k z_k'' \| \geq \| Q \sum a_k z_k'' \| \]
\[ = \| \sum a_k y'_k \| \]
\[ \geq A\| \sum a_k x_k \| \]

for some \( A > 0 \) which depends on \( y' \). It follows from (6) and (8) that \((z_k'')\) is equivalent to \((x_k)\). The basis constant of \((z_k'')\) is the constant \( A_1 \) of this equivalence. We also have that for some constant \( A_2 \), \((z_k'')\) is \( A_2 \)-equivalent to \((y'_k)\), so that from (7) we see that \( Q|_{[[z_k'']]} \) is an isomorphism from \([[(z_k'')]] \) onto \([[(y'_k)]] \). By Proposition 2.5 there is a constant \( A_3 \) such that any subsequence of \((y'_k)\) has span complemented by a projection \( P \) with \( \| P \| \leq A_3 \). Thus any subsequence of \((z_k'')\) is complemented by a projection \( Q|_{[[z_k'']]}^{-1}PQ \) of norm smaller than \( A_2A_3 \). Now
\[ \| z_k'' - z_{m_k} \| \leq \| z_k'' - z_{m_k}' \| + \| z_{m_k}' - z_{m_k} \| \]
\[ \leq \| P_{n_{m_k}+2}(z_{m_k}' - y) \| + \| z_{m_k}' - z_{m_k} \| \]
\[ < \varepsilon_k + \varepsilon_{m_k} < 2\varepsilon_k , \]
so that if \( k_l \) is chosen so that
\[ \sum \varepsilon_{k_l} < \frac{1}{16A_1A_2A_3} , \]
it follows from Proposition 2.2 that \((z_{m_k})\) is equivalent to \((x_l)\) and has complemented span. This completes the proof of Proposition 2.7 and Theorem 2.1.

Theorem 2.1 may be used to extend a result of Casazza, Lin, and Lohman [5].
COROLLARY 2.8. If \((z_n)\) is a spreading basic sequence in \(J\), then
(a) if \(z_n\) converges weakly to zero, then \((z_n)\) is equivalent to the unit vector basis of \(l_2\).
(b) If \(z_n \to 0\) weakly, then \((z_n)\) is equivalent to \((x_n)\).

PROOF. Since \((z_n)\) is a basic sequence, \((z_n)\) has no non-zero weak cluster point. Thus, by Theorem 2.1, if \((z_n)\) converges weakly to zero, \((z_n)\) has a subsequence equivalent to the unit vector basis of \(l_2\). Since \((z_n)\) is spreading, \((z_n)\) is itself equivalent to the \(l_2\) basis. In the case that \((z_n)\) is not weakly null, Theorem 2.1 implies that \((z_n)\) has a subsequence equivalent to \((x_n)\), from which it follows that \((z_n)\) is equivalent to \((x_n)\).

3. Subspaces of \(J\).

In this section we use Theorem 2.1 to obtain results concerning subspaces of \(J\).

THEOREM 3.1. If \(X \subset J\), then \(X\) is isomorphic to \(Y \oplus R\) where \(R\) is reflexive, \(Y\) is complemented in \(J\), and \(Y\) is either trivial or isomorphic to \(J\).

PROOF. If \(X\) is reflexive, we choose \(Y = \{0\}\) and \(R = X\).

Otherwise, there exists a sequence \((z_n) \subset X\), \(\|z_n\| = 1\) for all \(n\) such that \((z_n)\) has no weak cluster point in \(J\). By Theorem 2.1b, there is a subsequence \((z_{n_k})\) equivalent to \((x_k)\) with span complemented in \(J\) by a projection \(P\). We take \(Y = [(z_{n_k})]\), and \(R = (I - P)X\). Then \(Y\) is isomorphic to \(J\), and hence [5], \((I - P)J\) is reflexive. Since \(R \subset (I - P)J\), \(R\) is also reflexive.

Theorem 3.1 specializes to

COROLLARY 3.2. If \(X \subset J\) is non-reflexive, then there exists a subspace \(Y \subset X\) such that \(Y\) is isomorphic to \(J\) and \(Y\) is complemented in \(J\).

4. Spreading Models of \(J\).

In this section we show that if a Banach space \(X\) is a spreading model of \(J\), then either \(X\) is isomorphic to \(l_2\) or \(X\) is isomorphic to \(J\). In fact, if a basis \((f_n)\) is a fundamental sequence based on \((y_n) \subset J\) then either \((f_n)\) is equivalent to the unit vectors in \(l_2\) or \((f_n)\) is equivalent to the spreading basis \((x_n)\) for \(J\). We assume here that the fundamental sequence is Schauder basis.

We shall use a result of Guerre and Laprestè [7],
THEOREM 4.1. Let \((f_n)\) be the fundamental sequence of a spreading model of a
Banach space \(X\), based on a sequence \((y_n) \subset X\). Then
a. If \(f_n \to 0\) weakly, then \(y_n \to 0\) weakly.
b. If \((f_n)\) is weakly Cauchy but not weakly convergent, then \((y_n)\) is weakly
Cauchy but not weakly convergent.

We have then

THEOREM 4.2. Let a basis \((f_n)\) be the fundamental sequence of a spreading
model of \(J\), based on a sequence \((y_n) \subset J\). Then either \((f_n)\) is equivalent to the unit
vectors in \(l_2\) or \((f_n)\) is equivalent to the basis \((x_n)\) for \(J\).

PROOF. We consider three cases.

1. If \((f_n)\) converges weakly to zero, then the same is true of \((y_n)\) by Theorem
4.1a. By Theorem 2.1 we may pass to a subsequence \((y_{n_k})\) equivalent to the unit
vector basis in \(l_2\). But this implies that \((f_n)\) is equivalent to the unit vector basis
in \(l_2\).

2. If \((f_n)\) is weakly Cauchy, yet not weakly convergent, then the same is true
of \((y_n)\) by Theorem 4.1b. By Theorem 2.1 there is a subsequence \((y_{n_k})\) equivalent
to the spreading basis \((x_n)\) for \(J\). But this implies that \((f_n)\) is itself equivalent to
\((x_n)\).

3. Since \((f_n)\) is assumed to be a basis, the only remaining case is that when
\((f_n)\) is not weakly Cauchy. Since \((f_n)\) is spreading, it follows from a result of
Rosenthal [12] that \((f_n)\) is equivalent to the unit vector basis in \(l_1\). Thus it is
sufficient to show that \(l_1\) is not a spreading model of \(J\).

Now Proposition I.2 of [2] asserts that a Banach space \(X\) not containing \(l_1\)
has \(l_1\) as a spreading model if and only if \(X\) fails the Banach–Saks–Rosenthal
(BSR) property. Recall that \(X\) has BSR if every weakly null sequence has a
subsequence with norm convergent Cesaro means. Now \(l_1 \nsubseteq J\), and \(J\) has BSR
since every weakly null sequence in \(J\) has a subsequence equivalent to the unit
vectors in \(l_2\). It follows that \(J\) does not have \(l_1\) as a spreading model.

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