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THE CENTRAL LIMIT THEOREM FOR MARKOV CHAINS WITH THE SIMULTANEOUS TOTAL VARIATION CONVERGENCE OF THE CHAIN

SEPPO NIEMI

1. Introduction.

Let $(X)_{n\geqq 0}$ be an aperiodic positive recurrent Harris chain on a general measurable state space $(E,\mathscr{E})$ with n-step transition probabilities $P^n(x,A)$, $x\in E$, $A\in\mathscr{E}$, and with invariant probability measure π. Let f be a real-valued π-integrable function with $\pi(f)=\int f\,d\pi=0$. Write $S_n=\sum_{i=0}^n f(X_i)$.

Orey's convergence theorem ([4, Ch. I, Theorem 7.1]) states that

$$P^n(x,\cdot)\to\pi(\cdot) \tag{1}$$

in total variation norm, as $n\to\infty$, for all $x\in E$. Cogburn ([1, Cor. 4.2]) has shown that under suitable conditions

$$S_n/\sqrt{n}\to N(0,\sigma^2) \tag{2}$$

in distribution, as $n\to\infty$, for certain $0\leqq\sigma^2<\infty$.

The main aim of this paper is to combine these results to obtain a simultaneous version of these two convergence results. Moreover, we shall give, by using the splitting technique introduced in [2], an elementary proof for the asymptotic normality of S_n.

2. The main result.

We assume that the σ-field $\mathscr{E}$ is countably generated (see Remark 2). We shall denote by $\boldsymbol{P}_\mu$ the probability measure on the sample space $(E^\infty,\mathscr{E}^\infty)$ of the chain corresponding to the initial distribution μ. If $\mu=\varepsilon_x$ is the probability measure assigning unit mass to the point x, we write $\boldsymbol{P}_\mu=\boldsymbol{P}_x$. The corresponding expectation is denoted by $\boldsymbol{E}_\mu$ $(\boldsymbol{E}_x)$. We shall write $\|\cdot\|$ for the total variation of a signed measure on $(E,\mathscr{E})$ and Φ for the standard normal distribution function.

THEOREM. *Assume that* (2) *holds for some* $\sigma>0$. *Then for every real number* t

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$$\|\boldsymbol{P}_\mu\{S_n/\sigma\sqrt{n} \leqq t;\ X_n \in \cdot\} - \Phi(t)\pi(\cdot)\| \to 0, \quad as\ n \to \infty\,. \tag{3}$$

PROOF. Associate to every positive integer n a positive integer $\tilde{n} \leqq n$ such that

$$n - \tilde{n} \to \infty, \qquad \tilde{n}/n \to 1$$

and

$$(S_n - S_{\tilde{n}})/\sigma\sqrt{n} \to 0 \text{ in } \boldsymbol{P}_\mu\text{-probability}, \quad \text{as } n \to \infty\,. \tag{4}$$

This is possible due to the convergence in distribution of $S_n - S_{n-k}$ for any fixed k. Define

$$\Phi_n(t, A) = \boldsymbol{P}_\mu\{S_n/\sigma\sqrt{n} \leqq t\ ;\ X_n \in A\}\,,$$

$$\tilde{\Phi}_n(t, A) = \boldsymbol{P}_\mu\{S_{\tilde{n}}/\sigma\sqrt{n} \leqq t\ ;\ X_n \in A\}\,,$$

$$A_{M,\varepsilon} = \{x \in E \mid \|P^n(x, \cdot) - \pi(\cdot)\| < \varepsilon\ \forall m \geqq M\}\,.$$

Then for $n - \tilde{n} \geqq M$

$$\begin{aligned}
&|\tilde{\Phi}_n(t, A) - \Phi(t)\pi(A)| \\
&= |\boldsymbol{E}_\mu[(\boldsymbol{P}_{X_{\tilde{n}}}(X_{n-\tilde{n}} \in A) - \pi(A));\ S_{\tilde{n}}/\sigma\sqrt{n} \leqq t] \\
&\quad + \pi(A)(\boldsymbol{P}_\mu\{S_{\tilde{n}}/\sigma\sqrt{n} \leqq t\} - \Phi(t))| \\
&\leqq \varepsilon + \boldsymbol{P}_\mu\{X_{\tilde{n}} \in A^c_{M,\varepsilon}\} + |\boldsymbol{P}_\mu\{S_{\tilde{n}}/\sigma\sqrt{n} \leqq t\} - \Phi(t)|\,.
\end{aligned}$$

Hence by (1), (2) and (4)

$$\overline{\lim_{n\to\infty}} \sup_{A \in \mathscr{E}} |\tilde{\Phi}_n(t, A) - \Phi(t)\pi(A)| \leqq \varepsilon + \pi(A^c_{M,\varepsilon})\,, \tag{5}$$

which tends to 0, as first $M \uparrow \infty$ and then $\varepsilon \downarrow 0$. Thus, for every $\varepsilon > 0$,

$$\begin{aligned}
&\overline{\lim_{n\to\infty}} \sup_{A \in \mathscr{E}} |\Phi_n(t, A) - \Phi(t)\pi(A)| \\
&\leqq \varepsilon + \overline{\lim_{n\to\infty}} \sup_{A \in \mathscr{E}} |\Phi_n(t, A) - \tilde{\Phi}_n(t+\varepsilon, A) + \tilde{\Phi}_n(t+\varepsilon, A) - \Phi(t+\varepsilon)\pi(A)| \\
&\leqq \varepsilon + \overline{\lim_{n\to\infty}} \boldsymbol{P}_\mu\{|S_n - S_{\tilde{n}}|/\sigma\sqrt{n} > \varepsilon\} + 0 \\
&= \varepsilon \quad \text{by (4)}\,,
\end{aligned}$$

which proves (3).

REMARKS. 1. Due to the continuity of the function Φ the convergence in (3) is uniform in t.

2. We have assumed that the σ-field $\mathscr{E}$ is countably generated, which assumption is needed in the proof only to guarantee the measurability of the set $A_{M,\varepsilon}$. This assumption can be replaced by a somewhat milder assumption "E has a uniform subset A"; then the possibly non-measurable functions $x \mapsto \|P^m(x,\cdot)-\pi(\cdot)\|$ are bounded above by measurable functions $g_m \leqq 2$ converging to zero π-a.s., see [1, p. 511].

3. A proof for the central limit theorem for Markov chains.

In this section we shall give an elementary proof for (2) and give an expression for the parameter σ^2.

Due to the C-set theorem (see [6, Ch. 6, Lemma 1.1]) there is a function $h\colon E \to [0,1]$ with $\pi(h)>0$, an integer $k \geqq 1$ and a probability measure ν on $(E,\mathscr{E})$ such that

$$P^k(x,A) \geqq h\otimes\nu(x,A) = h(x)\nu(A), \qquad x \in E,\ A \in \mathscr{E}\,. \tag{6}$$

By using the splitting technique (see [2]) we can construct an increasing sequence of random times $0<N_1<N_2<\dots$ for the chain $(X_{nk})_{n\geqq 0}$ such that the law of the process $\{X_{N_ik}, X_{N_ik+1}, X_{N_ik+2},\dots\}$ is $\boldsymbol{P}_\nu$, independently of $\{X_0, X_1,\dots,X_{(N_i-1)k}\}$. Then the sums

$$Y_i = \sum_{n=N_ik}^{N_{i+1}k-1} f(X_n), \qquad i=1,2,\dots,$$

form a sequence of identically distributed random variables such that $\{Y_i\,;\, i \leqq n\}$ and $\{Y_i\,;\, i \geqq n+2\}$ are independent for all n. The mean of Y_i is

$$\begin{aligned} \boldsymbol{E}_\mu Y_i &= \sum_{m=0}^{\infty} \nu(P^k-h\otimes\nu)^m \sum_{n=0}^{k-1} P^n f \\ &= k\pi(h)^{-1}\pi(f) && \text{by [2, Theorem 3]}, \\ &= 0 && \text{by assumption}. \end{aligned}$$

If the variance $\alpha^2=\boldsymbol{E}_\mu Y_i^2$ is finite, then for any integer $l \geqq 2$ the sums

$$Z_m = \sum_{i=(m-1)l+1}^{ml-1} Y_i, \qquad m=1,2,\dots,$$

are independent identically distributed variables with mean 0 and variance

$$\boldsymbol{E}_\mu Z_m^2 = (l-1)\boldsymbol{E}_\mu Y_1^2 + 2(l-2)\boldsymbol{E}_\mu Y_1 Y_2 \overset{\text{def.}}{=} (l-1)\alpha^2 + 2(l-2)\alpha_{12}\,.$$

The central limit theorem for independent identically distributed random variables asserts that

$$\sum_{m=1}^{n} Z_m/\sqrt{n} \to N(0, (l-1)\alpha^2 + 2(l-2)\alpha_{12}) \text{ in distribution,} \quad \text{as } n \to \infty .$$

By a standard ergodic theorem argument

$$\psi(n)/n \overset{\text{def.}}{=} \max\{m \mid N_{ml+1} \leqq n/k\}/n \to \frac{\pi(h)}{kl}$$

almost surely, and hence in probability, as $n \to \infty$. Anscombe's theorem (e.g. [5, Ch. VIII, Theorem 7.2]) now implies that

$$\sum_{m=1}^{\psi(n)} Z_m/\sqrt{n} \to N\left(0, \frac{\pi(h)}{k}\left(\frac{l-1}{l}\alpha^2 + \frac{2(l-2)}{l}\alpha_{12}\right)\right) \tag{7}$$

in distribution, as $n \to \infty$. Similarly

$$\sum_{m=1}^{\psi(n)} Y_{ml}/\sqrt{n} \to N\left(0, \frac{\pi(h)}{k}\frac{\alpha^2}{l}\right) \text{ in distribution,} \quad \text{as } n \to \infty . \tag{8}$$

Choosing l great enough we obtain from (7) and (8) the following proposition.

PROPOSITION. *If* $\alpha^2 = \boldsymbol{E}_\mu Y_i^2$ *is finite, then*

$$S_n/\sqrt{n} \to N\left(0, \frac{\pi(h)}{k}(\alpha^2 + 2\alpha_{12})\right) \text{ in distribution,} \quad \text{as } n \to \infty .$$

In order to derive a more explicit expression for the variance $\sigma^2 = (\alpha^2 + 2\alpha_{12})\pi(h)/k$, we first write

$$U_i = \sum_{n=0}^{k-1} f(X_{ik+n}), \qquad i = 0, 1, \ldots ,$$

$$m^{(2)}(x) = \boldsymbol{E}_x U_0^2 ,$$

$$m(x, y) = \boldsymbol{E}_x(U_0 \mid X_k = y) ,$$

$$G(x, \cdot) = \sum_{i=0}^{\infty} (P^k - h \otimes \nu)^i \sum_{n=0}^{k-1} P^n(x, \cdot) .$$

Then

$$\begin{aligned} \alpha^2 + 2\alpha_{12} &= \boldsymbol{E}_\nu\left[\sum_{i=0}^{N_1-1} U_i^2 + 2\sum_{i=0}^{N_1-2} U_i \sum_{j=i+1}^{N_1-1} U_j\right] \\ &\quad + 2\boldsymbol{E}_\nu\left[\sum_{i=0}^{N_1-2} U_i \sum_{j=N_1}^{N_2-1} U_j + U_{N_1-1}\sum_{j=N_1}^{N_2-1} U_j\right] \\ &= \sum_{i=0}^{\infty} \nu(P^k - h \otimes \nu)^i m^{(2)} \end{aligned} \tag{9}$$

$$+2\sum_{i=0}^{\infty}\iint v(P^k-h\otimes v)^i(dx)(P^k-h\otimes v)(x,dy)m(x,y)Gf(y)$$

$$+0+2\sum_{i=0}^{\infty}\iint v(P^k-h\otimes v)^i(dx)h(x)v(dy)m(x,y)Gf(y)$$

$$=\pi(h)^{-1}\pi(m^{(2)})+2\pi(h)^{-1}\iint \pi(dx)P^k(x,dy)m(x,y)Gf(y)\ ,$$

provided that (9) is finite, when U_i and U_j are replaced by $|U_i|$ and $|U_j|$, respectively. This condition is obviously sufficient for the finiteness of α^2. Hence (9) holds with both sides finite, if $\pi(m^{(2)})$ is finite and the measure $E_\pi\{U_0\ ;\ X_k\in\cdot\}$ is $|f|$-regular (for the definition see [3, Def. 2.1]); especially if $\pi(m^{(2)})$ is finite and the function $|f|$ is special (for the definition see [6, Ch. 6. Def. 4.1]). In these cases

$$\sigma^2 = k^{-1}\pi(m^{(2)})+2k^{-1}\iint \pi(dx)P^k(x,dy)m(x,y)Gf(y)\ .$$

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UNIVERSITY OF HELSINKI
DEPARTMENT OF MATHEMATICS
HALLITUSKATU 15
SF-00100 HELSINKI 10
FINLAND