ON HSIAO’S CONJECTURE ON HECKE GROUPS

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1.
Let \( q \geq 3 \) be a rational integer, and put
\[
\zeta_q = \exp(\pi i/q), \quad \lambda_q = \zeta_q + \zeta_q^{-1} = 2\cos\frac{\pi}{q}.
\]
The Hecke group \( G(\lambda_q) \) is generated by the transformations
\[
U: \tau \mapsto \tau + \lambda_q \quad \text{and} \quad T: \tau \mapsto -\tau^{-1}
\]
of the upper half plane. These groups were introduced by E. Hecke [1, Nr. 33] when he found his famous correspondence between Dirichlet series with functional equation on the one hand and entire modular forms for \( G(\lambda_q) \) on the other hand. Recently Hong-Jen Hsiao [2] studied pairs of Dirichlet series which are related by a functional equation. He proved that these pairs correspond to entire modular forms for the subgroup \( H(\lambda_q) \) of \( G(\lambda_q) \) which is generated by \( U \) and
\[
L = TUT: \tau \mapsto \tau /(-\lambda_q \tau + 1).
\]
The relation \( TU^k = L^k T \) shows that \( H(\lambda_q) \) has index at most 2 in \( G(\lambda_q) \). If \( q \) is odd then the formula
\[
T = (TU)^q T = L(UL)^{q-1}/2 \in H(\lambda_q)
\]
proves \( H(\lambda_q) = G(\lambda_q) \). (This formula can be obtained from [1, p. 613].) Using some formulae from M. Knopp [3] Hsiao proved
\[
[ G(\lambda_q): H(\lambda_q) ] = 2
\]
if \( q = 4 \) or \( q = 6 \), and he conjectures this to be true for all even \( q \geq 4 \). In 2 below I give a simple proof of this conjecture. As in [2] this result yields examples of pairs of Dirichlet series \( \varphi \neq \psi \) which are related by a functional equation; this is shown in 3.
2.

It is convenient to use the matrices

\[
\mathcal{U} = \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix}, \quad \mathcal{T} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathcal{L} = \mathcal{T} \mathcal{U} \mathcal{T} = \begin{pmatrix} -1 & 0 \\ \lambda_q & -1 \end{pmatrix}.
\]

Let \( \mathcal{G}(\lambda_q) \) be the group which is generated by \( \mathcal{U} \) and \( \mathcal{T} \), and let \( \mathcal{H}(\lambda_q) \) be the group which is generated by \( \mathcal{U} \) and \( \mathcal{L} \). Then \( G(\lambda_q) \) and \( H(\lambda_q) \) are obtained from \( \mathcal{G}(\lambda_q) \) and \( \mathcal{H}(\lambda_q) \) by factoring out the center

\[
\left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.
\]

We introduce the set \( S_q \) which consists of all matrices

\[
\pm \begin{pmatrix} 1 + \lambda_q^2 p_1(\lambda_q^2) & \lambda_q p_2(\lambda_q^2) \\ \lambda_q p_3(\lambda_q^2) & 1 + \lambda_q^2 p_4(\lambda_q^2) \end{pmatrix}
\]

with arbitrary polynomials \( p_1, p_2, p_3, p_4 \) in \( \mathbb{Z}[X] \). Obviously, \( \mathcal{U} \) and \( \mathcal{L} \) belong to \( S_q \), and the product of any two matrices in \( S_q \) is again in \( S_q \). Therefore,

\[
\mathcal{H}(\lambda_q) \subset S_q.
\]

We assume that \( q \geq 4 \) is even, and we consider the field \( K_{2q} = \mathbb{Q}(\zeta_q) \) of \((2q)\)-th roots of unity and its maximal real subfield \( K_{2q}^* = K_{2q} \cap \mathbb{R} = \mathbb{Q}(\lambda_q) \). Let us suppose that \( \mathcal{T} \in S_q \). Then \( \lambda_q \cdot p_2(\lambda_q^2) = 1 \) for some \( p_2 \in \mathbb{Z}[X] \), whence \( \lambda_q \in \mathbb{Q}(\lambda_q^2) \), and \( K_{2q}^* = \mathbb{Q}(\lambda_q^2) \). But \( \mathbb{Q}(\lambda_q^2) = \mathbb{Q}(\zeta_q^2 + \zeta_q^{-2}) = K_q^* \) is the maximal real subfield of \( K_q = \mathbb{Q}(\zeta_q^2) \). From

\[
[K_{2q}: K_{2q}^*] = 2, \quad [K_q: K_q^*] = 2, \quad K_q \subset K_{2q}
\]

and \( K_q^* = K_{2q}^* \) we conclude that \( K_q = K_{2q} \). This is impossible since \( q \) is even. This contradiction shows that \( \mathcal{T} \notin S_q \), and hence that \( \mathcal{T} \notin \mathcal{H}(\lambda_q) \). Thus Hsiao’s conjecture (1) is proved for all even \( q \geq 4 \).

We note that the identity and \( T \) represent the cosets of \( H(\lambda_q) \) in \( G(\lambda_q) \). This will be used in the last section.

3.

The groups \( G(\lambda_q) \) and \( H(\lambda_q) \) meet the requirements in [4, chapter VIII]. Therefore ([4, p. 282]), for every even \( k \geq 4 \), the Eisenstein series \( E_k(\tau) \) and \( F_k(\tau) \) for \( G(\lambda_q) \) and \( H(\lambda_q) \), respectively, converge on the upper half plane and define entire modular forms of weight \( k \) for these groups. They are defined as follows. For any

\[
M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R}),
\]
put

\[ j_M(\tau) = \frac{d(M\tau)}{d\tau} = (c\tau + d)^{-2}. \]

The subgroups of translations in \( G(\lambda_q) \) and in \( H(\lambda_q) \) are both generated by \( U \); we denote this group by \( G_\infty \). Then

\[ E_k(\tau) = \sum_{M: G_\infty \setminus G(\lambda_q)} (j_M(\tau))^{k/2}, \quad F_k(\tau) = \sum_{M: G_\infty \setminus H(\lambda_q)} (j_M(\tau))^{k/2}, \]

where \( M \) runs through a complete set of representatives of cosets \( G_\infty M \) in \( G(\lambda_q) \) and in \( H(\lambda_q) \), respectively. These functions are not identically 0 because of \( E_k(i\infty) = F_k(i\infty) = 1 \).

Now, for \( k \geq 4 \) even, choose \( r = -i^k \) and define

\[ F_k^*(\tau) = r \left( \frac{1}{i} \right)^k F_k \left( -\frac{1}{\tau} \right) = -\sum_{M: G_\infty \setminus H(\lambda_q)} (j_{MT}(\tau))^{k/2}. \]

Then, by the remark at the end of section 2, \( F_k - F_k^* = E_k \neq 0 \). Let \( \varphi \) and \( \psi \) denote the Dirichlet series associated with \( F_k \) and \( F_k^* \), respectively. Then \( \varphi \neq \psi \), and the functional equation

\[(2\pi/\lambda_q)^{-s} \Gamma(s)\varphi(s) = r(2\pi/\lambda_q)^{-(k-s)} \Gamma(k-s)\psi(k-s)\]

holds, as explained in [2].

In closing this note I would like to call the reader's attention to [5] and [6] and related work on Hecke groups which is listed there.

I wish to thank the referee for his remarks which simplified and shortened the proof in section 2.

REFERENCES