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REFLEXIVE INVARIANT SUBSPACES OF $L^\infty(G)$ ARE FINITE DIMENSIONAL

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For G a compact or locally compact abelian group we shall show any closed left invariant reflexive subspace E of $L^\infty(G)$ is finite dimensional; for G compact abelian this is in marked contrast with the fact that $L^1(G)$ contains closed subspaces isomorphic to l^2 spanned by lacunary sets of characters. (Of course the question does not arise for other L^p.)

Actually the abelian result has been known for some time; it essentially follows from part of the proof of [4, 2.6] and the fact that an infinite subset of a discrete abelian group contains an infinite Sidon set [1, 37.18] (which together yield the final assertion of [4, 2.6] for $A=E^\perp$ the orthogonal subspace in $L^1(G)$). However it was not known to me when M. Hackman and I. Namioka brought the question to my attention some time ago, and I found the following more complicated proof which has the merit that it applies to G compact, a setting in which an infinite subset of the dual object need not contain an infinite Sidon set [1, p. 434], as well as to some other variants.

My proof is really almost a list of facts about weak topologies and elementary spectral synthesis. To begin, first with G locally compact abelian, the closed unit ball B of E is weakly compact, so is necessarily compact in the less fine w^* topology of $L^\infty(G)$, and both coincide on B. Because B is w^* closed in $L^\infty(G)$, E is also by the Krein–Smulian theorem, and thus E contains the spectrum of any f in E. (For if S is the hull of the orthogonal ideal $E^\perp$ in $L^1(G)$ then $E^\perp \subset k(S)$, the kernel of S [4], so the w^* closed span of S, $k(S)^\perp \subset E$. But $f \in E$ implies

$$\mathrm{sp}(f) = \bigcap \{\hat{h}^{-1}(0) : h \in L^1, h*f=0\} \subset \bigcap \{\hat{h}^{-1}(0) : h \in L^1, h*E=0\} = S ,$$

yielding the assertion.) Consequently if E contains only a finite set of characters each f in E is a linear combination of these by spectral synthesis for finite sets, and we are done; so we can assume E contains a sequence $\{\gamma_n\}$ of distinct characters. By Smulian's theorem [2] the weak compactness of B shows we can

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replace $\{\gamma_n\}$ by a subsequence (still called $\{\gamma_n\}$) which converges weakly to an element γ of $E \subset L^{\infty}(G)$, with $\gamma \neq \gamma_n$ for all n.

Now each $f \in E$ in fact lies in $C(G)$: since $x \to f_x$ is w^* continuous, and thus weakly continuous since both topologies coincide on $\|f\|B$, we have $x \to f_x$ strongly continuous by Mirkil's theorem [0, 2.8], and of course this says f is uniformly continuous. (For if $\{v_\delta\}$ is an approximate identity in L^1 it guarantees that $f = \lim v_\delta * f$ in L^{∞}; since $v_\delta * f \in L^1 * L^{\infty} \subset C(G)$, f is equivalent to the uniform limit of $\{v_\delta * f\}$, hence to a continuous function.) Thus our γ is continuous and evidently the weak convergence of $\{\gamma_n\}$ to γ in $E \subset C(G)$ implies pointwise convergence, so γ is multiplicative and therefore another character of G. But now the (uniformly) closed span F of $\{\gamma_n\}$ is a closed subspace of E which does not contain γ (since characters are orthonormal in L^2 of the Bohr group) and yet F must contain γ since a closed subspace is weakly closed by Mazur's theorem (which has already appeared as part of the basis of Mirkil's theorem). Hence our assumption that E contain infinitely many characters was false, and we are done.

It may be unnecessary to note that it is easy to produce infinite dimensional translation invariant weakly compact *subsets* of $L^{\infty}(G)$: the weak closure of the orbit $\mathcal{O}(f)$ of any $f \in \mathrm{Wap}(G) \subset L^{\infty}(G)$ which is not a trigonometric polynomial provides such a set. But the above argument using Mirkil's theorem shows an f lying in such a set must lie in Wap (G) (if one applies Mirkil's theorem to the (necessarily closed) subspace of functions f for which $x \to f_x$ is weakly continuous).

We can mimic the preceding argument if *G is a compact non-abelian group* (so compactness eliminates synthesis questions as usual and use of the Krein–Smulian Theorem): *then any (left) invariant reflexive subspace E of $L^{\infty}(G)$ must again be finite dimensional.* For exactly as before one has $E \subset C(G)$, and since trigonometric polynomials (i.e., finite linear combinations of entries in finite dimensional representations) are dense in $C(G)$ one can find, for $f \in E$ and $\varepsilon > 0$, a trigonometric polynomial v with $\|v * f - f\| < \varepsilon$; thus E is the closed span of finite dimensional left invariant subspaces, and so of minimal such subspaces (i.e., of minimal left ideals in $C(G)$ or $L^2(G)$). But [3, p. 158] each such minimal left ideal I generates a minimal 2-sided ideal which is spanned by the entries of an irreducible finite dimensional matricial unitary representation of G (and finitely many minimal left ideals), and elements of distinct minimal 2-sided ideals are pairwise orthogonal. Consequently if E is not just the span of finitely many left ideals I it contains a sequence $\{I_n\}$ from distinct 2-sided ideals, hence pairwise orthogonal in $L^2(G)$. But each such I_n contains a positive definite function χ_n (corresponding to a diagonal entry) which generates I_n, and so is non-zero. Normalizing χ_n so $\chi_n(e) = 1$, since $|\chi_n| \leq \chi_n(e) = 1$, by Smulian's theorem we can again assume $\chi_n \to \chi \in E \subset C(G)$ weakly. So $\chi(e) = \lim \chi_n(e) = 1$ and $\chi \neq 0$, while

$$0 = \int \chi_n \bar{\chi}_m \, dx \to \int \chi \bar{\chi}_m \, dx \,,$$

so χ is not in the uniformly closed span F of the χ_n, yet must be since F is weakly closed.

In fact the argument applies more generally. For example, suppose G *is compact* and E *is a left invariant reflexive Banach subspace of* $L^\infty(G)$ (i.e., a Banach space under a norm making the injection into L^∞ continuous) *while translations form an equicontinuous group of operators on* E (or equivalently, each $f \in E$ has a bounded orbit). *Then* E *is finite dimensional.*

Here the fact that $E \hookrightarrow L^\infty(G)$ is continuous guarantees that the restriction of the w^* topology of L^∞ to the ball B of E is less fine than the compact w topology (so again both coincide on B) and that an f in E for which $x \to f_x$ is strongly continuous into E has $x \to f_x$ strongly continuous into $L^\infty(G)$, so that $f \in C(G)$. Consequently, since for $f \in E$ we have its orbit under left translations contained in a dilate of B, the w^* continuity of $x \to f_x$ into $L^\infty(G)$ yields w continuity, hence strong continuity into E, and $E \subset C(G)$. But strong continuity of $x \to f_x$ into E certainly guarantees $U * f = \int U_y f_y \, dy = U \cdot \hat{f}(U)$ has entries in E for any irreducible matricial unitary representation of G so as before we obtain our contradiction if E is not finite dimensional.

In this setting compactness is essential: for example $E = L^2(\mathbf{Z})$ is a reflexive invariant Banach subspace of $L^\infty(\mathbf{Z})$ whose elements have bounded orbits, but without compactness we are unable to produce any characters in it.

Finally we might note that part of our argument (essentially that the weak and weak* topologies coincide on balls in E, so weak* continuity of $x \to f_x$ implies weak, hence strong) shows that any reflexive invariant subspace of $M(G)$ lies in $L^1(G)$ as a consequence of Plessner's theorem.

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