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EXTENSION OF LINEAR FORMS WITH STRICT DOMINATION ON LOCALLY COMPACT CONES

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Introduction.

In [1] and [2], we have studied extension theorems for continuous linear forms dominated by *hypolinear functionals* (i.e. sublinear functionals which may attain the value $+\infty$), defined on convex cones. The result can be stated in terms of lower semicontinuity of the largest hypolinear minorant of both the linear and the hypolinear functional. Unfortunately, continuity of the linear form and lower semicontinuity of the hypolinear domination functional do not guarantee the existence of a dominated (continuous) linear extension, even in a finite dimensional setting. However, an extension is possible, if the domination is *strict* (outside the origin) and the subspace is assumed to be finite dimensional [1,2.11]. In general, this last assumption cannot be dropped (cf. 1.6). The main argument used in the proof of [1,2.11] is the fact that a finite dimensional locally convex Hausdorff space is the cone generated by a compact *pseudo-base* (cf. 1.3), namely a unit sphere.

It turns out that convex cones with a compact pseudo-base are *locally compact*. Hustad [9] has proved that continuous linear forms on closed subspaces, strictly positive with respect to a locally compact convex cone, admit a positive continuous linear extension (cf. 3.3). Here the domination functional (0 on the negative cone) rather than the linear form has a locally compact domain.

Our main theorem 2.1 allows a unified treatment of both results. It deals with the existence of a continuous linear form, simultaneously strictly dominated by a finite number of hypolinear functionals, the domains of all but one being locally compact. As an immediate consequence, we get the following generalization of Hustad's theorem (cf. 2.6): A continuous linear form on a closed subspace of a locally convex space E which is strictly dominated by a lower semicontinuous hypolinear functional, defined on a locally compact convex cone Q, admits a dominated continuous linear extension to E, which is strictly dominated outside $Q \cap -Q$ if Q is closed.

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Finally, we apply our results to prove a generalized version of Boboc's affine sandwich theorem for extended convex functions [6] as well as separation theorems for convex cones, including a result recently obtained by Bair and Gwinner [3].

1. Preliminaries.

Let P be a *convex cone* in a real vector space E (i.e. $\mathsf{R}_+P=\{\lambda x : \lambda \geqq 0, x \in P\} \subset P$ and $P+P\subset P$). A mapping $p: P \to \bar{\mathsf{R}}=[-\infty,\infty]$ is called a *hypolinear functional* if p does not attain the value $-\infty$ and if p is positively homogeneous and subadditive (cf. [1, 1.1]). A numerical function $f: A \to \bar{\mathsf{R}}$ defined on a subset A of E is said to be *p-dominated* if $f(x)\leqq p(x)$ for $x \in A \cap P$. f is said to be *strictly p-dominated* on $B\subset A\cap P$ if $f(x)<p(x)$ for $x \in B$.

Unless otherwise stated, a locally convex (topological vector) space is not assumed to be Hausdorff.

We will use the following results of [1]:

1.1. *For a family $(p_i)_{i\in I}$ of hypolinear functionals p_i defined on convex subcones P_i of a locally convex space E, there exists a continuous linear form on E which is dominated by each p_i ($i \in I$) iff the largest hypolinear functional on E dominated by each p_i exists and is lower semicontinuous at the origin* ([1, § 1]).

Lower semicontinuity of hypolinear functionals is characterized by the following *approximation theorem* ([1, 3.4]):

1.2. *If $P\subset E$ is a convex cone and A is a subset of P with $0 \in A$, then a hypolinear functional $p: P \to \bar{\mathsf{R}}$ is lower semicontinuous at every point of A iff the upper envelope of all p-dominated continuous linear forms on E coincides with p on A.*

A subset K of a topological space E will be called *quasicompact* if each open cover of K has a finite subcover. K is *locally quasicompact*, if each point in K has a quasicompact neighbourhood in K. K will be called (*locally*) *compact*, if it is (locally) quasicompact and Hausdorff. If E is a topological vector space, each point of a locally quasicompact subset $K\subset E$ admits a fundamental system of quasicompact neighbourhoods in K.

1.3. Definition. Let Q be a convex cone in a topological vector space E. A subset $B\subset Q$ is called a *pseudo-base* of Q if $0 \notin \bar{B}$ and $\mathsf{R}_+B=Q$.

The following lemma generalizes a result in Köthe [11, § 25.4]:

1.4 LEMMA. *Let $Q \neq \{0\}$ be a convex cone in a topological vector space E. If Q has a quasicompact pseudo-base, then Q is locally quasicompact. Conversely, if* 0 *admits a compact neighbourhood in Q, then Q is Hausdorff and has a compact pseudo-base. If, moreover, E is Hausdorff, then Q is closed.*

PROOF. Suppose that B is a quasicompact pseudo-base of Q. Then there exists a closed circled neighbourhood U of 0 in E not meeting B. For $x \in Q$ there is $\lambda > 0$ such that $x \in \lambda \mathring{U}$. As $\lambda U \cap Q$ is contained in the quasicompact set $\{\mu b : 0 \leqq \mu \leqq \lambda,\ b \in B\}$, $\lambda U \cap Q$ is a quasicompact neighbourhood of x in Q.

Conversely, suppose that 0 has a compact neighbourhood in Q. Therefore, there exists a closed circled neighbourhood U of 0 in E such that $U \cap Q$ is compact. Then it is easy to see (and well-known) that Q is Hausdorff. Let U^* denote the boundary of U. Then $B = U^* \cap Q$ is a compact pseudo-base of Q. In fact, for $x \in Q$, the cone $\mathbf{R}_+ x$ is closed in Q, hence $\mathbf{R}_+ x \cap U \cap Q$ is compact and, therefore, if $x \neq 0$, then $\sup\{\lambda \geqq 0 : \lambda x \in U\} = \mu > 0$ is finite and $\mu x \in B$.

To prove that Q is closed if E is Hausdorff, consider a compact pseudo-base B of Q and an open circled neighbourhood U of 0 in E not meeting B. Let $x \in \bar{Q}$ and $\lambda > 0$ be such that $x \in \lambda U$. As

$$\lambda U \cap Q \subset K = \{\mu b : 0 \leqq \mu \leqq \lambda,\ b \in B\},$$

every neighbourhood of x meets the compact set K, hence $x \in \bar{K} = K \subset Q$.

1.5 REMARKS. (1) Without separation assumption, a nontrivial locally quasicompact convex cone need not have a quasicompact pseudo-base nor is it necessarily closed (cf. 2.9 (2)).

(2) If E is a locally convex Hausdorff space and $Q \neq \{0\}$ is a locally compact convex cone in E which is proper (i.e. $Q \cap -Q = \{0\}$), it is easily seen that Q even admits a compact *base*, i.e. a pseudo-base which is the intersection of Q with a closed hyperplane (cf. [11, § 25.4]).

If Q is not assumed to be proper, $Q \cap -Q$ is a locally compact, hence finite dimensional subspace of E. Therefore, $Q \cap -Q$ admits a topological supplement T, closed in E. As Klee [10, 2.3] remarked, this leads to the decomposition $Q = Q \cap -Q + S$ with the proper closed (hence locally compact) convex subcone $S = Q \cap T$ of Q. Therefore, if $S \neq \{0\}$, S admits a compact base B, hence there exists a continuous linear form g on T with $B \subset \{g = 1\}$. Defining $h(x+y) = g(y)$ $(x \in Q \cap -Q,\ y \in T)$ if $S \neq \{0\}$, and $h = 0$ if $S = \{0\}$, we get a continuous linear form h defined on E which is *positive on Q and strictly positive on $Q \setminus (-Q)$*. This result has also been proved by Klee [10, 2.2] using a separation theorem.

In [1, 2.11] we proved in particular that in a locally convex space E every (continuous) linear form which is defined on a finite dimensional subspace of E and strictly dominated (outside the origin) by a lower semicontinuous hypolinear functional p (defined on E) admits a p-dominated continuous linear extension to E. The following example shows that this result is no longer true if the subspace is assumed to be closed but not finite dimensional. Other examples of this kind are given in 2.9, 2.10 and [8, p. 405].

1.6. EXAMPLE (cf. [1, 1.6]). Let I and $J \subset I$ be different countable sets. Choose $E = \mathbf{R}^{(I)}$ and $F = \mathbf{R}^{(J)}$, respectively, to be the direct sum of $|I|$ and $|J|$ copies of the real line. Endow E with the finest locally convex topology. Then F, considered to be a subspace of E, is closed in E. Define the hypolinear functional p on E by

$$p(x) = -\sum_{\substack{i,k \in I \\ i \neq k}} \sqrt{x_i x_k} \qquad \text{for } x = (x_i)_{i \in I},\ x_i \geqq 0\ (i \in I)\,,$$

and $p(x) = \infty$, else. By [1, 1.6(3)], p is lower semicontinuous. For $j \in J$, choose $\alpha_j \leqq 0$ such that $\sum_{j \in J} (1-\alpha_j)^{-1} = 1$. By [1, 1.6(2)], the mapping

$$f\colon (x_j)_{j \in J} \mapsto \sum_{j \in J} \alpha_j x_j \qquad ((x_j)_{j \in J} \in F)$$

is a continuous linear form on F such that

$$f(x) < p(x) \qquad \text{for } x \in F \setminus \{0\}\,.$$

Let us prove that f does not admit any p-dominated linear extension to E. In fact, any linear extension g of f to E is of the form

$$g\colon (x_i)_{i \in I} \mapsto \sum_{i \in I} \alpha_i x_i$$

for some $\alpha_i \in \mathbf{R}$ $(i \in I \setminus J)$. If g were p-dominated, we would have $\alpha_i \leqq 0$ for all $i \in I$ and $\sum_{i \in I} (1-\alpha_i)^{-1} \leqq 1$, contradicting

$$1 = \sum_{j \in J} (1-\alpha_j)^{-1} < \sum_{i \in I} (1-\alpha_i)^{-1}\,.$$

2. Main results.

2.1. THEOREM. *For $i = 0, \ldots, n$ let Q_i be a convex cone in a locally convex space E and let q_i be a hypolinear functional on Q_i which is lower semicontinuous at every point of $Q_i \cap -\sum_{j=0, j \neq i}^{n} Q_j$. Assume Q_0 to be closed and $Q_1, \ldots, Q_n$ to be locally quasicompact. Denote by L the linear subspace*

$$L = \left\{ x \in \bigcap_{\substack{i=0 \\ Q_i \not\subset \overline{\{0\}}}}^{n} (Q_i \cap -Q_i) : q_i(x) = -q_j(-x) \text{ for } 0 \leqq i, j \leqq n,\ Q_i, Q_j \not\subset \overline{\{0\}} \right\}$$

and suppose that

$$\sum_{i=0}^{n} q_i(y_i) > 0 \quad \textit{for } (y_0,\ldots,y_n) \in \prod_{i=0}^{n} Q_i \setminus L^{n+1} \textit{ with } \sum_{i=0}^{n} y_i = 0\,.$$

Then there exists a continuous linear form on E which is simultaneously dominated by $q_0,\ldots,q_n$ and strictly dominated by q_i on $Q_i \setminus (-Q_i)$ for every $i \geqq 0$ with Q_i locally quasicompact and closed. If E is Hausdorff, it is sufficient to assume the cones to be only weakly locally compact rather than locally compact.

PROOF. (1) Let us first prove the theorem for a Hausdorff space E with $Q_1,\ldots,Q_n$ locally compact.

(i) Assume in addition that

$$\sum_{i=0}^{n} q_i(y_i) > 0 \quad \text{for } (y_0,\ldots,y_n) \in \prod_{i=0}^{n} Q_i \setminus \{(0,\ldots,0)\} \text{ with } \sum_{i=0}^{n} y_i = 0\,.$$

If $Q_1 = \ldots = Q_n = \{0\}$, the theorem reduces to our result [1, 1.8]. Therefore, we may assume $Q = Q_1 \times \ldots \times Q_n \neq \{0\}$. Then Q is a locally compact convex cone in E^n and $r\colon E \times Q \to \bar{\mathbf{R}}$, defined by

$$r(y) = q_0\left(y_0 - \sum_{i=1}^{n} y_i\right) + \sum_{i=1}^{n} q_i(y_i)$$

for $y = (y_0,\ldots,y_n) \in E \times Q$ with $y_0 - \sum_{i=1}^{n} y_i \in Q_0$, and by $r(y) = \infty$, else, is a hypolinear functional, lower semicontinuous at every point of $\{0\} \times Q$.

By 1.4, Q has a compact pseudo-base B. As r is strictly positive and lower semicontinuous on $\{0\} \times B$, we have $\varepsilon = \inf r(\{0\} \times B) > 0$. By 1.5 (2), there exists a continuous linear form h on E^n such that $Q \subset \{h \geqq 0\}$ and $Q \setminus (-Q) \subset \{h > 0\}$. In addition, we may assume $\sup h(B) < \varepsilon$. Define $r'\colon E \times Q \to \bar{\mathbf{R}}$ by

$$y = (y_0,\ldots,y_n) \mapsto r(y) - h(y_1,\ldots,y_n)\,.$$

Then r' is a hypolinear functional on $E \times Q$ which is dominated by r and lower semicontinuous at every point of $\{0\} \times Q$. Moreover, r' is strictly positive on $\{0\} \times B$. Hence, for $y \in \{0\} \times B$, there exists an open circled neighbourhood V_y of 0 in E such that

$$r'(z) > 0 \quad \text{for } z \in (y + V_y^{n+1}) \cap (E \times Q)\,.$$

By the compactness of $\{0\} \times B$, there is a finite set $Y \subset \{0\} \times B$ such that $\{0\} \times B \subset \bigcup_{y \in Y} (y + V_y^{n+1})$. Then $V = \bigcap_{y \in Y} V_y$ is a circled neighbourhood of 0 in E such that

$$r'(y) > 0 \quad \text{for } y \in V \times B .$$

This inequality also holds for $y \in V \times \{\lambda b : 1 < \lambda < \infty,\ b \in B\}$, as r' is positively homogeneous and V is circled. On the other hand,

$$K = \{\lambda b :\ 0 \leqq \lambda \leqq 1,\ b \in B\}$$

is compact with $r'(y) \geqq 0 > -1$ for $y \in \{0\} \times K$. Therefore, arguing as above, one can find a neighbourhood W of 0 in E such that

$$r'(y) > -1 \quad \text{for } y \in W \times K .$$

Hence, for the neighbourhood $U = V \cap W$ of 0 in E, we have

$$r'(y) > -1 \quad \text{if } y \in U \times Q .$$

This proves that the mapping $p' : E \to \bar{\mathsf{R}}$, defined by

$$x \mapsto \inf\{r'(x, y_1, \ldots, y_n) : y_i \in Q_i,\ i = 1, \ldots, n\} ,$$

is bounded below on the neighbourhood U of 0. It is easy to see that p' is a hypolinear functional, hence, by [1, 1.8], there exists a p'-dominated continuous linear form f on E. As

$$p : x \mapsto \inf\{r(x, y_1, \ldots, y_n) : y_i \in Q_i,\ i = 1, \ldots, n\} \qquad (x \in E)$$

is the largest hypolinear minorant of $q_0, \ldots, q_n$ (cf. [1, 1.5]), and since $p' \leqq p$, p' and hence f is simultaneously dominated by $q_0, \ldots, q_n$. Moreover, for $i = 1, \ldots, n$ and $x \in Q_i \setminus (-Q_i)$, we have

$$f(x) \leqq p'(x) \leqq r'(y) < r(y) = q_i(x) ,$$

where $y = (y_0, \ldots, y_n)$ with $y_0 = y_i = x$ and $y_j = 0$, else.

If, in addition, Q_0 is locally compact, then f can be chosen to be strictly dominated by q_0 on $Q_0 \setminus (-Q_0)$, too: This follows by application of the preceding result to the closed cone $\{0\}$ and the $n+1$ locally compact cones $Q_0, \ldots, Q_n$.

(ii) Let us now consider the case of a general exceptional set L in the Hausdorff space E. First note that L is a linear subspace of E. In fact, for $I = \{i : 0 \leqq i \leqq n,\ Q_i \neq \{0\}\}$, $M = \bigcap_{i \in I} (Q_i \cap -Q_i)$ is a subspace of E. L is a subspace of M as obviously $-L = L \subset M$, $\mathsf{R}_+ L \subset L$, and for $x, y \in L$, $i, j \in I$

$$\begin{aligned} q_i(x+y) &\leqq q_i(x) + q_i(y) = -q_i(-x) - q_i(-y) \leqq -q_i(-x-y) \\ &\leqq q_i(x+y) , \end{aligned}$$

hence

$$q_i(x+y) = q_i(x) + q_i(y) = -q_j(-x) - q_j(-y) = -q_j(-(x+y)) ,$$

which proves that $x + y \in L$.

We may again assume that $I\cap\{1,\ldots,n\}\neq\emptyset$. Then, by 1.4, M is locally compact, hence M and L are finite dimensional. Therefore, L has a closed topological supplement G in E. Replacing E by G, all cones by their intersections with G, and all hypolinear functionals by their corresponding restrictions, the assumptions of (i) are satisfied for G instead of E since $L=\bar{L}$ and $L\cap G=\{0\}$. Therefore, there exists a continuous linear form g on G simultaneously dominated by $q_0,\ldots,q_n$ and strictly dominated by q_i on $Q_i\cap G\setminus(-Q_i\cap G)$ for $i\in\{0,\ldots,n\}$ with Q_i locally compact.

By the definition of L, the mapping $l\colon y\mapsto -q_i(-y)$ is a (continuous) linear form on L, independent of $i\in I$. Therefore, the linear form f on E defined by

$$f(x+y) = g(x)+l(y) \qquad (x\in G,\ y\in L)\,,$$

is continuous. If $x+y\in Q_i$ ($i\in I$, $x\in G$, $y\in L$), then $x\in Q_i\cap G$ as $y\in L\subset -Q_i$. Hence

$$f(x+y) = g(x)-q_i(-y) \leqq q_i(x)-q_i(-y) \leqq q_i(x+y)\,.$$

As $Q_i=\{0\}$ for $i\notin I$, f is dominated by each q_i ($i=0,\ldots,n$). Suppose $i\in I$, Q_i locally compact and $x+y\in Q_i\setminus(-Q_i)$ with $x\in G$, $y\in L$. As $y\in Q_i\cap -Q_i$, we have $x\in Q_i\cap G\setminus(-Q_i\cap G)$, hence

$$f(x+y) = g(x)-q_i(-y) < q_i(x)-q_i(-y) \leqq q_i(x+y)\,.$$

This proves the strict domination conditions for f.

(2) Now assume E to be Hausdorff and $Q_1,\ldots,Q_n$ weakly locally compact. (In view of lemma 1.4, a locally compact convex cone in E is always weakly locally compact.) Note that, as a closed convex set, Q_0 is weakly closed. Therefore, if we assume $Q_1,\ldots,Q_n$ to be weakly locally compact, by (i) and (ii), it is sufficient to remark that the lower semicontinuity conditions for the q_i are also satisfied with respect to the weak topology on E. This follows from the approximation theorem (cf. 1.2).

(3) For the case of a general locally convex space E (not necessarily assumed to be Hausdorff), let $N=\overline{\{0\}}$. Then the quotient space $H=E/N$ is a locally convex Hausdorff space. Denote by $\pi\colon E\to H$ the (linear and continuous) canonical mapping $x\mapsto\dot{x}=x+N$.

We will use the following two lemmas, the proof of which will be given after the end of the proof of the theorem.

2.2. Lemma. (1) *A subset A of E is open (closed) iff $\pi^{-1}(\pi(A))=A$ and $\pi(A)$ is open (closed) in H.*

(2) *If P is a locally quasicompact convex cone in E, then $\pi(P)$ is a locally compact convex cone in H.*

2.3. LEMMA. *Let $p\colon P \to \bar{\mathbf{R}}$ be a hypolinear functional on a convex cone P in E, lower semicontinuous at 0. Then*

$$\dot{p}\colon \dot{x} \mapsto \inf_{y \in \dot{x} \cap P} p(y) \qquad (\dot{x} \in \pi(P))$$

is a hypolinear functional on $\pi(P)$.

If p is lower semicontinuous at $x \in P$, then $\dot{p}$ is lower semicontinuous at $\dot{x} = \pi(x)$, and $\dot{p}(\dot{x}) = p(x)$.

As π is linear, by 2.2, $\dot{Q}_0 = \pi(Q_0)$ is a closed and, for $i \in \{0,\dots,n\}$ with Q_i locally quasicompact, each $\dot{Q}_i = \pi(Q_i)$ is a locally compact convex cone in H. Note that

$$\dot{Q}_i \cap -\sum_{\substack{j=0\\ j\neq i}}^{n} \dot{Q}_j = \pi\left(Q_i \cap -\sum_{\substack{j=0\\ j\neq i}}^{n} Q_j\right)$$

since $\pi^{-1}(\pi(Q_0)) = Q_0$, by 2.2. Therefore, by 2.3, $\dot{q}_i$ is lower semicontinuous at every point of

$$\dot{Q}_i \cap -\sum_{\substack{j=0\\ j\neq i}}^{n} \dot{Q}_j \qquad (i=0,\dots,n)\,.$$

Since $\dot{Q}_i \subset \overline{\{\dot{0}\}}$ (i.e. $\dot{Q}_i = \{\dot{0}\}$) iff $Q_i \subset \overline{\{0\}}$, the corresponding exceptional set in H is

$$L_H = \left\{\dot{x} \in \bigcap_{i \in I} (\dot{Q}_i \cap -\dot{Q}_i) : \dot{q}_i(\dot{x}) = -\dot{q}_j(-\dot{x}) \text{ for } i,j \in I\right\},$$

where $I = \{i : 0 \leqq i \leqq n,\ Q_i \not\subset \overline{\{0\}}\}$. Let us prove that

$$\sum_{i=0}^{n} \dot{q}_i(\dot{y}_i) > 0 \quad \text{ for } (\dot{y}_0,\dots,\dot{y}_n) \in \prod_{i=0}^{n} \dot{Q}_i \setminus \overline{L_H}^{n+1} \text{ with } \sum_{i=0}^{n} \dot{y}_i = \dot{0}\,.$$

This is obvious, if I has at most one element, since $\dot{0} \in L_H$. Otherwise, each q_i ($i \in I$) is lower semicontinuous at every point of L, hence $\pi(L) \subset L_H$, by 2.3, and $\pi(\bar{L}) \subset \overline{L_H}$. As $\pi^{-1}(\pi(Q_0)) = Q_0$, there exists $(y_0,\dots,y_n) \in \prod_{i=0}^{n} Q_i \setminus \bar{L}^{n+1}$ with $\sum_{i=0}^{n} y_i = 0$ and $\pi(y_i) = \dot{y}_i$, hence $\dot{q}_i(\dot{y}_i) = q_i(y_i)$ by 2.3, and therefore $\sum_{i=0}^{n} \dot{q}_i(\dot{y}_i) = \sum_{i=0}^{n} q_i(y_i) > 0$.

By (1), there exists a continuous linear form $\dot{f}$ on H such that $\dot{f}$ is dominated simultaneously by $\dot{q}_0,\dots,\dot{q}_n$, strictly on $\dot{Q}_i \setminus (-\dot{Q}_i)$ for $i \in \{0,\dots,n\}$ with Q_i locally quasicompact. Then $f = \dot{f} \circ \pi$ is a continuous linear form on E such that

$$f(x) = \dot{f}(\dot{x}) \leqq \dot{q}_i(\dot{x}) \leqq q_i(x) \qquad \text{for } x \in Q_i \;\; (i=0,\dots,n)\,.$$

Moreover, if $0 \leqq i \leqq n$ and Q_i is locally quasicompact and closed, we have Q_i

$=\pi^{-1}(\pi(Q_i))$ and hence $\pi(Q_i \setminus (-Q_i)) = \dot{Q}_i \setminus (-\dot{Q}_i)$. This implies the strict domination condition for f.

PROOF OF LEMMA 2.2. (1) If $A \subset E$ is open or closed then $A + N \subset A$, hence $\pi^{-1}(\pi(A)) = A$. Therefore, (1) follows from the definition of the quotient topology.

(2) As π is linear, $\pi(P)$ is a convex cone. For $x \in P$ there is a closed neighbourhood U in E such that $U \cap P$ is quasicompact, hence $\pi(U \cap P)$ is compact. By (1), $U = \pi^{-1}(\pi(U))$, therefore $\pi(U) \cap \pi(P) = \pi(U \cap P)$ is a compact neighbourhood of $\pi(x)$ in $\pi(P)$.

PRROF OF LEMMA 2.3. If p is lower semicontinuous at $x \in P$ and $p(x) > \alpha$, then there exist $\beta \in \mathrm{R}$ and an open neighbourhood U of x in E such that $p(y) > \beta > \alpha$ for $y \in U \cap P$. Hence $\dot{p}(\dot{z}) \geqq \beta > \alpha$ for $\dot{z} \in \pi(U) \cap \pi(P)$, as

$$y \in \dot{z} \cap P \subset \pi^{-1}(\pi(U)) \cap P = U \cap P \quad \text{implies} \quad p(y) > \beta .$$

Obviously, $\dot{p}$ is positively homogeneous. As p is lower semicontinuous at 0, $\dot{p}$ is bounded below in some neighbourhood of $\dot{0}$, hence $\dot{p}(\dot{x}) > -\infty$ for $\dot{x} \in \pi(P)$. It is easy to see that $\dot{p}$ is subadditive.

2.4. REMARK. In particular, 2.1 implies (for $n=1$, $Q_0 = \overline{\{0\}}$, $q_0 = 0$, $Q_1 = -Q$, $q_1 = 0$) the following generalization of remark 1.5 (2): For every closed [weakly] locally quasicompact convex cone Q in a locally convex [Hausdorff] space E there exists a continuous linear form on E, positive on Q and strictly positive on $Q \setminus (-Q)$. This result will be extended considerably in proposition 3.3.

2.5. COROLLARY. *Let P and Q be convex cones in a locally convex [Hausdorff] space E such that P is closed and Q is [weakly] locally quasicompact. Let p and q be hypolinear functionals on P and Q, respectively, which are lower semicontinuous at every point of $P \cap Q$. If*

$$-q(y) \leqq p(y) \quad \textit{for } y \in P \cap Q$$

and

$$-q(y) < p(y) \textit{ for } y \in P \cap Q \setminus \{x \in -(P \cap Q) : p(-x) = -q(-x)\} ,$$

then there exists a continuous linear form g on E such that $-q$ is g-dominated (strictly on $Q \setminus (-Q)$, if Q is closed) and g is p-dominated (strictly on $P \setminus (-P)$, if P is [weakly] locally quasicompact).

PROOF. In theorem 2.1, choose $n=1$, $Q_0=P$, $Q_1=-Q$, $q_0=p$, and define $q_1\colon Q_1 \to \bar{\mathbb{R}}$ to be the hypolinear functional $q_1\colon x \mapsto q(-x)$. Observe that $p(y)+q(y)>0$ for $y \in P \cap Q$ if y does not belong to the exceptional set L of the theorem. In fact, for

$$L' = \{x \in P \cap -P \cap Q \cap -Q \;:\; p(-x)=-q(-x)\}$$

and $y \in L' \setminus L$ with $p(y)+q(y)=0$ we get

$$p(y) = -q(y) \leqq q(-y) = -p(-y) \leqq p(y)$$

in contradiction to $y \notin L$. As a linear form on E is q_1-dominated iff it dominates $-q$, the result follows from 2.1.

2.6. COROLLARY. *Let p be a hypolinear functional on a convex cone P in a locally convex [Hausdorff] space E and let f be a p-dominated continuous linear form on a subspace F of E. Suppose that F is closed and P is [weakly] locally quasicompact, or P is closed and F is locally quasicompact. If p is lower semicontinuous at every point of $P \cap F$ and if*

$$f(y) < p(y) \quad \textit{for } y \in P \cap F \setminus \{x \in P \cap -P \cap F \;:\; -p(-x)=p(x)\}\,,$$

then f can be extended to a p-dominated continuous linear form on E, which is strictly p-dominated on $P \setminus (-P)$ if P is [weakly] locally quasicompact and closed.

PROOF. In theorem 2.1, choose $n=1$ and $Q_0=F$, $Q_1=P$, $q_0=f$, $q_1=p$ in the first and $Q_0=P$, $Q_1=F$, $q_0=p$, $q_1=f$ in the second case. Note that $\{x \in P \cap -P \cap F : -p(-x)=p(x)\}$ is contained in the exceptional set L of the theorem and that every f-dominated linear form on E is an extension of f.

2.7. REMARKS. (1) If in corollary 2.6 we assume $F \subset P$, it can be shown that the continuity of f follows from the other assumptions. As every finite dimensional locally convex space is locally quasicompact, corollary 2.6 contains our theorem 2.11 in [1].

(2) The case of a one-dimensional subspace F is of particular interest: As an immediate consequence we get the approximation theorem [1, 3.4] for hypolinear functionals (cf. 1.2), important applications of which can be found in Lembcke's note [12] on Bauer's refined version [4] of the Choquet–Deny theorem and in an article of Bauer and Donner [5] on Korovkin theorems in $\mathscr{C}_0(X)$.

The following two examples show that corollary 2.6 (and hence 2.5 and 2.1) are no longer true if P is not assumed to be closed, even if E is finite dimensional. In the first example E is in addition a *Hausdorff* space, in the second P and F are *both locally compact*, but not closed. Moreover, the second example also disproves 2.6 if F is not assumed to be closed.

2.8. EXAMPLES. (1) Let $E=\mathsf{R}^2$ and choose $f=0$ on $F=\mathsf{R}\times\{0\}$. Let P be the convex cone

$$\{(x_1,x_2)\in E\;:\;x_1>0,\;x_2>0\}\cup\{(0,0)\}\;,$$

and define $p\colon P\to\bar{\mathsf{R}}$ by $p((x_1,x_2))=-\sqrt{x_1x_2}$. Then $P\cap F=\{(0,0)\}$, p is a continuous hypolinear functional on P (cf. [1, 1.6]), but there is no p-dominated linear extension g of f to E (otherwise, for $\lambda>0$, $g((0,1))=g((\lambda,0))+g((0,1))=g((\lambda,1))\leqq p((\lambda,1))=-\sqrt{\lambda}$).

(2) Denote by R_c the real numbers endowed with the coarsest topology. Let $E=\mathsf{R}_c\times\mathsf{R}$, $P=\mathsf{R}(1,1)$, $F=\{0\}\times\mathsf{R}$, define $p\colon P\to\mathsf{R}$ by $p(x)=0$ $(x\in P)$ and $f\colon F\to\mathsf{R}$ by $f(0,y)=y$ $(y\in\mathsf{R})$. Then f is obviously p-dominated and continuous. Furthermore, $P\cap F=\{(0,0)\}$, but the only p-dominated linear extension of f to E is $g\colon(x,y)\mapsto y-x$ $((x,y)\in E)$, which is not continuous.

In view of theorem 2.1 one might expect that the assumption in corollary 2.6 imposed on the subspace F to be locally quasicompact may be replaced by requiring F to be *generated* by a locally quasicompact convex cone. The following example shows that this conjecture is false, even if F is closed and E is Hausdorff.

2.9 EXAMPLE. Consider $\mathscr{C}([0,1])$ with the uniform topology. Let

$$B\;=\;\{\psi\in\mathscr{C}([0,1])\;:\;\psi(0)=1,\,|\psi(x)-\psi(y)|\leqq|x-y|\;\text{ for }\;x,y\in[0,1]\}\;.$$

Then B is convex and closed. Moreover, by Ascoli's theorem, B is relatively compact and hence compact. As $0\notin B$, $Q=\mathsf{R}_+B$ is a locally compact convex cone with (pseudo-) base B (cf. 1.4). Let F be the subspace of $\mathscr{C}([0,1])$ generated by Q, and define $\varrho\colon[0,1]\to\mathsf{R}$ by $\varrho(x)=1/x$ $(x\neq0)$ and $\varrho(0)=0$. Then $\varrho\notin F$.

Let $E=F\oplus\mathsf{R}\varrho$ be endowed with the direct sum topology of the Hausdorff spaces F and $\mathsf{R}\varrho$. Then F is closed and Q is a locally compact subcone of E. It is easy to verify that

$$E_+\;=\;\{\psi\in E\;:\;\psi(x)\geqq0\;\text{ for }\;x\in[0,1]\}$$

is a closed convex cone, hence the hypolinear functional p on E, equal to 0 on $-E_+$ and to ∞, else, is lower semicontinuous. The Lebesgue integral

$$f\colon\psi\mapsto\int_0^1\psi(x)\,dx\qquad(\psi\in F)$$

is a strictly p-dominated continuous linear form on F. Let us prove that, nevertheless, there is no p-dominated (i.e. positive) linear form (continuous or not) on E extending f. First note that F consists of all Lipschitz continuous functions on $[0,1]$. Therefore, the function ψ_n: $[0,1] \to \mathbf{R}$ ($n \in \mathbf{N}$), defined by $\psi_n(x)=n^2x$ for $0 \leqq x \leqq 1/n$ and by $\psi_n(x)=\varrho(x)$ for $1/n \leqq x \leqq 1$, belong to F. For $n \in \mathbf{N}$, we have

$$f(\psi_n) \geqq \int_{1/n}^{1} \frac{1}{x}\, dx ,$$

hence $\lim f(\psi_n)=\infty$. As $\psi_n \leqq \varrho$ for $n \in \mathbf{N}$, f does not admit a positive linear extension to E.

In a final example we show that even in a Banach space E the compactness condition imposed on F in corollary 2.6 cannot be replaced by weak compactness of the unit ball of F (i.e. reflexivity).

2.10. Example. Let E be a separable real Hilbert space with complete orthonormal system $\{e_n : n \in \mathbf{N}\}$. Let P and F be the closure of the subspaces generated by $\{e_{2n} : n \in \mathbf{N}\}$ and by $\{b_n : n \in \mathbf{N}\}$, respectively, where $b_n = e_{2n} + n^{-1}e_{2n-1}$ for $n \in \mathbf{N}$.

Then the 0-functional p on P is a (lower semi-) continuous (hypo-) linear functional. There is a unique continuous linear form f on F such that $f(b_n) = 1/n$ for $n \in \mathbf{N}$. Obviously, f is strictly p-dominated on the empty set $P \cap F \setminus \{0\}$. Suppose, g is a p-dominated continuous linear extension of f to E. Then for $n \in \mathbf{N}$

$$g(e_{2n-1}) = nf(b_n) - ng(e_{2n}) \geqq nf(b_n) - np(e_{2n}) = 1 ,$$

which contradicts the continuity of g. Therefore, there is no p-dominated continuous linear extension of f to E.

3. Applications.

The following propositions 3.1 and 3.2 generalize known separation and support properties for convex cones. Let us first give a new proof of a theorem recently obtained by Bair and Gwinner [3, Théorème 1] which generalizes a result of Klee [10, 2.5] and, as an immediate consequence, implies a separation theorem for convex cones (cf. [3, Théorème 2]). Note that in the Hausdorff case it is sufficient to assume one of the cones to be *weakly* locally compact.

3.1. Proposition. *Let P and Q be closed convex cones in a locally convex [Hausdorff] space E such that $P \cap Q$ is a linear subspace of E. If Q is [weakly] locally quasicompact, there exists a continuous linear form g on E such that*

$$Q \subset \{g \geqq 0\},\ Q \setminus (-Q) \subset \{g > 0\},\ P \subset \{g \leqq 0\}$$

(*and* $P \setminus (-P) \subset \{g < 0\}$ *if* P *is* [*weakly*] *locally quasicompact*).

PROOF. In corollary 2.5, choose $p=0$ and $q=0$.

3.2. PROPOSITION. *Let F be a finite dimensional subspace of a locally convex Hausdorff space E. If P is a closed convex cone in E such that $P+F \neq E$ (e.g. $F \neq E$ and P proper) and $P \cap F$ is a linear subspace, then there exists a closed hyperplane in E containing F and supporting P.*

PROOF. Let $z \in E \setminus (P+F)$ and apply theorem 2.1 to

$$Q_0 = P,\ q_0 = 0,\ Q_1 = F,\ q_1 = 0,\ Q_2 = -\mathbf{R}_+ z + P \cap F,\ q_2 = 0\,.$$

Then $L \supset P \cap F$ and

$$\{(y_0, y_1, y_2) \in Q_0 \times Q_1 \times Q_2 :\ y_0 + y_1 + y_2 = 0\} \subset L^3\,.$$

Therefore, the assumptions of 2.1 are satisfied. Since $-z \in Q_2 \setminus (-Q_2)$, there exists a continuous linear f on E such that $P \subset \{f \leqq 0\}$, $F \subset \{f=0\}$, and $f(-z)<0$, hence $\{f=0\}$ is a closed hyperplane in E containing F and supporting P.

The following extension theorem for positive linear forms generalizes a result due to Hustad ([9, p. 64, cor.]).

3.3. PROPOSITION. *Let E be a locally convex [Hausdorff] space, preordered by a [weakly] locally quasicompact convex cone E_+. Then every positive continuous linear form f, defined on a closed subspace F of E, which is strictly positive on $F \cap E_+ \setminus (-E_+)$, admits a positive and continuous linear extension to E, which is strictly positive on $E_+ \setminus (-E_+)$ if E_+ is closed.*

PROOF. Apply corollary 2.6 to the hypolinear functional $p=0$ on $P=-E_+$.

As a further application of theorem 2.1 we will finally prove a slight generalization of Boboc's sandwich theorem for extended (i.e. $]-\infty,\infty]$-valued) convex functions ([6, lemme 2]), following the ideas that led to our previous results [1, 4.4, 4.6].

3.4. PROPOSITION. *Let K_1 and K_2 be convex subsets of a locally convex [Hausdorff] space F and let k_1 and k_2 be extended convex functions on K_1 and K_2, respectively, lower semicontinuous at every point of $K_1 \cap K_2$. Suppose that*

K_1 is [weakly] quasicompact, K_2 is closed and that k_1, k_2 are locally bounded below. If

$$-k_1(x) < k_2(x) \quad \textit{for } x \in K_1 \cap K_2 \, ,$$

then there exists a continuous affine function a on F such that

$$-k_1(x) < a(x) \textit{ for } x \in K_1 \quad \textit{and} \quad a(x) < k_2(x) \textit{ for } x \in K_2 \, .$$

PROOF. If $K_1 \cap K_2 \neq \varnothing$, the mapping $x \mapsto k_1(x)+k_2(x)$ $(x \in K_1 \cap K_2)$ attains its (strictly positive) infimum. Hence, in any case, we may replace k_1 by $k_1-\alpha$ and k_2 by $k_2-\alpha$ for some $\alpha>0$ without changing the assumptions. Therefore, it is sufficient to prove the existence of a continuous affine function a on F such that

$$-k_1(x) \leqq a(x) \ \ (x \in K_1) \quad \text{and} \quad a(x) \leqq k_2(x) \ \ (x \in K_2) \, .$$

As in [1, 4.2], define $E = \mathsf{R} \times F$, $P_0 = \mathsf{R}_+(\{1\} \times K_2)$ and $Q = \mathsf{R}_+(\{1\} \times K_1)$. Then P_0, $P = \overline{P_0}$ and Q are convex cones with P closed and Q locally quasicompact (1.4). By [1, 4.2], the mappings

$$p_0 \colon (\lambda, \lambda x) \mapsto \lambda k_2(x) \qquad (\lambda \in \mathsf{R}_+, \ x \in K_2)$$

and

$$q \colon (\lambda, \lambda x) \mapsto \lambda k_1(x) \qquad (\lambda \in \mathsf{R}_+, \ x \in K_1)$$

are hypolinear functionals on P_0 and Q, respectively. Define the hypolinear functional $p \colon P \to \bar{\mathsf{R}}$ by $p(x) = p_0(x)$ for $x \in P_0$ and by $p(x) = \infty$ for $x \in P \setminus P_0$. Then p and q are lower semicontinuous at every point of $P \cap Q$ ([1, 4.2]), as $P \setminus P_0 \subset \{0\} \times F$ implies

$$P \cap Q \; = \; P_0 \cap Q \; = \; \{0\} \cup \mathsf{R}_+(\{1\} \times K_1 \cap K_2) \, .$$

Moreover, for $y = (\lambda, \lambda x) \in P \cap Q \setminus \{0\}$ we have

$$p(y)+q(y) \; = \; \lambda(k_2(x)+k_1(x)) > 0 \, .$$

Hence, by corollary 2.5, there is a continuous linear form g on E with

$$-q(y) \leqq g(y) \ \ (y \in Q) \quad \text{and} \quad g(y) \leqq p(y) \ \ (y \in P) \, .$$

Therefore, the continuous affine function $a \colon x \mapsto g(1,x)$ on F has the required properties.

Finally, if F is a Hausdorff space and $K_1 \subset F$ is weakly compact, then $\{1\} \times K_1$ is weakly compact in E (cf. [7, ch. II, § 6. prop. 8]). Therefore, 2.5 may also be applied to this situation.

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