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ANALYTIC CONTINUATION OF FUNCTIONS DEFINED BY MEANS OF CONTINUED FRACTIONS

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1. Introduction.

We begin with a brief review of the definition of a continued fraction, with variable elements, as well as the notation to be used. Let D be a region in the complex plane and let two sequences of complex valued functions $\{a_n(z)\}$, $\{b_n(z)\}$, $n \geqq 1$, $z \in D$, be given. Then we define

$$s_n(z,w) = \frac{a_n(z)}{b_n(z)+w}, \qquad n \geqq 1 ,$$

and

$$S_N^{(n)}(z,w) = s_{n+1}(z, S_N^{(n+1)}(z,w)), \qquad 0 \leqq n \leqq N-1 ,$$
$$S_N^{(N)}(z,w) = w .$$

We then have, using one of the standard notations,

$$S_N^{(n)}(z,w) = \frac{a_{n+1}(z)}{b_{n+1}(z)} + \ldots + \frac{a_N(z)}{b_N(z)+w} .$$

Instead of $S_N^{(0)}(z,w)$ we shall usually write $S_N(z,w)$.

The *continued fraction algorithm* $\mathbf{K}$,

$$\mathop{\mathbf{K}}_{n=1}^{\infty} \left(\frac{a_n(z)}{b_n(z)}\right) = \frac{a_1(z)}{b_1(z)} + \frac{a_2(z)}{b_2(z)} + \ldots + \frac{a_n(z)}{b_n(z)} + \ldots$$

then is the function that associates with the *sequences of elements* $\{a_n(z)\}$. $\{b_n(z)\}$ the *sequence of approximants* $\{S_N(z,0)\}$. The notation $\mathbf{K}_{n=1}^{\infty}(a_n(z)/b_n(z))$ is also used for $\lim_{N\to\infty} S_N(z,0)$, if it exists.

In this paper we shall consider only *limit periodic continued fractions*, that is those for which

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$$\lim_{n\to\infty} a_n(z) = a(z) \quad \text{and} \quad \lim_{n\to\infty} b_n(z) = b(z)$$

exist for all $z \in D$. The fixed points of the mapping

$$s(z,w) = \frac{a(z)}{b(z)+w},$$

considered as a function of w, are the solutions of the quadratic equation

$$x^2 + b(z)x - a(z) = 0 .$$

Let one of the solutions be $x(z)$, then the other one is $-(b(z)+x(z))$.

If in the course of an argument, as for example in Section 2, the variable z is held constant or does not occur we shall simply write

$$a_n,\ b_n,\ a,\ b,\ x$$

and

$$s_n(w) = \frac{a_n}{b_n+w}, \quad S_N^{(n)}(w) = s_{n+1}(S_N^{(n+1)}(w))$$

$$S_N^{(N)}(w) = w, \quad S_N^{(0)}(w) = S_N(w) .$$

This article is in a way a continuation of our investigation [7]. The ideas are similar but the emphasis has shifted. In the earlier paper our aim was to find a modification of the approximants of the continued fraction that would lead to a substantial acceleration of the convergence of the continued fraction under consideration. Here we seek a modification which will lead to an analytic continuation of the function to which the continued fraction converges in the region D into a region $D^* \supset D$, even though the convergence of the modified sequence may be quite slow, at least in $D^* \sim D$. Again, the modification can be considered as a method of summability applied to the sequence of approximants of the continued fraction with a view to having the modified sequence converge for values of z for which the original sequence does not converge or does not converge to the "right" value. As in our earlier paper there is here also a substantial overlap with work of Gill, in particular his article [1], where he indicates the possibility of using the repulsive fixed point to obtain analytical continuation.

The modified sequences we are concerned with are

$$\{S_N(z, x(z))\} \quad \text{and} \quad \{S_N(z, -(b(z)+x(z)))\} .$$

Here $x(z)$ as well as $-(b(z)+x(z))$ will, for certain values of z, be the repulsive fixed point of the mapping $s(z,w)$. In order to insure convergence of the modified sequence in this case it will be desirable (and possibly necessary) to require that $a_n(z)$ and $b_n(z)$ approach their respective limits geometrically.

In Section 2 we shall establish a general result on boundedness of modified sequences. Making use of a recent result of Jones and Thron [3], which establishes a connection between boundedness and convergence of sequences of functions that correspond (see definition in Section 3) to a formal power series, we then give two applications. In Section 3 we extend earlier results of Waadeland and Hovstad to general T-fractions. In Section 4 we present an application to regular C-fractions, it yields among others one of the rare results on the location of singular points of analytic functions defined in terms of continued fractions. In this case the exact location as well as the nature of the singularity can be specified. In a way the result on general T-fractions can also be interpreted as a result on singularities in that it gives regions in which there can be no singularities, other than poles, of the functions represented by the general T-fractions.

Our method can undoubtedly be applied to other types of continued fractions, but for the time being we restrict ourselves to the ones given here.

Instead of relying on the result of Jones and Thron to make the transition from boundedness to convergence, the convergence of the "wrong modification", that is the sequence $\{S_N(z, x(z))\}$, where $x(z)$ is the repulsive fixed point, can also be established directly. This approach, more straight forward but possibly more tedious than the one presented here, will be taken up in a separate paper. It will give an illustration of the role of the geometric approach of a_n to a and b_n to b for the convergence of the wrong modification.

2. Boundedness of sequences of modified approximants.

The main result of this section is the following.

LEMMA 2.1. *Set*

$$\begin{aligned} \delta_n &= a_n - a, \quad & d_n &> \max_{\nu \geqq n} |\delta_\nu|\,, \\ \eta_n &= b_n - b, \quad & e_n &> \max_{\nu \geqq n} |\eta_\nu|\,, \end{aligned} \tag{2.1}$$

and let x *be one of the solutions of the equation*

$$x^2 + bx - a = 0\,.$$

Finally, set

$$P = \left|\frac{x}{b+x}\right| < \infty\,. \tag{2.2}$$

If there exists a Q

$$0 < Q < 1,\ QP < 1\,, \tag{2.3}$$

so that

$$d_n + (|x| + |b+x|)e_n = k_n Q^n, \quad n \geqq 1\,, \tag{2.4}$$

where $\sum_{n=1}^{\infty} k_n$ *converges, then there is an* m_0, *independent of* N, *such that*

$$|S_N^{(m)}(x) - x| \leqq \frac{Q^{m+1}}{|b+x|} \sum_{n=1}^{\infty} k_n, \quad m \geqq m_0\,. \tag{2.5}$$

PROOF. Keep N fixed, we may think of it as being very large, and set

$$\varrho_n = S_N^{(n)}(x) - x, \quad 0 \leqq n \leqq N\,.$$

Since $a = x(b+x)$ we have in terms of the notation introduced in (2.1)

$$\begin{aligned}\varrho_n &= \frac{x(b+x)+\delta_{n+1}}{b+\eta_{n+1}+x+\varrho_{n+1}} - x, \quad 0 \leqq n \leqq N-1\,,\\ &= \frac{\delta_{n+1} - x\eta_{n+1} - x\varrho_{n+1}}{(b+x)+\varrho_{n+1}+\eta_{n+1}}\,.\end{aligned}$$

To obtain a bound r_n for $|\varrho_n|$ for all $n \geqq m$, where m is a sufficiently large number to be determined, it would be desirable to have the following two inequalities satisfied

$$r_n < |b+x| - e_n, \quad m \leqq n \leqq N\,, \tag{2.6}$$

and

$$\frac{d_{n+1} + |x|e_{n+1} + |x|r_{n+1}}{|b+x| - (r_{n+1} + e_{n+1})} < r_n, \quad m \leqq n \leqq N-1\,. \tag{2.7}$$

If (2.6) holds then relation (2.7) is equivalent to

$$d_{n+1} + |x|e_{n+1} + r_n e_{n+1} + |x|r_{n+1} < |b+x|r_n - r_n r_{n+1}\,.$$

This will surely be satisfied if

$$(d_{n+1} + |x|e_{n+1} + |b+x|e_{n+1}) + |x|r_{n+1} < |b+x|r_n - r_n r_{n+1}$$

holds. Using (2.4) we arrive at

$$k_{n+1}Q^{n+1} + |x|r_{n+1} < |b+x|r_n - r_n r_{n+1}\,. \tag{2.8}$$

This inequality will be satisfied by

$$r_n = \frac{Q^{n+1}}{|b+x|} \sum_{\nu=n+1}^{\infty} k_\nu\,, \tag{2.9}$$

for $n>m$ provided m is large enough. This we shall show now. Substituting (2.9) into (2.8) leads to

$$k_{n+1}Q^{n+1}+(PQ)Q^{n+1}\left(\sum_{\nu=n+2}^{\infty} k_\nu\right) < Q^{n+1}\left(k_{n+1}+\sum_{\nu=n+2}^{\infty} k_\nu\right)-r_n r_{n+1}$$

or

$$r_n r_{n+1} < Q^{n+1}(1-PQ)\sum_{\nu=n+2}^{\infty} k_\nu . \tag{2.10}$$

We note that it follows from (2.3) that

$$p = 1-PQ > 0 .$$

Substituting (2.9) into (2.10) leads to

$$Q^{n+2}\sum_{\nu=n+1}^{\infty} k_\nu < |b+x|^2 p . \tag{2.11}$$

Set $B=\sum_{n=1}^{\infty} k_n$ and $q=1/Q>1$. Then (2.11) will hold if

$$q^{n+2} > \frac{B}{p|b+x|^2}$$

or

$$\begin{aligned} n &> \log_q B-2\log_q|b+x|-\log_q p-2 \\ &= m_1(a,b,Q,B) . \end{aligned}$$

To have (2.6) satisfied it suffices that

$$\frac{Q^{n+1}B}{|b+x|}+\frac{Q^n B}{|b+x|} < |b+x| , \tag{2.12}$$

since

$$(|x|+|b+x|)e_n \leqq k_n Q^n \leqq Q^n B$$

and hence

$$e_n < \frac{Q^n B}{|x|+|b+x|} < \frac{Q^n B}{|b+x|} .$$

Inequality (2.12) is satisfied provided

$$2Q^n B < |b+x|^2$$

or

$$n > \log_q 2B - 2\log_q|b+x| = m_2(a,b,Q,B)$$

holds. Hence, for $m = \max(m_1, m_2) + 1$, (2.6) and (2.7) are both satisfied. From this (2.5), with $m_0 = \max(m_1, m_2) + 1$, follows and Lemma 2.1 is proved.

3. A result for general T-fractions.

We begin by introducing some further concepts, definitions and formulas which shall be used in the sequel. Since $S_N^{(m)}(z,w)$ is a linear fractional transformation in w one easily proves that

$$S_N^{(m)}(z,w) = \frac{A_N^{(m)}(z) + wA_{N-1}^{(m)}(z)}{B_N^{(m)}(z) + wB_{N-1}^{(m)}(z)}, \quad 0 \leqq m \leqq N-1, \tag{3.1}$$

where

$$\begin{aligned} A_N^{(m)}(z) &= b_N(z)A_{N-1}^{(m)}(z) + a_N(z)A_{N-2}^{(m)}(z), \\ B_N^{(m)}(z) &= b_N(z)B_{N-1}^{(m)}(z) + a_N(z)B_{N-2}^{(m)}(z), \\ A_m^{(m)}(z) &= 0, \quad A_{m+1}^{(m)}(z) = a_{m+1}(z), \\ B_m^{(m)}(z) &= 1, \quad B_{m+1}^{(m)}(z) = b_{m+1}(z), \end{aligned} \qquad m+2 \geqq N, \tag{3.2a}$$

and

$$A_N^{(m)}(z)B_{N-1}^{(m)}(z) - A_{N-1}^{(m)}(z)B_N^{(m)}(z) = (-1)^{N-m-1} \prod_{n=m+1}^{N} a_n(z). \tag{3.2b}$$

For $m=0$ these formulas are well known and can be found for example in [6]. For general m the proof is analogous.

If in particular $a_n(z) = F_n z$, $b_n(z) = 1 + G_n z$ then

$$B_N^{(m)}(z) = 1 + \ldots + z^{N-m} \prod_{n=m+1}^{N} G_n. \tag{3.3}$$

Next, let

$$L = \sum_{n=k}^{\infty} c_n z^n, \quad c_k \neq 0,$$

be a formal power series (not necessarily convergent) then we define $\lambda(L) = k$. Further, $L_0(R)$ shall be the Taylor series expansion of R at $z=0$, provided it exists. $\lambda(L_0(R))$ is then defined by our first rule.

Analogously we introduce for

$$L^* = \sum_{n=k^*}^{-\infty} c_n^* z^n, \quad c_k^* \neq 0,$$

$\lambda_\infty(L^*)=k^*$. $L_\infty(R)$ shall be the Laurent series of R at $z=\infty$, provided it exists.

A sequence $\{R_N(z)\}$ of functions holomorphic at $z=0$ will be said to *correspond to a formal power series* $L=\sum_{n=0}^{\infty} c_n z^n$ *at* $z=0$ if

$$\lim_{N\to\infty} \lambda(L_0(R_N)-L) = \infty .$$

Similarly, a sequence $\{R_N(z)\}$ of functions holomorphic in a deleted neighborhood of ∞ will be said to *correspond to a formal Laurent series* $L^* =\sum_{n=0}^{-\infty} c_n^* z^n$ *at* $z=\infty$ if

$$\lim_{N\to\infty} \lambda_\infty(L_\infty(R_N)-L^*) = -\infty .$$

The ideas described above were introduced and discussed in greater detail in [3].

A general T-fraction

$$\underset{n=m+1}{\overset{\infty}{\mathrm{K}}} \left(\frac{F_n z}{1+G_n z}\right), \quad F_n \neq 0,\ G_n \neq 0, \text{ for } n\geqq m+1 , \tag{3.4}$$

is known to correspond to a formal power series $L_m=\sum_{n=1}^{\infty} c_n^{(m)} z^n$ at $z=0$. By this we mean that the sequence $\{S_N^{(m)}(z,0)\}$ of approximants of (3.4) corresponds to L_m at $z=0$. The continued fraction (3.4) also corresponds at $z=\infty$ to a series $L_m^*=\sum_{n=0}^{-\infty} {}^*c_n^{(m)} z^n$. These results can be found in [4].

At this stage it may be useful to introduce a concept, which can be defined more precisely than we are about to do here, but which is quite suggestive. Let $1<R$ and let $\alpha(R,f'(0))$ be a positive valued function satisfying certain conditions (which we shall not state here) then $f(z)$ is said to be *very bounded at* $z=0$ if $f(z)$ is holomorphic for $|z|<R$ and

$$|f(z)-f(0)-zf'(0)| < \alpha \quad \text{for } |z|<R .$$

Similarly one calls $g(z)$ *very bounded at* $z=\infty$ if there is a $\varrho<1$ and a positive valued function $\beta(\varrho,a)$ such that $g(z)$ is holomorphic for all $|z|>\varrho$ and

$$|g(z)-a| < \beta \quad \text{for } |z|>\varrho \text{ and a certain } a .$$

Our Theorem 3.1, to be stated presently, descended from a series of results going back to 1964. In that year Waadeland [9] proved that every function $f(z)$ very bounded at $z=0$ had a limit period T-fraction expansion

$$f(0)+(f'(0)-1)z+\underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{z}{1+d_n z}\right),$$

with $\lim d_n=-1$, which converges to $f(z)$ for all $|z|<1$. In 1966 [10] he showed that the modification $\{S_n(z,z)\}$ converges to $f(z)$ in a larger disk $|z|<R'$, $R'>1$.

Hovstad [2] was able to improve this result to $R' = R - \varepsilon$, where R is the radius of the disk in which $f(z)$ is very bounded and $\varepsilon > 0$ is arbitrary small. Hovstad also showed that

$$|d_n + 1| < K/R^n .$$

These results are also discussed, in terms of fixed points, by Gill. See for instance [1].

In two recent papers [11, 12] Waadeland studied the behavior of general T-fractions

$$\mathop{\mathrm{K}}_{n=1}^{\infty}\left(\frac{F_n z}{1+G_n z}\right), \quad F_n \neq 0,\ G_n \neq 0,\ n \geqq 1 , \tag{3.5}$$

and obtained results analogous to the ones for ordinary T-fractions. In the first paper he proved that if $L = L_0(f)$, $L^* = L_\infty(g)$, where $f(z)$ and $g(z)$ are very bounded at $z=0$ and $z=\infty$, respectively, then there exists a general T-fraction which corresponds to L at $z=0$ and to L^* at $z=\infty$. The general T-fraction satisfies

$$\lim_{n\to\infty} F_n = \lim_{n\to\infty} (-G_n) = F \neq 0 ,$$

and converges to $f(z)$ for $|z| < 1/|F|$ and to $g(z)$ for $|z| > 1/|F|$.

The second result is the following. Given $1 < R' < R$ there exists a $K(R, R') > 0$ such that if (here we have normalized F to be 1)

$$|F_n - 1| < K/R^n, \quad |G_n + 1| < K/R^n, \quad n \geqq 1 , \tag{3.6}$$

then the general T-fraction (3.5) whose elements satisfy (3.6) corresponds at $z=0$ to a $L = L_0(f)$, where $f(z)$ is holomorphic for $|z| < R'$ and corresponds at $z=\infty$ to a $L^* = L_\infty(g)$, where $g(z)$ is holomorphic for $|z| > 1/R'$.

Whether the result of Hovstad can be extended to general T-fractions, that is whether very boundedness of $f(z)$ at $z=0$ and of $g(z)$ at $z=\infty$ (with $\varrho = 1/R$) together implies (3.6) is still an open question. Our main result for general T-fractions is the following.

THEOREM 3.1. *Let $1 < R' < R$. Let a general* T-*fraction*

$$\mathop{\mathrm{K}}_{n=1}^{\infty}\left(\frac{F_n z}{1+G_n z}\right), \quad F_n \neq 0,\ G_n \neq 0,\ n \geqq 1 ,$$

whose elements satisfy

$$|F_n - 1| < K/R^n, \quad |G_n + 1| < K/R^n, \quad n \geqq 1$$

for some $K > 0$ be given. Then there exists an $m(R', R, K)$ such that the sequence

$$\{S_N^{(m)}(z,z)\}$$

of modified approximants of the general T-*fraction*

$$\underset{n=m+1}{\overset{\infty}{\mathrm{K}}}\left(\frac{F_n z}{1+G_n z}\right) \tag{3.7}$$

converges to a holomorphic function $f_m(z)$, *uniformly on compact subsets, for* $|z|<R'$.

The power series

$$L_m = L_0(f_m)$$

is the series to which the general T-*fraction* (3.7) *corresponds at* $z=0$. *The function* $f_m(z)$ *is also the limit of the sequence of approximants of* (3.7) *for* $|z|<1$.

Similarly, there exists an $r(R',R,K)$ *such that the sequence*

$$\{S_N^{(r)}(z,-1)\}$$

of modified approximants of

$$\underset{n=r+1}{\overset{\infty}{\mathrm{K}}}\left(\frac{F_n z}{1+G_n z}\right) \tag{3.8}$$

converges to a holomorphic function $g_r(z)$, *uniformly on compact subsets, for* $|z|>1/R'$. *The Laurent series*

$$L_r^* = L_\infty(g_r)$$

is the series to which (3.8) *corresponds at* $z=\infty$. *The function* $g_r(z)$ *is also the limit of the sequence of approximants of* (3.8) *for* $|z|>1$.

PROOF. We first consider the case where $|z|<R'$. The results of Lemma 2.1 apply to (3.5). We can set

$$x = z, \qquad -(b+x) = -1\,.$$

Then

$$P = |z|\,.$$

Further

$$d_n = K/R^n, \qquad e_n = K/R^n\,,$$

and if we introduce

$$R'' = \frac{R+R'}{2}$$

then

$$k_n Q^n = K(1+|z|+1)/R^n = K(2+|z|)\left(\frac{R''}{R}\right)^n/(R'')^n .$$

We thus can choose $Q = 1/R''$ and

$$k_n = (2+|z|)K\left(\frac{R''}{R}\right)^n .$$

It follows that $QP = |z|/R''$ and hence for all $|z| < R'$

$$p = 1 - QP = 1 - \frac{|z|}{R''} > \frac{R''-R'}{R''} = \frac{R-R'}{R+R'}$$

and

$$B = (2+|z|)K\frac{R''}{R} \cdot \frac{1}{1-\dfrac{R''}{R}} \leqq (2+R')K\frac{R+R'}{R-R'} .$$

Hence an m for which

$$\left|S_N^{(m)}(z,z) - z\right| \leqq \frac{Q^{m+1}B}{|b+x|} \leqq \left(\frac{2}{R+R'}\right)^m \frac{2K(2+R')}{R-R'}$$

is valid, can be chosen independent of z provided $|z| < R'$. This leads to

$$|S_N^{(m)}(z,z)| < R' + \frac{2K(2+R')}{R-R'}\left(\frac{2}{R+R'}\right)^m, \quad |z| < R',\ N > m .$$

Thus the sequence $\{S_N^{(m)}(z,z)\}$ is uniformly bounded for $|z| < R'$.

To prove convergence of the sequence we use a recent result of Jones and Thron [3] to the effect that if a sequence $\{R_N(z)\}$ of functions holomorphic for $z \in D$ is uniformly bounded on compact subsets of the region D, if further $0 \in D$ and if finally the sequence corresponds to a formal power series L at $z = 0$, then $\{R_N(z)\}$ converges, uniformly on compact subsets of D, to a holomorphic function $f(z)$ with $L = L_0(f)$.

It is known [4] that $\{S_N^{(m)}(z,0)\}$, the sequence of approximants of (3.7), corresponds to a formal power series L_m at $z=0$. We shall show that $\{S_N^{(m)}(z,z)\}$ also corresponds to L_m at $z=0$. Using (3.1) one obtains

$$S_N^{(m)}(z,z) - S_N^{(m)}(z,0) = \frac{z(-1)^{N-m}\prod_{n=m+1}^{N}(F_n z)}{B_N^{(m)}(z)(B_N^{(m)}(z) + zB_{N-1}^{(m)}(z))} .$$

Since, by (3.3) $B_N^{(m)}(0) = 1$, $N \geqq m$, it then follows that

$$\lambda\left(L_0(S_N^{(m)}(z,z) - S_N^{(m)}(z,0))\right) = N - m + 1$$

and hence $\{S_N^{(m)}(z,z)\}$ corresponds to L_m at $z=0$.

That there exists an r, independent of z, so that $\{S_N^{(r)}(z,-1)\}$ is bounded for $|z|>1/R'$ is proved in an analogous manner. In this case $x=-1$, $-(b+x)=z$ so that $P=1/|z|$ and

$$p = 1-QP = 1-1/(R''|z|) > 1-R'/R'' = \frac{R-R'}{R+R'} \quad \text{for } |z| > 1/R' .$$

To establish the correspondence of the sequence $\{S_N^{(r)}(z,-1)\}$ at $z=\infty$ we consider

$$S_N^{(r)}(z,-1)-S_N^{(r)}(z,0) = \frac{(-1)^{N-r-1}\prod_{n=r+1}^{N}(F_n z)}{B_N^{(r)}(z)(B_N^{(r)}(z)-B_{N-1}^{(r)}(z))} .$$

Since no G_n vanishes $B_N^{(r)}(z)$ and $B_{N-1}^{(r)}(z)$ are polynomials of exact degrees $N-r$ and $N-r-1$, respectively. It follows that

$$\lambda_\infty(L_\infty(S_N^{(r)}(z,-1)-S_N^{(r)}(z,0))) = -(N-r) .$$

That the sequence of approximants of (3.7) converges for $|z|<1$ and that of (3.8) for $|z|>1$ can be seen in a number of ways. One can refer to classical theorems on limit periodic continued fractions or to the recent result of the authors [7], since, for $|z|<1, z$ is the attractive fixed point of

$$\frac{z}{1-z+w}$$

and, for $|z|>1$, -1 is the attractive fixed point. Hence for those values of z the limit of the modified approximants equals the limit of the approximants. This completes the proof of the theorem.

If we now look at the sequence

$$\{S_N(z,z)\} \tag{3.9}$$

we see that for each R', $1<R'<R$, a "tail" converges to a holomorphic function for $|z|<R'$. The sequence (3.9) itself thus converges to a meromorphic function, which does not have a pole at $z=0$, for all $|z|<R'$. Since the functions obtained for different R' are all analytic continuations of each other we have one meromorphic function to which the sequence (3.9) converges for all $|z|<R$. It is also easily seen that the Taylor series of this function at $z=0$ is the power series to which the general T-fraction (3.5) corresponds at $z=0$.

A similar argument holds at $z=\infty$ so that the following result has been proved.

THEOREM 3.2. *Let the general* T-*fraction*

$$\mathop{\mathrm{K}}_{n=1}^{\infty}\left(\frac{F_n z}{1+G_n z}\right), \qquad F_n \neq 0,\ G_n \neq 0,\ n \geqq 1$$

satisfy

$$|F_n - 1| < K/R^n, \qquad |G_n + 1| < K/R^n, \qquad n \geqq 1 ,$$

for some $R > 1$ *and* $K > 0$. *Let* L *and* L^* *be the formal power series to which the* T-*fraction corresponds at* 0 *and* ∞, *respectively. Then*

$$L = L_0(f) ,$$

where $f(z)$ *is meromorphic for all* $|z| < R$, *and*

$$f(z) = \lim_{N\to\infty} S_N(z,z), \qquad |z| < R .$$

Similarly,

$$L^* = L_\infty(g) ,$$

where $g(z)$ *is meromorphic for* $|z| > 1/R$, *and*

$$g(z) = \lim_{N\to\infty} S_N(z,-1), \qquad |z| > 1/R .$$

4. An application to regular C-fractions.

A regular C-fraction is a continued fraction of the form

$$\mathop{\mathrm{K}}_{n=1}^{\infty}\left(\frac{\alpha_n z}{1}\right), \qquad \alpha_n \neq 0,\ n \geqq 1 . \tag{4.1}$$

Its sequence of approximants $\{S_N(z,0)\}$ corresponds to a power series

$$P = \sum_{n=1}^{\infty} c_n z^n$$

at $z=0$. These and other results on regular C-fractions can be found in Perron [5]. In terms of the notation of Section 1 we have $a_n(z) = \alpha_n z$, $b_n(z) = 1$. As before we shall assume that the continued fraction is limit periodic, that is

$$\lim_{n\to\infty} \alpha_n = \alpha \neq 0 , \tag{4.2}$$

and that the approach to the limit is fast enough so that

$$|\alpha_n - \alpha| = \gamma_n Q^n, \qquad n \geqq 1,\ 0 < Q < 1,\ \sum_{n=1}^{\infty} \gamma_n = B < \infty . \tag{4.3}$$

By $+\sqrt{v}$ we shall mean the root in the right half plane (including the positive imaginary axis but excluding the negative imaginary axis). In terms of this notation

$$x_1(z) = -\tfrac{1}{2}+\sqrt{\tfrac{1}{4}+\alpha z}$$

is the attractive fixed point of $s(w)=\alpha z/(1+w)$ and

$$x_2(z) = -\tfrac{1}{2}-\sqrt{\tfrac{1}{4}+\alpha z}$$

is the repulsive fixed point of $s(w)$. If we now introduce

$$\omega^2 = 1+4\alpha z \tag{4.4}$$

then

$$x(z) = -\tfrac{1}{2}(1-\omega)$$

shall equal $x_1(z)$ for ω in the right half plane and shall be $x_2(z)$ for ω in the left half plane. Thus

$$P = \left|\frac{x(z)}{1+x(z)}\right| = \left|\frac{\omega-1}{\omega+1}\right| .$$

In terms of ω and $q=1/Q$ the condition $PQ<1$ becomes $\omega \in \Omega$, where

$$\Omega = \left[\omega : \left|\frac{\omega-1}{\omega+1}\right| < q\right] \quad q>1 . \tag{4.5}$$

Since $\omega=0 \in \Omega$, it is clear that Ω is the outside of the circle with center and radius given by

$$-\frac{1+q^2}{q^2-1} \quad \text{and} \quad \frac{2q}{q^2-1} ,$$

respectively. The boundary of Ω intersects the real axis at the points

$$-\left(\frac{q-1}{q+1}\right) \quad \text{and} \quad -\left(\frac{q+1}{q-1}\right) .$$

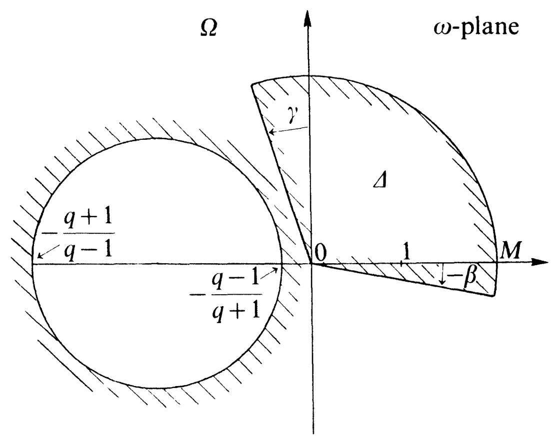
Next, we define

$$\Delta = \Delta(\beta,\gamma,M) = [\omega : \ -\beta<\arg\omega<\pi/2+\gamma, \ |\omega|<M] ,$$

where $\gamma+\beta<\pi/2$, $\beta>0$, $\gamma>0$, $M>1$. In addition γ is to be sufficiently small so that $\Delta\subset\Omega$. This is the case if

$$\gamma < \cos^{-1}\frac{2q}{1+q^2} .$$

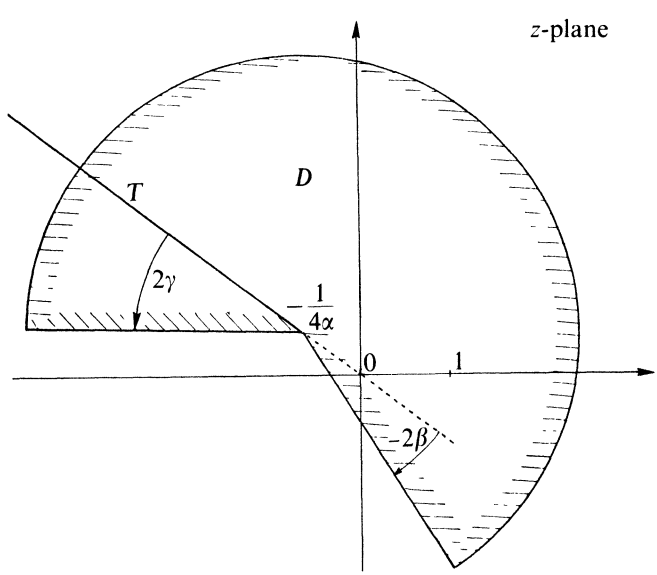
The figure below shows the two regions Ω and Δ.



To the region Δ for ω corresponds the region

$$D = \frac{1}{4\alpha}(\Delta^2 - 1) = \left[z : z = \frac{\omega^2 - 1}{4\alpha}, \ \omega \in \Delta \right] \tag{4.6}$$

in the z-plane. The region D is sketched below.



It was already known to Van Vleck in 1904 [8] that a limit periodic C-fraction converges for all z not on the ray

$$T = [z : z = -t/4\alpha,\ 1 \leqq t < \infty]$$

to a meromorphic function.

For the modified approximants $\{S_N^{(m)}(z, x(z))\}$ of the tails of (4.1), satisfying (4.3), it follows from Lemma 2.1 that they satisfy

$$|S_N^{(m)}(z, x(z)) - x(z)| < \frac{Q^{m+1}B}{|1 + x(z)|}$$

for all $z \in D$ and some m depending only on Q, B, β, γ, M. Hence

$$|S_N^{(m)}(z, x(z))| < \tfrac{1}{2} + M + \frac{Q^{m+1}B}{\frac{1}{2}\cos\gamma}, \qquad z \in D,\ N > m\,.$$

Next we show that at $z = 0$ the sequence $\{S_N^{(m)}(z, x(z))\}$ corresponds to the same power series to which

$$\underset{n=m+1}{\overset{\infty}{\mathrm{K}}} \left(\frac{\alpha_n z}{1}\right) \tag{4.7}$$

corresponds.

Using the binomial expansion of $(1 + 4\alpha z)^{\frac{1}{2}}$ one obtains

$$\begin{aligned} x(z) &= -\tfrac{1}{2} + \tfrac{1}{2} \sum_{n=0}^{\infty} \frac{\prod_{k=0}^{n-1} (\frac{1}{2} - k)}{n!} (4\alpha z)^n \\ &= \alpha z + \tfrac{1}{2} \sum_{n=2}^{\infty} \frac{\prod_{k=0}^{n-1} (\frac{1}{2} - k)}{n!} (4\alpha z)^n\,. \end{aligned}$$

From (3.1)

$$S_N^{(m)}(z, x(z)) - S_N^{(m)}(z, 0) = \frac{x(z)(-1)^{N-m} \prod_{n=m+1}^{N} (\alpha_n z)}{B_N^{(m)}(z)(B_N^{(m)}(z) + x(z) B_{N-1}^{(m)}(z))}\,,$$

so that, since in this case also $B_N^{(m)}(0) = 1$,

$$\lambda\big(L_0(S_N^{(m)}(z, x(z)) - S_N^{(m)}(z, 0))\big) = N - m + 1\,,$$

and our assertion follows. Using the result of Jones and Thron [3] again one can then conclude that $\{S_N^{(m)}(z, x(z))\}$ converges uniformly on compact subsets of D to a holomorphic function $f_m(z)$ which is such that P_m, the series to which the regular C-fraction (4.7) corresponds at $z = 0$, satisfies

$$P_m = L_0(f_m)\,.$$

Hence, in particular, $f_m(z)$ provides an analytic continuation beyond T of the function to which the regular C-fraction (4.7) converges in $\mathbf{C} \sim T$.

A completely analogous argument can be made with Δ replaced by its conjugate $\bar{\Delta}$ and D replaced by

$$\hat{D} = \frac{1}{4\alpha}((\bar{\Delta})^2 - 1).$$

Hence analytic continuation of f_m across the ray T "from below" is also possible. We have now proved the following result.

THEOREM 4.1. *Let a regular C-fraction satisfying*

$$|\alpha_n - \alpha| = \gamma_n Q^n, \quad n \geqq 1,\ 0 < Q < 1,\ \sum_{n=1}^{\infty} \gamma_n = B < \infty$$

be given and let

$$D = \left[z : z = \frac{\omega^2 - 1}{4\alpha}, -\beta < \arg \omega < \frac{\pi}{2} + \gamma, |\omega| < M \right],$$

where

$$0 < \gamma < \cos^{-1}\frac{2Q}{Q^2+1}, \ \beta > 0,\ \beta + \gamma < \frac{\pi}{2},\ M > 1.$$

Then there exists an m depending on Q, B, D such that the sequence

$$\{S_N^{(m)}(z, x(z))\}$$

of modified approximants of

$$\underset{n=m+1}{\overset{\infty}{\mathrm{K}}}\left(\frac{\alpha_n z}{1}\right) \tag{4.7}$$

converges uniformly on all compact subsets of D to a function f_m holomorphic for all $z \in D$. Further

$$P_m = L_0(f_m),$$

where P_m is the power series to which (4.7) *corresponds at $z=0$. Finally, $f_m(z)$ is the analytic continuation across T "from above" of the function to which the regular C-fraction* (4.7) *converges in* $\mathbb{C} \sim T$. *An analogous statement holds for*

$$\hat{D} = \frac{1}{4\alpha}((\bar{\Delta})^2 - 1)$$

and for analytic continuation across T "from below".

As in Section 3 there is in this case also a version of this result which asserts the convergence of the sequence of modified approximants of the whole C-fraction to a meromorphic function in the region

$$D^{+} = \left[z : z = (\omega^2 - 1)/4\alpha, \ -\pi/2 < \arg\omega - \cos^{-1}\frac{2Q}{Q^2+1} < \pi/2 \right],$$

which provides an analytic continuation across T of the meromorphic function to which the C-fraction is known to converge in $\mathbf{C} \sim T$.

We conclude this section by showing that there is a singularity of $f_m(z)$ at $z = -1/4\alpha$ and that it is a branch point of order 2.

Let $z \notin T$, $z \neq 0$ and put

$$l_i^{(m)}(z) = \lim_{N\to\infty} S_N^{(m)}(z, x_i(z)), \qquad l_i(z) = l_i^{(0)}(z), \qquad i = 1, 2 .$$

We shall show that

$$l_1(z) \neq l_2(z) .$$

Since z will be kept fixed in the argument we shall delete it from now on.

From Theorem 2.1 it follows that

$$\lim_{m\to\infty} l_i^{(m)} = x_i, \qquad i = 1, 2 .$$

Since $x_1 \neq x_2$ it then follows that for m sufficiently large $l_1^{(m)} \neq l_2^{(m)}$. Let k be a natural number such that $l_1^{(k)} \neq l_2^{(k)}$.

The linear fractional transformation

$$S_k(w) = \frac{A_k + wA_{k-1}}{B_k + wB_{k-1}}$$

is non-singular, since

$$A_k B_{k-1} - B_k A_{k-1} = (-1)^{k-1} z^k \prod_{\nu=1}^{k} \alpha_\nu \neq 0 ,$$

and hence it maps the Riemann sphere $1-1$ onto itself. Now

$$l_i = \lim_{N\to\infty} S_N(x_i) = \lim_{N\to\infty} S_k(S_N^{(k)}(x_i)) = S_k(l_i^{(k)}) .$$

Since $S_k(w)$ is $1-1$ and $l_1^{(k)} \neq l_2^{(k)}$ it follows that $l_1 \neq l_2$.

Hence continuation across T changes the value of the function so there must be a singular point at $z = -1/4\alpha$. As T is crossed $x_1(z)$ changes into $x_2(z)$ and if one continues in a path around $-1/4\alpha$ $x_2(z)$ *becomes* $x_1(z)$ again and hence the singularity is a branch point of order 2. This completes the proof of the following theorem.

THEOREM 4.2. *Let a regular C-fraction*

$$\underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{\alpha_n z}{1} \right)$$

satisfy

$$|\alpha_n - \alpha| = \gamma_n Q^n, \quad n \geqq 1,\ 0 < Q < 1,\ \sum_{n=1}^{\infty} \gamma_n = B < \infty .$$

Then the meromorphic function to which it converges in $\mathbf{C} \sim T$ *has a branch point of order* 2 *at* $z = -1/4\alpha$.

That limit periodicity alone is not sufficient for the conclusions of Theorem 4.2 to be valid is illustrated by

$$\log(1+z) = \frac{z}{1 + \mathbf{K}_{n=1}^{\infty}(\beta_n z/1)},$$

where

$$\beta_{2n+1} = \frac{1}{4} + \frac{1}{8n+4}, \quad n \geqq 0,$$

$$\beta_{2n} = \frac{1}{4} - \frac{1}{8n+4}, \quad n \geqq 1,$$

which has a logarithmic branch point at $z = -1$.

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