DERIVATIONS OF JORDAN C*-ALGEBRAS

HARALD UPMEIER*

0. Introduction.

The classical results of Kadison [13, 14] reveal a close connection between
the geometric structure (group of isometries, state space) of a C*-algebra and
its Jordan algebraic (quantum mechanical) structure. This feature is also
typical of the more general class of Jordan C*-algebras (JB*-algebras)
introduced by Kaplansky [30]. By complexification, JB*-algebras correspond
exactly to the JB-algebras investigated by Alfsen, Shultz and Størmer [1]. JB-
algebras are of interest in functional analysis, spectral theory and algebra
(formally real Jordan algebras). A promising application of JB-algebras is to be
found in complex analysis, based on the 1–1 correspondence between JB*-algebras
and bounded symmetric domains in complex Banach spaces with tube
realization [18,7].

Derivations of JB-algebras, which have been thoroughly studied in special
cases (for C*-algebras and for finite dimensional JB-algebras), are of particular
importance to the holomorphic automorphism group G of a bounded
symmetric domain Δ of tube type and to its Lie algebra g = aut (Δ) consisting
of all complete holomorphic vector fields on Δ. More precisely, g has a Cartan
decomposition g = f ⊕ p into the subalgebra f of all infinitesimal isometries and
the subspace p of all vector fields \((x - \langle xa^*z\rangle)\partial / \partial z\), where \(\langle xa^*z\rangle\) denotes the
triple product of the JB*-algebra Z associated with Δ [16, 17]. Further, the
self-adjoint part X of Z induces a decomposition f = im ⊕ aut (X), where m
consists of all Jordan multiplications by elements of X and aut (X) is the Lie
algebra of all derivations of X. The summands p and im of g are well-known; p
and Z are isomorphic as real Banach spaces and m is related to derivations of
self-dual Hilbert cones [8,3]. However, for JB-algebras X in general, little
seems to be known about the structure of aut (X). Therefore the purpose of this
paper is to study derivations of JB-algebras and to indicate their applications
to complex analysis.

Our first objective is to describe derivations of Jordan operator algebras
(JC-algebras) in terms of derivations of C*-algebras which are well-
understood. The restriction to JC-algebras is justified by the structure theory

* Research supported by the Deutsche Forschungsgemeinschaft.
Received June 25, 1979.
developed in [1], as indicated in section 1. The principal result of section 2 shows that for reversible JC-algebras $X$ each derivation of $X$ can be extended to a $*$-derivation of the C*-algebra generated by $X$, and is therefore implemented by a Hilbert space operator. This is a generalization of a theorem of Sinclair [24] on Jordan derivations of C*-algebras. For infinite dimensional spin factors the above extension property is not valid.

Our second problem is to clarify the relationship between $\text{aut}(X)$ and its ideal $\text{int}(X)$ of inner derivations. This problem is of relevance to complex analysis since $\text{int}(X)$ determines the subalgebra $[p,p]$ of $\mathfrak{f}$ generated by Lie products of vector fields in $p$. If $\dim(X) < \infty$, then it is well-known that $\text{aut}(X) = \text{int}(X)$ and therefore $\mathfrak{f} = [p,p]$. In section 3 we characterize those JBW-algebras (i.e. JB-algebras with predual) $X$ satisfying $\text{aut}(X) = \text{int}(X)$. It is shown in particular that purely exceptional JBW-algebras and reversible JW-algebras have only inner derivations. If $X$ is a JB-algebra, then $\text{int}(X)$ need not be uniformly dense in $\text{aut}(X)$, but it is shown in section 4 that $\text{aut}(X)$ is the strong operator closure of $\text{int}(X)$. Consequently each infinitesimal isometry of a bounded symmetric domain of tube type can be approximated pointwise by vector fields in $[p,p]$.

1. Preliminaries.

1.1. Definition. A JB-algebra is a real Banach Jordan algebra $X$ with unit $e$ and product $x \circ y$ such that $\|x \circ y\| \leq \|x\| \|y\|$ and $\|x\|^2 \leq \|x^2 + y^2\|$ whenever $x, y \in X$. A derivation of $X$ is a linear map $D: X \to X$ satisfying $D(x \circ y) = Dx \circ y + x \circ Dy$ for all $x, y \in X$. Let $\text{aut}(X)$ denote the Lie algebra of all derivations of $X$.

The set $\mathcal{C}(S, X)$ of all continuous maps from a compact space $S$ to a JB-algebra $X$ is a JB-algebra with pointwise algebraic operations and supremum norm. Each $x \in X$ generates a JB-subalgebra isomorphic to $\mathcal{C}(T, R)$, where $T$ is a compact space. Therefore the proof for C*-algebras [21; Lemma 4.1.3] can be modified to show that all derivations of a JB-algebra $X$ are bounded. If $\text{aut}(X)$ is equipped with the operator norm, there is an isomorphism of Lie algebras

\[(1.2) \quad \text{aut} \mathcal{C}(S, X) \approx \mathcal{C}(S, \text{aut} X)\]

(clear, if $\dim(X) < \infty$; in the general case, (1.2) follows from [29; Cor. 1.10]).

1.3. Definition. Let $H$ be a complex Hilbert space and let $\mathcal{H}(H)$ denote the JB-algebra of all bounded hermitian operators on $H$ with the product $x \circ y = (xy + yx)/2$. A uniformly (weakly) closed unital subalgebra of $\mathcal{H}(H)$ is called
a JC-algebra (JW-algebra) on $H$, respectively. The exceptional Jordan algebra of all self-adjoint $3 \times 3$-matrices with octonion entries is denoted by $\mathcal{H}_3(\mathcal{O})$.

By the deep results of Alfsen, Shultz and Størmer [1, 23], the study of JB-derivations can frequently be reduced to the case of JC-algebras: The second dual $X''$ of a JB-algebra $X$ is a JB-algebra with the Arens product and for each $D \in \text{aut} (X)$ the second transpose $D''$ is a derivation of $X''$. Further, each JB-algebra $X$ with predual has a decomposition

\begin{equation}
X = X_{sp} \oplus X_{ex},
\end{equation}

such that $X_{sp}$ can be realized as a JW-algebra and $X_{ex} \cong \mathcal{C}(S, \mathcal{H}_3(\mathcal{O}))$, where $S$ is a compact space [23; Th. 3.9]. Since the center of $X$ vanishes under aut ($X$), it follows from (1.2) and (1.4) that

\begin{equation}
\text{aut} (X) \cong \text{aut} (X_{sp}) \oplus \mathcal{C}(S, \text{aut} \mathcal{H}_3(\mathcal{O})).
\end{equation}

Finally, aut $\mathcal{H}_3(\mathcal{O})$ is a well-known classical Lie algebra (a real form of $F_4$ [11, p. 411]).

2. Extension of derivations of JC-algebras.

The most important examples of JC-algebras are provided by C*-algebras. Henceforth, derivations of associative *-algebras commuting with the involution are called *-derivations. C*-algebras $Z$ have the fundamental property, that each *-derivation $D$ of $Z$ is implemented by a Hilbert space operator $w$ lying in the weak closure of $Z$ [21; Cor. 4.1.7]:

\begin{equation}
Dz = [w, z] := wz - zw \quad \text{for all } z \in Z.
\end{equation}

Sinclair [24] has shown that each Jordan derivation of the self-adjoint part $X$ of $Z$ is actually induced by a *-derivation of $Z$. In this section this result is extended to a large class of JC-algebras $X$, if $Z$ is replaced by the C*-algebra generated by $X$. As a byproduct we obtain a new proof of the theorem of Sinclair.

2.2. Definition. A JC-algebra $X$ on a complex Hilbert space $H$ is said to have the extension property, if each derivation of $X$ can be extended to a *-derivation of the C*-algebra generated by $X$ on $H$.

Not all JC-algebras have the extension property, as the counterexample of infinite dimensional spin factors shows:

2.3. Example. If $X$ is a real Hilbert space of dimension $> 2$ and $e \in X$ is a unit vector with orthogonal complement $Y$, then $X = \mathbb{R}e \oplus Y$ with the product
\[(r_1 e + y_1) \circ (r_2 e + y_2) := (r_1 r_2 + (|y_1| y_2))e + r_1 y_2 + r_2 y_1\]

is a JB-algebra called a spin factor. The derivations of \(X\) are exactly the skew-adjoint operators on \(Y\). Now suppose that \(\dim (X) = \infty\) and consider a (faithful) JC-representation \(X = \mathcal{H}(H)\). By the universal property of the Clifford representation of \(X\), the C*-algebra \(Z\) generated by \(X\) on \(H\) can be identified with the Clifford C*-algebra of \(X\) [22]. Each derivation of the simple C*-algebra \(Z\) [22; Prop. 1] is inner [21; Th. 4.1.11]. Hence it follows from [2; Th. 4] that \(D \in \text{aut} (X)\) can be extended to a *-derivation of \(Z\) if and only if \(D\) is a trace-class operator.

**Reversible** JC-algebras \(X\) defined by the property
\[x_1, \ldots, x_n \in X \Rightarrow x_1 \cdot \ldots \cdot x_n + x_n \cdot \ldots \cdot x_1 \in X\]

and thoroughly studied by Størmer [26, 27] are complementary to spin factors in the following sense: By [26; Th. 6.4 and Th. 6.6], each JW-algebra \(X\) has a decomposition
\[(2.4) \quad X = X_{\text{rev}} \oplus X_2 ,\]

such that \(X_{\text{rev}}\) is reversible and \(X_2\) is a direct sum of JW-algebras isomorphic to \(L^\infty(S, U)\), where \(S\) is a measure space and \(U\) is a spin factor [33; Th. 2]. Further, a JW-factor of dimension \(\pm 3, 4, 6\) is a spin factor if and only if it is not reversible [26; Th. 7.1].

The following theorem is the main result of this section.

**2.5. Extension Theorem.** *Every reversible JC-algebra \(X\) on a complex Hilbert space \(H\) has the extension property, i.e. each derivation \(D\) of \(X\) can be extended to a *-derivation of the C*-algebra \(Z\) generated by \(X\) on \(H\).*

For the proof we need some facts on derivations of JW-factors. Throughout, let \(K\) be one of the skew fields \(R, C\) and \(H\) of real, complex and quaternion numbers, respectively. If \(E\) and \(F\) are \(K\)-Hilbert spaces (with scalar multiplication on the right), \(\mathcal{L}(E, F)\) denotes the Banach space of \(K\)-linear continuous maps from \(E\) to \(F\). Let \(z^* \in \mathcal{L}(F, E)\) be the adjoint of \(z \in \mathcal{L}(E, F)\). Then \(u^* v \in K\) is the inner product of \(u, v \in E = \mathcal{L}(K, E)\). Put
\[\mathcal{L}(E) := \mathcal{L}(E, E), \quad \mathcal{H}(E) := \{x \in \mathcal{L}(E) : x^* = x\}\]

and
\[\mathcal{S}(E) := \{w \in \mathcal{L}(E) : w^* = -w\} .\]

If \(r\) is a positive integer, define \(\mathcal{H}_r(K):= \mathcal{H}(K^r)\).
2.6. **Lemma.** Let $E$ be a $K$-Hilbert space. Then each derivation $D$ of $\mathcal{H}(E)$ has the form $Dx = [w, x]$ for all $x \in \mathcal{H}(E)$, where $w \in \mathcal{L}(E)$ satisfies $4\|w\| \leq 5\|D\|$.\\

**Proof of 2.6** (included for the sake of completeness). Put $r := \dim_K(E)$. If $r = 2$, then $\mathcal{H}(E)$ is a spin factor. Choose a spin system $e_1, \ldots, e_n$ [26; p. 181] and define

\begin{equation}
8w := \sum_{v=1}^{n} [De_v e_v].
\end{equation}

If $r \geq 3$, then $D$ is induced by a $*$-derivation, again denoted by $D$, of $\mathcal{L}(E)$ [11; p. 143, Cor. 3]. As in [15; Lemma 2] choose an orthogonal decomposition $E = K \oplus F$ and write the operators $z \in \mathcal{L}(E)$ as matrices

\[
z = \begin{pmatrix} a & f \\ v & A \end{pmatrix}, \text{ where } a \in K, v \in F, f \in \mathcal{L}(F, K) \text{ and } A \in \mathcal{L}(F).
\]

Define

\[
(a) := \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \quad (v) := \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix} \quad \text{and} \quad (A) := \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}.
\]

Applying $D$ to the identities $(1)^2 = (1), (a)(1) = (a) = (1)(a)$ and $(v)(1) = (v), (1)(v) = 0$, we get for some $u \in F$:

\[
D(1) = \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix}, \quad D(a) = \begin{pmatrix} \tilde{a} & au^* \\ ua & 0 \end{pmatrix} \quad \text{and} \quad D(v) = \begin{pmatrix} -u^*v & 0 \\ \tilde{v} & vu^* \end{pmatrix},
\]

where $a \mapsto \tilde{a}$ is a $*$-derivation of $K$ and $\|\tilde{v}\| \leq \|D\| \|v\|$. Hence, $\tilde{a} = [s, a]$ for all $a \in K$, where $s \in K$ is skew-adjoint. From $(v)(a) = (va)$ and $(v_1)(v_2) = (v_1^*v_2)$ it follows that $S := \tilde{v} + vs$ defines a bounded operator $S \in \mathcal{L}(F)$. Finally, $(1)(A) = 0 = (A)(1)$ and $(A)(v) = (Av)$ imply that

\[
D(A) = \begin{pmatrix} 0 & -u^*A \\ -Au & [S, A] \end{pmatrix}.
\]

Hence,

\[
w := \begin{pmatrix} s & -u^* \\ u & S \end{pmatrix} \in \mathcal{L}(E)
\]

satisfies $Dx = [w, x]$ for all $x \in \mathcal{H}(E)$.\\

In the complex case it is possible to choose $\|w\| \leq \|D\|$ by [25; Th. 4]. If $K \neq \mathbb{C}$, we may assume that $r = 2$ by considering a 2-dimensional subspace containing $h \in E$ and $wh$. From (2.7) it follows that $2\|w\| \leq \|D\|$ if $K = \mathbb{R}$ and $4\|w\| \leq 5\|D\|$ if $K = \mathbb{H}$. 

PROOF OF 2.5. For each pure state \( f \) of \( Z \), let \( \pi_f : Z'' \to \mathcal{L}(K_f) \) be the canonical \( \mathcal{W}^* \)-representation associated with the normal state \( f \) of the second dual \( Z'' \) [21; p. 41]. Let \( \sigma \) be the weak operator topology on \( \mathcal{L}(K_f) \). Then

\[
\pi_f(Z'') = \pi_f Z^\sigma = \mathcal{L}(K_f).
\]

If \( X'' \) is embedded in \( Z'' \), it can be proved as in [21; Prop. 1.16.2] that the unit ball of \( \pi_f(X'') \) is \( \sigma \)-compact. Hence it follows from the Kaplansky density theorem for Jordan algebras that

\[
Y := \pi_f(X'') = \pi_f X
\]
is reversible. By (2.9) and the proof of [21; Lemma 4.1.4], there is a commutative diagram

\[
\begin{array}{ccc}
X & \overset{D}{\longrightarrow} & X \\
\downarrow \pi_f & & \downarrow \pi_f \\
Y & \overset{\delta}{\longrightarrow} & Y
\end{array}
\]

where \( \delta \in \text{aut}(Y) \) satisfies \( \| \delta \| \leq \| D \| \). Since the complex algebra \( \mathbb{C}[X] \) generated by \( X \) is uniformly dense in \( Z \), it follows from (2.8) that \( \mathbb{C}[\pi_f X] \) is \( \sigma \)-dense in \( \mathcal{L}(K_f) \). Thus \( Y \) is a JW-factor acting irreducibly on \( K_f \). Hence \( Y \) is of type I by [27; Th. 4.1]. Let \( r \) be the degree of \( Y \). If \( r = 2 \), then \( Y \) is a spin factor of dimension \( \leq 6 \) by [26; Th. 7.1]. If \( r \geq 3 \), it follows from [26; Th. 3.9] that there exists a \( K \)-Hilbert space structure \( E \) on \( K_f \) such that \( Y = \mathcal{H}(E) \). From (2.7) and 2.6 we can deduce that

\[
\delta y = [w_f, y] \quad \text{for all } y \in Y,
\]

where \( w_f \in \mathcal{S}(K_f) \) satisfies \( 4\| w_f \| \leq 5\| D \| \). Denote by \( K \), \( \pi \) and \( w \) the direct sum (over all pure states \( f \) of \( Z \)) of \( K_f \), \( \pi_f \) and \( w_f \), respectively. Then (2.10) and (2.11) imply that \( \pi(Dx) = [w, \pi x] \) for all \( x \in X \). Hence \( [w, \pi z] \in \pi Z \) for all \( z \in Z \) and thus \( \pi(Dz) = [w, \pi z] \) defines a \( * \)-derivation of \( Z \) extending \( D \), since \( \pi : Z \to \mathcal{L}(K) \) is faithful.

2.12. COROLLARY (Sinclair [24]). Each Jordan derivation of a \( C^* \)-algebra is already a derivation of the associative product.

3. Inner derivations of JBW-algebras.

If \( B \) and \( C \) are subsets of a Lie algebra with bracket \( [\cdot, \cdot] \), let \( [B, C] \) denote the set of all finite sums of elements \( [b, c] \), where \( b \in B \) and \( c \in C \).

3.1. DEFINITION. Let \( X \) be a JB-algebra and put \( M := \{ xM : x \in X \} \), where \( xM \) denotes the multiplication operator defined by \( (xM)y := xy \) for all
The elements of the ideal \( \text{int} (X) := [m, m] \) of \( \text{aut} (X) \) are called \textit{inner derivations} of \( X \).

If \( \Delta \) is the bounded symmetric domain associated with a JB-algebra \( X \) (i.e. the open unit ball of the JB*-algebra \( Z := X \oplus iX \)) and if \( g = \mathfrak{t} \oplus \mathfrak{p} \) is the Cartan decomposition of \( g = \text{aut} (\Delta) \), then the map \( \lambda \mapsto ((\lambda e)M, \lambda - (\lambda e)M) \) induces decompositions

\[
[p, p] = \text{im} \oplus \text{int} (X) \subset \mathfrak{t} = \text{im} \oplus \text{aut} (X).
\]

Since the infinitesimal isometries in \([p, p]\) are explicitly known, the problem arises, to what extent \( \mathfrak{t} \) or \( \text{aut} (X) \) is determined by \([p, p]\) or \( \text{int} (X) \), respectively. If \( \dim (X) < \infty \), then the answer is well-known [6; p. 281, Satz 3.1]:

\[
(3.3) \quad \text{aut} (X) = \text{int} (X) \text{ and therefore } \mathfrak{t} = [p, p].
\]

For the self-adjoint part \( X \) of a C*-algebra, \( \text{aut} (X) \) does not agree with \( \text{int} (X) \) in general (see (4.1)). Therefore we consider in this section JB-algebras with predual (JBW-algebras) and determine completely those JBW-algebras which have only inner derivations. Recall that by (1.4) and (2.4), each JBW-algebra \( X \) has a canonical decomposition

\[
X = X_{\text{ex}} \oplus X_2 \oplus X_{\text{rev}},
\]

such that

(i) \( X_{\text{ex}} \) is a purely exceptional JBW-algebra,

(ii) \( X_2 \) is a JW-algebra of type I_\(2\) isomorphic to \( \oplus_{j \in J} L^\infty (S_j, U_j) \), where \( S_j \neq \varnothing \)

is a measure space and \( U_j \) is a spin factor for each \( j \in J \),

(iii) \( X_{\text{rev}} \) is a reversible JW-algebra.

3.5. THEOREM. Suppose \( X \) is a JBW-algebra with canonical decomposition (3.4). Then \( \text{aut} (X) = \text{int} (X) \) if and only if \( \sup_{j \in J} \dim U_j < \infty \).

The proof of 3.5 consists of several steps giving more precise information in special cases.

3.6. PROPOSITION. Each purely exceptional JBW-algebra \( X \) has only inner derivations.

\textbf{Proof.} Since \( X \approx C(S, \mathcal{M}_3 (\mathcal{O})) \) for some compact space \( S \), the assertion follows easily from (1.2).
3.7. Example. Let $X = \Re \oplus Y$ be a spin factor. Then int $(X)$ consists of finite rank operators since $[xM, yM]$ has rank $\leq 2$ for all $x, y \in X$. Conversely, if $D \in \text{aut} (X)$ has finite rank and if $e_1, \ldots, e_n$ is a Hilbert basis of $DX$, then

$$2D = \sum_{v=1}^{n} [(De_v)^*M, (e_v)M] \in \text{int} (X).$$

If a JB-algebra $X$ is given as in (3.4.ii), these remarks imply that $\text{aut} (X) = \text{int} (X)$ if and only if $\sup_{j \in J} \dim U_j < \infty$.

Each JW-algebra $X$ has a type decomposition of the form

$$I_{\text{fin}} \oplus I_{\infty} \oplus II_1 \oplus II_{\infty} \oplus III$$

(cf. [28; Th. 13]). $X$ is called properly non-modular, if its modular part $I_{\text{fin}} \oplus II_1$ vanishes. A real W*-algebra on a complex Hilbert space $H$ is a weakly closed self-adjoint real subalgebra $W$ of $\mathcal{L}(H)$. If $d$ is a cardinal number, let $\mathcal{M}_d(W)$ be the set of all bounded operators on the Hilbert sum of $d$ copies of $H$ which are matrices with entries in $W$.

3.8. Theorem. Let $X$ be a properly non-modular JW-algebra. Then each $D \in \text{aut} (X)$ is the sum of 6 commutators $[xM, yM]$, where $x, y \in X$. In particular, $\text{aut} (X) = \text{int} (X)$.

Proof. $X$ is reversible by [26; Th. 6.4 and Th. 6.6]. Applying [26; Lemma 6.1], [27; Lemma 2.3 and Th. 2.4] and the extension theorem 2.5, we may assume that $X$ is the self-adjoint part of a properly infinite real W*-algebra $W$ and that $Dx = [w, x]$ for all $x \in X$, where $w \in W$ is skew-adjoint. It follows from [4; p. 103, Th. 1] that there is a spatial *-isomorphism $\varphi: W \to \mathcal{M}_2(W)$. Define

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} := \varphi(w) \text{ and } A := \varphi^{-1}\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}.$$ 

Similarly, we may identify $W$ and $\mathcal{M}_{\mathcal{K}_0}(W)$ by a spatial *-isomorphism, such that

$$A = \begin{bmatrix} a & 0 & \ldots \\ b_1 & 0 & \ldots \\ b_2 & 0 & \ldots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$
Then $A=[P,Q]$ by [20; p. 512], where

$$
P := \begin{bmatrix}
0 & \cdots & 0 \\
1 & 0 & \cdots \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\vdots & \ddots & \vdots
\end{bmatrix}, \quad Q := \begin{bmatrix}
b_1 & -a & 0 & \cdots \\
b_2 & 0 & -a & \cdots \\
b_3 & 0 & 0 & -a \\
\vdots & \ddots & \ddots & \ddots
\end{bmatrix}.
$$

Put $2x_1 := P + P^*$. Now it can be verified that $P - P^* = 2[x_2, x_3]$, where

$$
x_2 := \begin{bmatrix}
0 \\
1 \\
0 \\
1 \\
0 \\ \vdots
\end{bmatrix}, \quad 2x_3 := \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & -1 & 0 \\
-1 & 0 & 1 & 0 \\
0 & \cdots & \cdots & \cdots
\end{bmatrix}.
$$

By the Jacobi identity, $A = [x_1, Q] + [x_2, [x_3, Q]] + [x_3, [Q, x_2]]$. Applying a similar argument to

$$
B := \varphi^{-1} \left( \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix} \right),
$$

we obtain $w = \sum_{v=1}^6 [x_v, w_v]$, where $x_v \in X$ and $w_v \in W$ satisfy $\|x_v\| \cdot \|w_v\| \leq 4\|w\|$. Put $2y_v := w_v + w_v^*$ for all $v$. Then

$$
D = 4 \sum_{v=1}^6 [x_v, M, y_v, M].
$$

3.9. Theorem. Each derivation $D$ of a reversible JW-algebra $X$ of type $I_{\text{fin}}$ can be written as $D = 4 \sum_{v=1}^5 [x_v, M, y_v, M]$, where $x_v, y_v \in X$ satisfy $\|x_v\| \cdot \|y_v\| \leq 5\|D\|$.

Proof. Suppose that $X = \mathcal{H}_r(K)$, where $r$ is a positive integer. Then it follows from 2.6 that $Dx = [w, x]$ for all $x \in X$, where $w$ is a skew-adjoint $r \times r$-matrix over $K$ satisfying $4\|w\| \leq 5\|D\|$. One can assume that $\|w\| \leq 1$.

If $K = \mathbb{R}$, we have

$$
\begin{bmatrix}
0 & -a \\
a & 0
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
0 & a
\end{bmatrix}, \quad \begin{bmatrix}
0 & a \\
0 & 0
\end{bmatrix}
$$

for all $a \in \mathbb{R}$

which shows that $w = [x, y]$, where $x, y \in X$ satisfy $\|x\| \cdot \|y\| \leq 1$. If $K \neq \mathbb{R}$, one can assume that $w$ is a diagonal matrix with diagonal entries $a_1, \ldots, a_r \in i\mathbb{R}$. If
trace \((w) = 0\), one can further assume that for each \(k \leq r\) the partial sums \(s_k := \sum_{j=1}^{k} a_j\) have modulus \(\leq 1\). Define \(P\) similar as in the proof of 3.8 and let \(Q\) be the \(r \times r\)-matrix with \(Q_{k,k+1} := -s_k\) for \(k < r\) and with zero entries elsewhere. Then \(w = [P, Q]\). Now argue as in the proof of 3.8. If \(p, q \in \mathcal{H}\) are skew-adjoint and \(\alpha \in \mathbb{R}\) satisfies \(2\alpha^2 = |p|^2 + |q|^2\), it follows that

\[
\begin{pmatrix}
0 & p \\
-p & 0
\end{pmatrix}
\begin{pmatrix}
0 & -q \\
q & 0
\end{pmatrix} =
\begin{pmatrix}
s & 0 \\
0 & 0
\end{pmatrix}
\text{ and }
\begin{pmatrix}
0 & p & \alpha \\
-p & 0 & -q \\
\alpha & q & 0
\end{pmatrix}
\begin{pmatrix}
\alpha & -q & 0 \\
q & 0 & -p \\
0 & p & -\alpha
\end{pmatrix} =
\begin{pmatrix}
s & 0 & 0 \\
0 & 2s & 0 \\
0 & 0 & s
\end{pmatrix},
\]

where \(s := [p, q]\). Applying these formulae to the remaining case of a scalar matrix \(w\), it is easy to show that 3.9 is true if \(X = \mathcal{H}_r(K)\).

Now suppose that \(X\) is a reversible JW-algebra of type \(I_{\text{fin}}\). Decomposing \(X\) into homogeneous JW-algebras with finite faithful normal trace and applying [12; Satz 36], we may assume that \(X = \mathcal{C}(S, U)\), where \(S\) is a Stone space and \(U\) is a reversible spin factor or \(U = \mathcal{H}_r(K)\) and \(r \geq 3\) is finite. Each \(D \in \text{aut}(X)\) can be viewed as a map \(D \in \mathcal{C}(S, \text{aut} U)\) by (1.2). If \(s \in S\), the above reasoning implies that

\[
D(s) = 4 \sum_{v=1}^{5} [x_v(s)M, y_v(s)M],
\]

where the maps \(x_v, y_v : S \rightarrow U\) satisfy

\[
\|x_v(s)\| \|y_v(s)\| \leq 5\|D\| \quad \text{for all } s \in S.
\]

By [9; Th. 2.5], one can assume that \(x_v, y_v \in X\) for all \(v\).

3.10. **Theorem.** Let \(X\) be a reversible JW-algebra. Then each derivation of \(X\) is inner, i.e. \(\text{aut}(X) = \text{int}(X)\).

**Proof.** By 3.8 and 3.9, it suffices to consider JW-algebras without type I summand. From the extension theorem 2.5 and [21; Th. 4.1.6] it follows that each \(D \in \text{aut}(X)\) has the form \(Dx = [w, x]\) for all \(x \in X\), where \(w = -w^*\) lies in the complex \(W^*\)-algebra \(W\) generated by \(X\). It has been shown in [31, 32], that each operator having central trace 0 in a finite \(W^*\)-algebra \(W\) is a sum of 10 commutators in \(W\) (the separability condition assumed in [31; Th. 4.1] is not necessary). Using this result and [20; p. 512], we may assume that \(w \in [W, W]\). By [26; Lemma 6.1] and [27; Lemma 2.3 and Th. 2.4], we may further assume that \(X\) is the self-adjoint part of a real \(W^*\)-algebra \(V\) and \(w \in [V, V]\). Since \(X\) is supposed to have no type I summand, \(V\) is of continuous type. Hence \(V\) is \(*\)-
isomorphic to \( \mathcal{M}_2(A) \), where \( A \) is a real \(*\)-algebra [4; p. 121], since \( w^* = -w \), the following Lemma implies that \( w \in [X, X] \) and therefore \( D \in \text{int}(X) \).

3.11. **Lemma.** Let \( A \) be an associative real \(*\)-algebra with unit 1 and denote by \( V \) the \(*\)-algebra of all \( 2 \times 2 \)-matrices over \( A \). Put

\[
S := \{ v \in V : v^* = -v \} \quad \text{and} \quad X := \{ v \in V : v^* = v \}.
\]

Then \([S, S] \subset [X, X]\).

**Proof.** If

\[
s = \begin{pmatrix} a & -c^* \\ c & b \end{pmatrix} \in [S, S],
\]

then \( a + b \) is a finite sum of commutators \([\alpha_1, \alpha_2]\) and \([\beta_1, \beta_2]\), where \( \alpha_1, \alpha_2 \in A \) are self-adjoint and \( \beta_1, \beta_2 \in A \) are skew-adjoint. Hence the assertion follows from the formulae

\[
\begin{pmatrix} 0 & -c^* \\ c & 0 \end{pmatrix} = \begin{bmatrix} (0, 0), (0, c^*) \\ (0, 1), (c, 0) \end{bmatrix},
\]

\[
2 \begin{pmatrix} a - b & 0 \\ 0 & b - a \end{pmatrix} = \begin{bmatrix} (0, a - b), (0, 1) \\ (b - a, 0), (1, 0) \end{bmatrix},
\]

\[
\begin{pmatrix} [\alpha_1, \alpha_2] & 0 \\ 0 & [\alpha_1, \alpha_2] \end{pmatrix} = \begin{bmatrix} (\alpha_1, 0), (\alpha_2, 0) \\ (0, \alpha_1), (0, \alpha_2) \end{bmatrix},
\]

\[
\begin{pmatrix} [\beta_1, \beta_2] & 0 \\ 0 & [\beta_1, \beta_2] \end{pmatrix} = \begin{bmatrix} (0, -\beta_2), (0, -\beta_1) \\ (\beta_2, 0), (\beta_1, 0) \end{bmatrix}.
\]

4. **Approximation by inner derivations.**

For JB-algebras in general, the solution of the problem, how to describe \( \text{aut}(X) \) in terms of \( \text{int}(X) \), requires topological arguments. If \( E \) is a Banach space, let \( \tau_E \) and \( \sigma_E \) denote the strong operator topology and the weak operator topology, respectively, on the algebra \( \mathcal{L}(E) \) of all bounded operators on \( E \). It is proved in this section that \( \text{aut}(X) \) is the closure of \( \text{int}(X) \) in the topology \( \tau_X \) of simple convergence on \( X \). By 3.7, \( \text{int}(X) \) is not uniformly dense in \( \text{aut}(X) \) if \( X \) is a spin factor of infinite dimension. Another example of this type is the following

4.1. **Example.** Let \( X \) be the self-adjoint part of the \( C^* \)-algebra

\[
Z := \{ c1_H + z : c \in C, \ z \ \text{compact operator on} \ H \}
\]
acting on a Hilbert space $H$ of dimension $\aleph_0$. Then

$$\text{aut} (X) = \{ x \mapsto [w, x] : w \in \mathcal{S} (H) \}.$$ 

Obviously $w \in \mathcal{S} (H)$ induces a (Jordan) inner derivation if and only if

$$w \in [X, X] \mod \mathbb{C} 1_H,$$

which by [19; Th. 1] is equivalent to $w \in Z$. Therefore $\text{int} (X)$ is uniformly closed but different from $\text{aut} (X)$.

Since JB-derivations are automatically bounded, the pointwise limit of a net of inner derivations lies in $\text{aut} (X)$. Thus $\tau_X$ seems to be a natural topology on $\text{aut} (X)$.

4.2. Approximation Theorem. For every JB-algebra $X$, $\text{aut} (X)$ is the closure of $\text{int} (X)$ in the strong operator topology $\tau_X$ (i.e. the topology of simple convergence on $X$).

Proof. Suppose that $X$ is a JW-algebra of type $I_2$. Choose $D \in \text{aut} (X)$ and $x_1, \ldots, x_m \in X$. We may assume that each $x_\mu$ has vanishing central trace (cf. [28; p. 40]). Put $x_m + \mu := Dx_\mu$ for all $\mu \leq m$. By [33; Th. 2], $X$ is a direct sum of JW-algebras isomorphic to $L^\infty (S, U)$, where $S$ is a measure space and $U$ is a spin factor. By the orthonormalization process, there exist $e_1, \ldots, e_{2m} \in X$ such that

$$x_v = \sum_{\mu = 1}^{2m} e_\mu \circ (e_\mu \circ x_v)$$

and $e_\mu \circ x_v$ are central for all $\mu, v \leq 2m$. This implies for all $v \leq m$:

$$2Dx_v = \sum_{\mu = 1}^{2m} [(D e_\mu) M, (e_\mu) M] x_v.$$

If $X$ is a JB-algebra with predual, the above argument together with 3.5 shows that $\text{aut} (X) = \text{int} (X)^{\tau_X}$. By [1; Prop. 3.9], the unit ball of a JB-algebra $X$ is strongly dense in the unit ball of $X''$. Since each $D \in \text{aut} (X)$ can be extended to a derivation of $X''$ and multiplication is jointly strongly continuous on bounded subsets of $X''$ by [1; Prop. 3.7], it follows that

$$\text{aut} (X) = \text{int} (X)^{\sigma_X} = \text{int} (X)$$

by [5; p. 77, Prop. 11].

4.3. Corollary. If $A$ is a bounded symmetric domain of tube type and if $g = \mathfrak{t} \oplus \mathfrak{p}$ is the Cartan decomposition of $g := \text{aut} (A)$, then $\mathfrak{t}$ is the closure of $[\mathfrak{p}, \mathfrak{p}]$ in the topology of simple convergence.
DERIVATIONS OF JORDAN C*-ALGEBRAS

REFERENCES

18. M. Koecher, An elementary approach to bounded symmetric domains, Rice University, Houston, 1969.