ON THE EQUATION \( a(x^m - 1)/(x - 1) = b(y^n - 1)/(y - 1) \)

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1.

The equation

\[
\frac{x^m - 1}{x - 1} = \frac{y^n - 1}{y - 1}
\]

has finitely many solutions in integers \( m > 1, n > 1, x > 1, y > 1 \) with \( x \neq y \) if

(a) \( x, y \) fixed (Thue [10], Siegel [9], Baker [1])
(b) \( x, n \) fixed (Siegel [8], Schinzel [7], Coates [3])
(c) \( m, n \) fixed and \( (m - 1, n - 1) > 2 \) (Schinzel [6]).
(d) \( m, n \) fixed (Davenport, Lewis and Schinzel [4]).

Let \( P \geq 2 \). Denote by \( S \) the set of all positive integers composed of primes not exceeding \( P \). Then we shall generalise (a) as follows:

**Theorem 1. The equation**

\[
\frac{a}{x^m - 1} = \frac{b}{y^n - 1}
\]

has finitely many solutions in integers \( m > 1, n > 1, x > 1, y > 1, a \geq 1, b \geq 1 \) with \( x, y, a, b \) in \( S \), \( (a, b) = 1 \) and \( a(y - 1) + b(x - 1) \). Further bounds for \( m, n, x, y, a \) and \( b \) can be determined explicitly in terms of \( P \).

Thus there are only finitely many integers with all the digits equal to \( a \) in their \( x \)-adic expansions and all the digits equal to \( b \) in their \( y \)-adic expansions. For integers with digits equal to 0 and 1 that occur periodically in their \( g \)-adic expansions, theorem 1 gives the following:

**Corollary.** Let \( g > 1 \) and \( g_1 > 1 \) be multiplicatively independent integers. Then the equation

\[
\frac{g^{uv} - 1}{g^u - 1} = \frac{g_1^{uv_1} - 1}{g_1^{u_1} - 1}
\]

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has finitely many solutions in integers $u \geq 1$, $u_1 \geq 1$, $v > 1$, $v_1 > 1$. Further bounds for $u, u_1, v, v_1$ can be given explicitly in terms of the greatest prime factor of $gg_1$.

2.

The proof of theorem 1 depends on linear forms in logarithms. Let $\alpha_1, \ldots, \alpha_n$ be non-zero rational numbers of heights not exceeding $A_1, \ldots, A_n$ respectively, where we assume that $A_j \geq 3$ for $1 \leq j \leq n$. (The height of a rational number $m/n$ with $(m,n)=1$ is defined as $\max(|m|,|n|)$). Put

$$\Omega' = \prod_{j=1}^{n-1} \log A_j \quad \text{and} \quad \Omega = \Omega' \log A_n.$$ 

**Theorem A (Baker [2]).** There exist effectively computable absolute constants $c_1 > 0$ and $c_2 > 0$ such that the inequalities

$$0 < |\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1| < \exp \left( - (c_1n)^{c_2n} \Omega \log \Omega' \log B \right)$$

have no solution in rational integers $b_1, \ldots, b_n$ with absolute values at most $B$ ($\geq 2$).

**Theorem B (van der Poorten [5]).** Let $p > 0$ be a prime number. There exist effectively computable absolute constants $c_3 > 0$ and $c_4 > 0$ such that the inequalities

$$\infty > \operatorname{ord}_p (\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1) > (c_3n)^{c_4n} \Omega (\log B)^2 \frac{p}{\log p}$$

have no solution in rational integers $b_1, \ldots, b_n$ with absolute values at most $B$ ($\geq 2$).

We note that we could have used older results in place of theorem A and B. In fact we shall apply theorem A with $n, A_1, \ldots, A_{n-1}$ fixed and theorem B with $n, p, A_1, \ldots, A_n$ fixed. Further we remark that we shall use theorem A thrice and theorem B twice in the proof of theorem 1.

3.

In this section, we shall prove theorem 1. Let $m, n, x, y, a, b$ be as in the theorem. Assume that they satisfy (1). Write

$$x = p_1^{s_1} \cdots p_s^{s_1}, \quad y = p_1^{s_1} \cdots p_s^{s_2},$$

$$a = p_1^{s_1} \cdots p_s^{s_n}, \quad b = p_1^{b_1} \cdots p_s^{b_n}.$$
Here $p_1, \ldots, p_s$ are primes $\leq P$ and the exponents of $p_1, \ldots, p_s$ in the factorisation of $x, y, a, b$ are non negative integers not exceeding $2 \log x, 2 \log y, 2 \log a, 2 \log b$ respectively. It is no loss of generality to assume that $m \geq n$. Further the equation (1) gives

$$ax^{m-1} < nby^{n-1}, \quad by^{n-1} < max^{m-1}.$$  \hspace{1cm} (2)

Denote by $c_5, c_6, \ldots$ effectively computable positive constants depending only on $P$. Then we have:

**Lemma 1.** $\max (\log a, \log b) \leq c_5 (\log (m \log x))^2$.

**Proof.** First we prove the inequality for $\log b$. Suppose that a prime $p$ divides $b$. Then, by (1), we have

$$\text{ord}_p (b) \leq \text{ord}_p \left( a \frac{x^{m-1}}{x-1} \right) \leq \text{ord}_p (x^{m-1}) = \text{ord}_p (p_1^{mx_1} \cdots p_s^{mx_s} - 1).$$

Now we apply theorem B with $n= s \leq P, \ A_1 = A_2 = \ldots = A_s = P$ and $B = 2m \log x$ to the right hand side of the above inequality. We obtain

$$\text{ord}_p (b) \leq c_6 (\log (m \log x))^2,$$

and hence

$$\log b = \sum_{p \mid b} \text{ord}_p (b) \log p \leq c_7 (\log (m \log x))^2.$$

Similarly

$$\log a \leq c_8 (\log (n \log y))^2.$$

In view of (2), the lemma follows immediately.

**Lemma 2.**

$$\min (\log x, \log y) \leq c_9 \log m.$$  \hspace{1cm} (3)

**Proof.** We prove the lemma when $x \leq y$. The proof is similar for the case $x \geq y$. Let $\delta$ be the smallest positive integer such that $ax^{m-\delta} + by^{n-\delta}$. Observe that $\delta \leq n$, since $(a, b) = 1$. Now it follows from (1) and (2) that

$$0 < |ax^{m-\delta} - by^{n-\delta}| \leq max^{m-1-\delta} + nby^{n-1-\delta} \leq 2m^2ax^{m-1-\delta}.$$

Thus we have

$$0 < |p_1^{\delta_1} \cdots p_s^{\delta_s} - 1| \leq 2m^2x^{-1}.$$  \hspace{1cm} (4)
where \( u_i = b_i - a_i + (n - \delta)y_i - (m - \delta)x_i \) for \( i = 1, \ldots, s \). By lemma 1 and (2), we find that the integers \( |u_i| \) do not exceed \( c_{10}m \log x \). Now apply theorem A with \( n = s \leq P, A_1 = A_2 = \ldots = A_s = P \) and \( B = c_{10}m \log x \) to obtain

\[
|p_1^{u_1} \cdots p_s^{u_s} - 1| > (m \log x)^{-c_{11}}.
\]

Now by combining (5) and (4), we get (3). This completes the proof of lemma 2.

**Lemma 3.** \( \max (\log x, \log y) \leq c_{12}(\log m)^2 \).

**Proof.** We prove the lemma when \( x \leq y \). The proof is similar for the case \( x \geq y \). From (1), lemma 1 and (2), we have

\[
0 = \frac{ax^m}{x-1} - by^n = \frac{a}{x-1} + b(y^{n-2} + \ldots + 1)
\leq a + nby^{n-2} \leq y^{n-2}\exp(c_{13}(\log(n\log y))^2).
\]

Thus

\[
0 \geq (p_1^{u_1} \cdots p_s^{u_s}(x-1)^{-1} - 1) \leq y^{-1}\exp(c_{13}(\log(n\log y))^2),
\]

where \( u_i = a_i - b_i + mx_i - (n-1)y_i \) for \( i = 1, \ldots, s \). From lemma 1 and (2), we observe that the absolute values of \( u_i \) with \( i = 1, \ldots, s \) do not exceed \( c_{14}m \log x \) which, in view of lemma 2, is less than \( c_{15}m \log m \). Now apply theorem A with \( n = s + 1 \leq P + 1, A_1 = A_2 = \ldots = A_s = P, A_{s+1} = x \leq m c_{15} \) and \( B = c_{15}m \log m \) to conclude that

\[
(p_1^{u_1} \cdots p_s^{u_s}(x-1)^{-1} - 1) \geq \exp(-c_{16}(\log m)^2).
\]

Observe that we have used lemma 2 for a bound for \( A_{s+1} \). Now the lemma follows immediately from (6) and (7).

**Proof of theorem 1.** From (1) and lemma 1, we have

\[
0 \geq \left| \frac{ax^m}{x-1} - \frac{by^n}{y-1} \right| = \left| \frac{a}{x-1} - \frac{b}{y-1} \right| \leq \exp(c_{17}(\log(m\log x))^2).
\]

Thus

\[
0 < \left| p_1^{w_1} \cdots p_s^{w_s} \frac{x-1}{y-1} - 1 \right| \leq c_{18}x^{-m+1},
\]

where \( w_i = b_i - a_i + ny_i - mx_i \). Observe that the absolute values of the integers \( w_i \) do not exceed \( c_{19}m \log m \). Now apply theorem A with \( n = s + 1 \leq P + 1, A_1 = A_2 = \ldots = A_s = P, B = c_{19}m \log m \) and \( A_{s+1} = \max(x-1, y-1) \leq \exp(c_{12}(\log m)^2) \).
The last inequality follows from lemma 3. We obtain

\begin{equation}
\left| p_1^{w_1} \ldots p_s^{w_s} \frac{x-1}{y-1} - 1 \right| > \exp \left( -c_{20}(\log m)^3 \right).
\end{equation}

Combining (9) and (8), we find that \( m \leq c_{21} \). Now the proof of the theorem is complete in view of lemma 2 and lemma 1.

 Remark. The proof of theorem 1 depends on three approximations that the equation (1) provides. If \( a = 1, b = 1 \), it is sufficient to use only one approximation. Rewriting the equation

\[ x \frac{x^{m-1} - 1}{x-1} = y \frac{y^{n-1} - 1}{y-1}, \]

observe that \( \text{ord}_p(x) = \text{ord}_p(y) \) for every prime \( p \) dividing \( (x, y) \). Thus we can write \( x = dx_1, y = dy_1 \) with \( (x_1, y_1) = (d, x_1) = (d, y_1) = 1 \). From the equation

\[ x^{m-1} + \ldots + x^n = (y-x) \left( \frac{y^{n-1} - x^{n-1}}{y-x} + \ldots + 1 \right) \]

and theorem B, we show that \( \log d \leq c_{22}(\log \log y)^2 \). Now using the equation

\[ x_1 \frac{x_1^{m-1} - 1}{x-1} = y_1 \frac{y_1^{n-1} - 1}{y-1} \]

and theorem B along with the above bound for \( \log d \), we find that \( \log y \leq c_{23}(\log m)^2 \). This is lemma 3. Now use an approximation, as in theorem 1, to complete the proof.

REFERENCES

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