A II$_1$ FACTOR ANTI-ISOMORPHIC TO ITSELF BUT WITHOUT INVOLUTORY ANTIAUTOMORPHISMS

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Abstract.

We construct a type II$_1$ factor $\mathcal{A}$ which is anti-isomorphic to itself but has no involutory anti automorphisms. The proof uses an invariant $\kappa(\theta)$ for elements $\theta$ in Connes' group $\chi(\mathcal{A})$. We use $\kappa(\theta)$ to show that if $\mathcal{M}$ is a II$_1$ factor without non-trivial hypercentral sequences and $\theta$ is an element of $\chi(\mathcal{M})$, then $\gamma(\theta) = \pm 1$, where $\gamma$ is Connes' invariant for elements of $\text{Out} \, \mathcal{M}$. We give an example which shows that $\gamma(\theta)$ can be $-1$ for $\theta$ in $\chi(\mathcal{M})$.

1. Introduction.

An antiisomorphism of a von Neumann algebra is a vector space isomorphism $\Phi: \mathcal{M} \rightarrow \mathcal{M}$ with $\Phi(a^*) = \Phi(a)^*$ and $\Phi(ab) = \Phi(b)\Phi(a)$.

If $\mathcal{M}$ is a factor, there are many mathematical objects associated with $\mathcal{M}$, say $\mathcal{O}_{\mathcal{M}}$, satisfying the following type of theorem: "If $\psi: \mathcal{O}_{\mathcal{M}} \rightarrow \mathcal{O}_{\mathcal{N}}$ is an isomorphism preserving the structure in $\mathcal{O}_{\mathcal{M}}$, there is an isomorphism or antiisomorphism $\varphi: \mathcal{M} \rightarrow \mathcal{N}$ inducing $\psi". Examples for $\mathcal{O}$ are: the projection lattice [7], the unitary group [7], the Lie algebra [12], the Jordan algebra [11] and the automorphism group [9]. It was important to know whether all von Neumann algebras were antiisomorphic to themselves or not. This question was answered in the negative by Connes in [2] and [3] where he gave first a type III factor and then a type II$_1$ factor which were not antiisomorphic to themselves.

If $\mathcal{O}$ is a functor as above, there will be an exact sequence $1 \rightarrow \text{Aut} \mathcal{M} \rightarrow \text{Aut} \mathcal{O}_{\mathcal{M}} \rightarrow \mathcal{H} \rightarrow 1$, where $\mathcal{H}$ is $\mathbb{Z}_2$ when $\mathcal{M}$ is antiisomorphic to itself and 1 otherwise. It is natural to ask the question whether this exact sequence splits when $\mathcal{H} = \mathbb{Z}_2$. This is equivalent to the question "Does $\mathcal{M}$ possess an involutory antiautomorphism?". In this paper we answer this question in the negative by exhibiting a II$_1$ factor $\mathcal{A}$ which is antiisomorphic to itself but has no involutory antiautomorphisms. The method we use is a development of [3] so we begin by fixing notation and explaining some terms used in [3].

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If $\text{Aut} \mathcal{M}$ is the automorphism group of a $\text{II}_1$ factor $\mathcal{M}$ (always supposed to have separable predual, trace $\tau$ and trace-norm $\| - \|_2$), give it the topology of pointwise $\| - \|_2$ convergence on $\mathcal{M}$. The subgroup $\text{Int} \mathcal{M}$ of inner automorphisms is not necessarily closed so let $\overline{\text{Int} \mathcal{M}}$ be its closure. An automorphism $\psi \in \text{Aut} \mathcal{M}$ is said to be centrally trivial if, given a $\| - \|$-bounded sequence $(x_n)$ in $\mathcal{M}$ with $\|[x_n, y]\|_2 \to 0$ as $n \to \infty$ for every $y \in \mathcal{M}$ (i.e. a central sequence), we have $\lim_{n \to \infty} \|\psi(x_n) - x_n\|_2 = 0$. Centrally trivial automorphisms form a normal subgroup $\text{Ct} \mathcal{M}$ of $\text{Aut} \mathcal{M}$ and if $\epsilon: \text{Aut} \mathcal{M} \to \text{Out} \mathcal{M}$ is the quotient map onto $\text{Aut} \mathcal{M}/\text{Int} \mathcal{M}$, we define $\chi(\mathcal{M}) = \epsilon(\text{Ct} \mathcal{M} \cap \overline{\text{Int} \mathcal{M}})$. In [4] Connes shows that $\chi(\mathcal{M})$ is abelian. It is a powerful isomorphism invariant for $\mathcal{M}$ and can be just about any abelian group.

For the most familiar $\text{II}_1$ factors (e.g. the hyperfinite one and those associated with free groups), $\chi(\mathcal{M})$ is trivial. The same is true for tensor products (finite or infinite) of these factors. The easiest way to construct factors $\mathcal{M}$ with non-trivial $\chi(\mathcal{M})$ is using crossed products by actions of finite groups. This is the method we shall employ.

In section 2 we shall define a conjugacy (in $\text{Out} \mathcal{M}$) invariant $\chi(\theta)$ for elements $\theta$ of $\chi(\mathcal{M})$, which is a complex number of modulus 1 when $\mathcal{M}$ has no non-trivial hypercentral sequences (a central sequence $(y_n)$ is called hypercentral when $\lim_{n \to \infty} \|[y_n, x_n]\|_2 = 0$ for every central sequence $(x_n)$). This invariant $\chi$ is sensitive to conjugation by antiautomorphisms and will be used to prove that $\mathcal{A}$ has no non-trivial antiautomorphisms. It is defined using the action (or rather the lack of action) of $\theta$ on central sequences. We could have defined it for arbitrary factors, but we prefer to keep technical simplicity by supposing that $\mathcal{M}$ is a $\text{II}_1$ factor without non-trivial hypercentral sequences.

The group $\chi(\mathcal{M})$, being a subgroup of $\text{Out} \mathcal{M}$, gives an example of a Q-kernel in the sense of [8] or [16]. Thus one can associate an element of $H^3(\chi(\mathcal{M}), \mathbb{T})$ with $\chi(\mathcal{M})$, where $\mathbb{T}$ is the circle group and the action of $\chi(\mathcal{M})$ on $\mathbb{T}$ is trivial. This is done as follows. For each element $\theta$ of $\chi(\mathcal{M})$ choose an $\alpha_\theta \in \text{Aut} \mathcal{M}$ with $\epsilon(\alpha_\theta) = \theta$. Once this choise is made, we have

$$\alpha_\theta \alpha_v = \text{Ad} u(\theta, v) \alpha_{\theta v}$$

for $\theta, v \in \chi(\mathcal{M})$ and unitaries $u(\theta, v)$ in $\mathcal{M}$. Associativity implies

$$\text{Ad} (u(\theta, v)u(\theta v, \eta)) = \text{Ad} (\alpha_\theta(u(v, \eta))u(\theta, v\eta)) .$$

Thus

$$u(\theta, v)u(\theta v, \eta) = \gamma(\theta, v, \eta)\alpha_{\theta}(u(v, \eta))u(\theta, v\eta)$$

for some $\gamma(\theta, v, \eta) \in \mathbb{T}$. The function $\gamma: \chi(\mathcal{M}) \times \chi(\mathcal{M}) \times \chi(\mathcal{M}) \to \mathbb{T}$ is seen to be a 3-cocycle and its cohomology class $\Omega_\mathcal{M}$ in $H^3(\chi(\mathcal{M}), \mathbb{T})$ does not depend on the choices made. Hence there is a further invariant $\Omega_\mathcal{M}$ for $\mathcal{M}$. 

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It is not yet known which elements of $H^3(A,T)$ can occur as $\Omega_{M}$ for $A = \chi(M)$ and some fixed abelian group $A$, but there are severe restrictions. For example, when $\chi(M)$ is cyclic, $\Omega_{M}$ is just the invariant $\gamma(\theta)$ of $\theta$ for a generator $\theta$ of $\chi(M)$. The interplay between $\chi(\theta)$ and $\gamma(\theta)$ allows us to show that $\gamma(\theta) = \pm 1$, whereas a priori one might have thought that $\gamma(\theta)$ could be any $n$th root of unity. So in this case $\Omega_{M}$ can take at most two values. It also follows that $\gamma(\theta)$ is insensitive to conjugation by antiautomorphisms and is useless for distinguishing between an algebra and its opposite algebra, at least for $\gamma(\theta)$.

In section 6 we give an example showing that $\gamma(\theta)$ can be $-1$ for $\theta \in \chi(M)$, and hence that $\Omega_{M}$ can be non-zero.

A further word about central sequences and ultrafilters is in order. Let $\mathfrak{I}_{\infty}$ be the ideal of $l^\infty(N,M)$ consisting of all bounded sequences $(x_n)$ with $\lim_{n \to \infty} \|x_n\|_2 = 0$. The quotient C*-algebra $l^\infty(N,M)/\mathfrak{I}_{\infty}$ is written $M^\infty$. Note that $M$ is embedded in $M^\infty$ as constant sequences. The commutant $M' \cap M^\infty$ is denoted $M_\infty$ and is the algebra of central sequences. One may repeat this same process for a free ultrafilter $\omega$ on $N$ instead of $\infty$ (i.e. the Frechet filter). One obtains algebras $M^\omega$ with $M \subset M^\omega$ and $M_\omega$ as $M' \cap M^\omega$. The advantage of an ultrafilter is that $M^\omega$ and hence $M_\omega$ are von Neumann algebras. Automorphisms $\alpha$ of $M$ induce automorphisms of $M^\infty$, $M^\omega$, $M_\infty$, and $M_\omega$, which will all also be written $\alpha$. If $(x_n)$ is a bounded sequence in $M$, $[x_n]$ will denote its image in $M^\infty$ or $M^\omega$. It is important to note that if $\alpha = \lim_{n \to \infty} \text{Ad} u_n$ for $\alpha \in \text{Aut} M$ and unitaries $u_n$ in $M$, and if $U = [u_n]$, then $\alpha = \text{Ad} U$ on $M \subset M^\infty$, i.e. if $X \in M \subset M^\infty$, $\alpha(X) = UXU^*$. The same for $\omega$ in the place of $\infty$.

If $M$ is an algebra, $Z(M)$ will denote its centre. To say that a sequence $(x_n)$ is hypercentral is just to say that $[x_n] \in Z(M_\omega)$. To say that $M$ has no non-trivial hypercentral sequences means that if $[x_n] \in Z(M_\omega)$, then $[x_n] = [\lambda_n]$ for a bounded sequence of scalars $\lambda_n \in \mathbb{C}$. If $\omega$ is an ultrafilter, such a sequence has a unique limit an $n \to \omega$, so that “$M$ has no non-trivial hypercentral sequences” is the same as saying “$M_\omega$ is a factor” for some (and hence all) free ultrafilter $\omega$ on $N$.

To say that an automorphism $\alpha$ is centrally trivial is the same as saying $\alpha = \text{id}$ on $M_\infty$ (or $M_\omega$).

2. An invariant for elements of $\chi(M)$.

In this section we shall define an invariant $\kappa(\theta)$ for elements $\theta \in \chi(M)$ and examine its relationship with the invariant $\gamma(\theta)$ of [6]. We also show how $\kappa(\theta)$ behaves under conjugation by antiautomorphisms. We develop the definition of $\kappa$ in three lemmas.
Lemma 2.1. Let $\alpha \in \text{Int} \, \mathcal{M} \cap \text{Ct} \, \mathcal{M}$ and let $\alpha = \text{Ad} \, U$ for $U \in \mathcal{M}^\infty$ on $\mathcal{M} \subset \mathcal{M}^\infty$. Then $U^* \alpha(U) \in Z(\mathcal{M}_\infty)$.

Proof. Let $X \in \mathcal{M} \subset \mathcal{M}^\infty$. Then $X = \alpha(U^*XU) = \alpha(U^*)\alpha(X)\alpha(U) = \alpha(U^*)UXU^*\alpha(U)$. Thus $U^*\alpha(U) \in \mathcal{M}^\infty = \mathcal{M}' \cap \mathcal{M}^\infty$. Moreover if $Y \in \mathcal{M}^\infty$, $YU^*\alpha(U) = U^*(UYU^*)\alpha(U) = U^*\alpha(U)Y$, since $UYU^* \in \mathcal{M}^\infty$ and $\alpha \in \text{Ct} \, \mathcal{M}$. Thus $YU^*\alpha(U) = U^*\alpha(U)Y$ for all $Y \in \mathcal{M}^\infty$, and $U^*\alpha(U) \in Z(\mathcal{M}_\infty)$.

Lemma 2.2. Let $\alpha$ and $U$ be as in lemma 2.1 and suppose $\mathcal{M}$ has no non-trivial hypercentral sequences. Then there is a $\lambda_\alpha \in \mathbb{T}$ such that $\alpha(U) = \lambda_\alpha U$. This $\lambda_\alpha$ does not depend on the choice of $U$ with $\alpha = \text{Ad} \, U$ on $\mathcal{M} \subset \mathcal{M}^\infty$.

Proof. Since there are no hypercentral sequences, we know that there is a bounded sequence $\lambda_n$ with $\lim_{n \to \infty} \| \lambda_n 1 - u_n^* \alpha(u_n) \|_2 = 0$, where $(u_n)$ is a representing sequence for $U$. We shall show that $\lambda_n$ converges by showing that it has at most one accumulation point.

Suppose $\lambda$ and $\mu$ are two accumulation points for the sequence $(\lambda_n)$. Then let $(n_i)$ and $(m_i)$ be infinite sequences with $\lim_{i \to \infty} \lambda_{n_i} = \lambda$ and $\lim_{i \to \infty} \lambda_{m_i} = \mu$. Let $V$ and $W \in \mathcal{M}^\infty$ be defined by $V = [u_{n_i}]$ and $W = [u_{m_i}]$. Clearly $\alpha(V) = \lambda V$ and $\alpha(W) = \mu W$ so that $\alpha(VW^*) = \lambda \mu VW^*$. But $\text{Ad} \, V = \text{Ad} \, W = \text{Ad} \, U$ on $\mathcal{M} \subset \mathcal{M}^\infty$ so that $VW^* \in \mathcal{M}^\infty$. Since $\alpha = \text{id}$ on $\mathcal{M}^\infty$, $\lambda \mu = 1$ so $\lambda = \mu$.

The same argument shows that $\lambda_\alpha$ does not depend on $U$ with $\alpha = \text{Ad} \, U$ on $\mathcal{M} \subset \mathcal{M}^\infty$.

Lemma 2.3. If $\alpha$, $U$ and $\mathcal{M}$ are as in lemma 2.2, and $v$ is a unitary in $\mathcal{M}$, $\lambda_\alpha = \lambda_{\text{Ad} \, v \alpha}$.

Proof. We have $\text{Ad} \, v \alpha = \text{Ad} \, (vU)$ on $\mathcal{M} \subset \mathcal{M}^\infty$. By definition,

$$\lambda_{\text{Ad} \, v \alpha} \alpha(U) = \text{Ad} \, v \alpha(U) = v \alpha(U^*U) = vU^*U^*\alpha(U)v^* = \lambda_\alpha vU$$

Hence $\lambda_\alpha = \lambda_{\text{Ad} \, v \alpha}$.

Definition 2.4. Let $\mathcal{M}$ be a II$_1$ factor without non-trivial hypercentral sequences and $\theta \in \chi(\mathcal{M})$. Then $\alpha(\theta) = \lambda_\alpha$ for any $\alpha$ with $\varepsilon(\alpha) = \theta$. This means that if $u_n$ are unitaries with $\alpha = \lim_{n \to \infty} \text{Ad} \, u_n$, $\lim_{n \to \infty} \| \alpha(u_n) - \lambda_\alpha u_n \|_2 = 0$. (Such a sequence of unitaries exists since $\alpha \in \text{Int} \, \mathcal{M}$.)

Each automorphism $\beta \in \text{Aut} \, \mathcal{M}$ determines an automorphism $\beta$ of $\chi(\mathcal{M})$ by conjugation, i.e. $\beta(\theta) = \varepsilon(\beta) \theta \varepsilon(\beta^{-1})$. The next lemma shows that $\varepsilon$ is a conjugacy invariant.
Lemma 2.5. If $\beta \in \text{Aut } \mathcal{M}$ and $\theta \in \chi(\mathcal{M})$, then $\kappa(\theta) = \kappa(\beta(\theta))$.

Proof. If $\alpha$ is such that $\varepsilon(\alpha) = \theta$ and $\alpha = \text{Ad } U$ for $U \in \mathcal{M}^\infty$ on $\mathcal{M} \subset \mathcal{M}^\infty$, then $\beta \alpha \beta^{-1} = \text{Ad } \beta(U)$ on $\mathcal{M} \subset \mathcal{M}^\infty$, and $\beta \alpha \beta^{-1}(\beta(U)) = \lambda_\alpha \beta(U)$.

Each antiautomorphism $\Phi$ of $\mathcal{M}$ determines an automorphism $\tilde{\Phi}$ of $\chi(\mathcal{M})$ by conjugation, i.e. $\tilde{\Phi}(\theta) = \varepsilon(\Phi \alpha \Phi^{-1})$ where $\varepsilon(\alpha) = \theta$.

Lemma 2.6. If $\Phi$ is an antiautomorphism of $\mathcal{M}$ and $\theta \in \chi(\mathcal{M})$, then $\kappa(\theta) = \kappa(\tilde{\Phi}(\theta))$.

Proof. Exactly as for 2.5 except that $\Phi \alpha \Phi^{-1} = \text{Ad } (\Phi(U^*))$.

We will also be interested in $\kappa(\theta^{-1})$.

Lemma 2.7. If $\theta \in \chi(\mathcal{M})$, $\kappa(\theta^{-1}) = \kappa(\theta)$.

Proof. If $\varepsilon(\alpha) = \theta$ and $\alpha = \text{Ad } U$ on $\mathcal{M} \subset \mathcal{M}^\infty$, $\alpha^{-1} = \text{Ad } U^*$ and $\alpha^{-1}(U^*) = \lambda_\alpha U^*$.

Remember that for $\theta \in \text{Out } \mathcal{M}$, $\gamma(\theta)$ is defined by $\alpha(v) = \gamma(\theta)v$ where $\varepsilon(\alpha) = \theta$ and $\alpha^n = \text{Ad } v$ with $n =$ period of $\theta$.

Lemma 2.8. Let $\theta \in \chi(\mathcal{M})$ have period $n$ and suppose $\mathcal{M}$ has no hypercentral non-trivial sequences. Then $\gamma(\theta) = \kappa(\theta)^n$.

Proof. If $\varepsilon(\alpha) = \theta$, $\alpha = \text{Ad } U$ on $\mathcal{M} \subset \mathcal{M}^\infty$ and $\alpha^n = \text{Ad } v$ on $\mathcal{M}$, then $\alpha(v*U^n) = v*U^n$. But also $\alpha(v*U^n) = \alpha(v*)\kappa(\theta)^n U^n = \gamma(\theta)\kappa(\theta)^n v*U^n$. Hence $\gamma(\theta) = \kappa(\theta)^n$.

Lemma 2.9. With $\theta$ and $\mathcal{M}$ as in 2.8, $\gamma(\theta) = \kappa(\theta)^{-n}$.

Proof. If $\varepsilon(\alpha) = \theta$, $\alpha = \text{Ad } U$ on $\mathcal{M} \subset \mathcal{M}^\infty$ and $\alpha^n = \text{Ad } v$,

$$\gamma(\theta)v = \alpha(v) = UvU^* = UvU^*v^*v = U\alpha^n(U^*)v = \kappa(\theta)^{-n}v.$$ Thus $\gamma(\theta) = \kappa(\theta)^{-n}$.

Corollary 2.10. If $\theta$ and $\mathcal{M}$ are as in 2.8, $\gamma(\theta) = \pm 1$.

We shall see in section 6 that $\gamma(\theta)$ can be $-1$. 
Remark 2.11. If $\omega$ is a free ultrafilter on $\mathbb{N}$, and $\text{Ad} \, U = \alpha$ on $\mathcal{M} \subset \mathcal{M}^{\omega}$, then as above $U^* \alpha(U) \in \mathcal{Z}(\mathcal{M}_\omega)$. By hypothesis $\mathcal{Z}(\mathcal{M}_\omega)$ is just the scalars so that $\lambda_\omega = U^* \alpha(U)$. This shows that if $\lambda_\omega$ had been defined with an ultrafilter, the result does not depend on the ultrafilter.

3. Definition of $\mathcal{A}$. Let $F_{24}$ be the free group on the 24 generators $g_i, i = 1, 2, \ldots, 24$. Let $\lambda(g_i)$ be the unitaries of the left regular representation of $F_{24}$, and let $UF_{24}$ be the von Neumann algebra generated by the $\lambda(g_i)$. $UF_{24}$ is a II$_1$ factor. For each 25th root of unity define the automorphisms $\zeta_\mu^\gamma: UF_{24} \to UF_{24}$ by $\zeta_\mu^\gamma(\lambda(g_i))^\gamma = \mu^\gamma \lambda(g_i)^\gamma$. By [13], $\zeta_\mu^\gamma$ is outer if $\mu \neq 1$ and the period of $\zeta_\mu^\gamma$ is the order of $\mu$ as a root of unity. Also $\zeta_\mu^\gamma \zeta_\nu^\gamma = \zeta_{\mu \nu}^\gamma$.

Let $\sigma = e^{2\pi i/25}$ and $\gamma = \sigma^\alpha$. Let $\mathcal{P}$ be the crossed product $W^*(UF_{24}, \mathbb{Z}_5)$, where $\mathbb{Z}_5$ acts on $UF_{24}$ by $n(x) = \zeta_\gamma^\gamma(x)$ for $n \in \mathbb{Z}_5$. Write the elements of $\mathcal{P}$ as sums of the form $\sum_{i=0}^4 a_i u^i$ where $a_i \in UF_{24}$ and $u$ is a unitary, $u^5 = 1$ and $\text{Ad} \, u = \zeta_\gamma$ on $UF_{24}$. Define $r_5^\gamma \in \text{Aut} \, \mathcal{P}$ by

$$r_5^\gamma \left( \sum_{i=0}^4 a_i u^i \right) = \sum_{i=0}^4 \gamma^{-i \zeta_\gamma^\gamma(a_i)} u^i.$$ 

Then $(r_5^\gamma)^5 = \text{Ad} \, u$ and $r_5^\gamma(u) = \gamma u$. (This is just the construction of [8, theorem 2.1] in the cyclic case.)

Now choose an automorphism $s_5^\gamma$ of the hyperfinite II$_1$ factor $R$ with $(s_5^\gamma)^5 = \text{Ad} \, w$ and $s_5^\gamma(w) = \gamma w$. (See [6, proposition 1.6].) Let $\mathcal{B}$ be the tensor product $\mathcal{P} \otimes R$ and define $r_5^\gamma \otimes s_5^\gamma$ on $\mathcal{B}$. Then $(r_5^\gamma \otimes s_5^\gamma)^5 = \text{Ad} \, (u \otimes w)$ and $r_5^\gamma \otimes s_5^\gamma(u \otimes w) = u \otimes w$ is in the centre of the fixed point algebra $\mathcal{L}$ of $r_5^\gamma \otimes s_5^\gamma$. Choose a 5th root of $(u \otimes w)^5$ in $\mathcal{Z}(\mathcal{L})$ and let $\psi = \text{Ad} \, t(r_5^\gamma \otimes s_5^\gamma)$. Then $\psi^5 = \text{id}$. The von Neumann algebra $\mathcal{A}$ is the crossed product $W^*(\mathcal{B}, \mathbb{Z}_5)$ where $\psi$ determines the action of $\mathbb{Z}_5$ on $\mathcal{B}$. Since $\psi^i$ is outer for $0 < i < 5$, $\mathcal{A}$ is a type II$_1$ factor.

Before going on to prove that $\mathcal{A}$ is antiisomorphic to itself, we prove a fact about $\mathcal{P}$ which allows us to control central sequences in $\mathcal{P} \otimes R$.

Lemma 34. There is a $K > 0$ such that if $x \in \mathcal{P}$ satisfies $\| [x, \lambda_\omega(g_k)] \|_2 < \varepsilon$ for $k = 1, 2, \ldots, 24$, then $\| x - \tau(x) \|_2 < K \varepsilon$

Here $\tau$ denotes the trace both on $UF_{24}$ and $\mathcal{P}$. Thus $\tau(\sum_{i=0}^4 x_i u^i) = \tau(x_0)$.

Proof. Write $x \in \mathcal{P}$ in the form $x = \sum_{i=0}^4 x_i u^i$ with $x_i \in UF_{24}$. For $k = 5$ and 10, $[\lambda_\omega(g_k), u] = 0$ so that for these values of $k$,
\[ \| [x_i, \lambda(g_k)] \|_2^2 = \sum_{i=0}^{4} \| [x_i, \lambda(g_k)] \|_2^2 < \epsilon^2. \]

Thus for each \( i \), \( \|[x_i, \lambda(g_k)]\|_2 < \epsilon \). Hence by \([14, \text{lemma 4.3.3}]\), \( \|x_i - \tau(x_i)\|_2 < 14\epsilon \). But if \( y = \sum_{i=0}^{4} \tau(x_i)u^i \), then \( \|x - y\|_2 < \sqrt{5}14\epsilon \), so that
\[ \| [y, \lambda(g_k)] \|_2 < \sqrt{5}28\epsilon + \epsilon \quad \text{for} \ k = 1, 2, \ldots, 24. \]

In particular
\[ \| [y, \lambda(g_1)] \|_2^2 = \sum_{i=0}^{4} |\tau(x_i)|^2 |1 - \gamma|^2 < (28\sqrt{5} + 1)^2\epsilon^2 \]
(since we know how \( u \) commutes with \( \lambda(g_1) \): \( u\lambda(g_1)u^* = \gamma\lambda(g_1) \)). Thus for \( i \neq 0 \),
\[ |\tau(x_i)| < \left( (28\sqrt{5} + 1)/|1 - \gamma| \right) \epsilon \]
and
\[ \|x - \tau(x)\|_2 \leq \|x - y\|_2 + \|y - \tau(x)\|_2 < 14\sqrt{5}\epsilon + 2(28\sqrt{5} + 1)/|1 - \gamma|\epsilon. \]

Remark 3.2. Lemma 3.1 implies (by \([1, \text{lemma 2.11 and corollary 3.6}]\)), that \( \mathcal{P} \) is full, i.e. \( \overline{\text{Int} \mathcal{P}} = \text{Int} \mathcal{P} \).

4. \( \mathcal{A} \) is antiisomorphic to itself.

To prove this assertion we use the following special case of a result which must be known to many authors.

Lemma 4.1. Let \( \mathcal{M} \) be a \( II_1 \) factor and \( G \) a (discrete) group of automorphisms of \( \mathcal{M} \). If \( \Psi \) is an anti-automorphism of \( \mathcal{M} \) such that \( \Psi g\Psi^{-1} = \phi(g) \) for some automorphism \( \phi \) of \( G \), then the formula
\[ \Phi \left( \sum_{g \in G} \Phi(g)u_g \right) = \sum_{g \in G} \phi(g)^{-1} \Psi \left( \Phi(g^{-1})u_{\phi(g)^{-1}} \right) = \sum_{g \in G} \psi^{-1}(\phi(\Phi(g)))u_{\phi(g)^{-1}} \]
defines an anti-automorphisms of \( W^*(\mathcal{M}, G) \).

Here \( \{u_g\} \) is the usual unitary representation in the crossed product implementing \( G \) on \( \mathcal{M} \).

Proof. Since \( \Phi \) preserves the trace there is no problem extending it from finite sums to all of \( W^*(\mathcal{M}, G) \). Thus we only need to verify the relations \( \Phi(xy) = \Phi(y)\Phi(x) \) and \( \Phi(x^*) = \Phi(x)^* \). By linearity it suffices to do this for a pair \( au_g \) and \( bu_h \). Now
\[ \Phi(au_k) = q(g)^{-1} \Psi(a) u_{q(h)^{-1}} \]
\[ \Phi(bu_k) = q(h)^{-1} \Psi(b) u_{q(h)^{-1}} \]

so that
\[ \Phi(bu_k) \Phi(au_k) = q(h)^{-1} \Psi(b) q(h)^{-1} q(g)^{-1} \Psi(a) u_{q(h)^{-1}} \]
\[ = \Psi(h^{-1} b) \Psi((gh)^{-1} a) u_{q(h)^{-1}} \]
\[ = \Psi((gh)^{-1} (ag(b))) u_{q(h)^{-1}} \]
\[ = \Phi(\text{ag}(b) u_{q(h)}) \]
\[ = \Phi(au_k bu_k) . \]

Moreover
\[ \Phi(au_k)^* = u_{q(h)}^{-1} q(g)^{-1} \Psi(a^*) = \Psi(a^*) u_{q(h)} \]
and
\[ \Phi((au_k)^*) = \Phi(g^{-1} (a^*) u_{q^{-1} (h)} - 1) = \Psi(a^*) u_{q(h)} . \]

**Theorem 4.2.** \( \mathcal{A} \) is antiisomorphic to itself.

**Proof.** We want to use lemma 4.1 so we begin by constructing appropriate antiautomorphisms of \( \mathcal{P} \) and \( R \).

Define the involutory antiautomorphism \( \Lambda \) of \( UF_{24} \) by \( \Lambda(\lambda(g)) = \lambda(g)^{-1} \) for all \( g \in F_{24} \); For a 25th root of unity \( \mu \), \( \Lambda \zeta_{\mu} \Lambda^{-1} = \zeta_{\mu}^{-1} \). Now define the automorphism \( \pi: UF_{24} \rightarrow UF_{24} \) by the permutation of the generators \( \pi(\lambda(g_i)) = \lambda(g_{2i \mod 25}) \). One checks that \( \pi^{-1} \zeta_{\mu} \pi = \zeta_{\mu}^{-2} \) so that \( \Psi = \pi^{-1} \Lambda \) is an antiautomorphism and \( \Psi \zeta_{\mu} \Psi^{-1} = \zeta_{\mu}^2 \). Thus by lemma 4.1, the formula
\[ \Phi \left( \sum_{i=0}^{4} a_i u^i \right) = \sum_{i=0}^{4} \zeta_{\gamma}^{-2i} \Psi(a_i) u^{-2i} \]
defines an antiautomorphism \( \Phi \) of \( \mathcal{P} = W^*(UF_{24}, \mathbb{Z}_2) \). Also
\[ \Phi r^*_S \Phi^{-1}(au) = \Phi r^*_S (\Psi^{-1}\zeta_{\gamma}^{-1}(a) u^{2i}) \quad \text{(see below)} \]
\[ = \Phi(y^{-2i} \zeta_{\sigma} \Psi^{-1} \zeta_{\gamma}^{-1}(a) u^{2i}) \]
\[ = \gamma^{-2i} \zeta_{\gamma} \Psi \zeta_{\sigma} \Psi^{-1} \zeta_{\gamma}^{-1}(a) u^{i} \]
\[ = \gamma^{-2i} \zeta_{\sigma} (a) u^{i} = (r^*_S)^2 (au) . \]

Thus \( \Phi r^*_S \Phi^{-1} = (r^*_S)^2 \).

(To calculate \( \Phi^{-1}(au) \), note that \( \Phi(au) = \zeta_{\gamma}^{-2i} \Psi(a) u^{-2i} \) so that
\[ \Phi^{-1}(\zeta_{r^{-2i}} \Psi(a) u^{-2i}) = a u^i. \]

Put \( a' = \zeta_{r^{-2i}} \Psi(a) \) and \( j = -2i \). Then \( a = \Psi^{-1} \zeta_{r^{-j}} (a') \) and \( i = 2j \mod 5 \). Thus \( \Phi^{-1}(a' u^j) = \Psi^{-1} \zeta_{r^{-j}} (a') u^{2j} \).

Now by [6, theorem 1.11] there is an anti-automorphism \( \Gamma \) of \( R \) such that \( \Gamma s^2 \Gamma^{-1} = (s^2)^2 \).

Remember that \( \mathcal{A} \) is the crossed product of \( \mathcal{P} \otimes R \) by \( \psi = \text{Ad } t (r_2^2 \otimes s_2^2) \). If we define the anti-automorphism \( \Phi \otimes \Gamma \) of \( \mathcal{P} \otimes R \), then

\[ \Phi \otimes \Gamma ( \text{Ad } t (r_2^2 \otimes s_2^2) ) \Phi^{-1} \otimes \Gamma^{-1} = \text{Ad}(\Phi \otimes \Gamma (t^*)) (r_2^2 \otimes s_2^2)^2. \]

But this means that the automorphisms \( (\Phi \otimes \Gamma) \psi (\Phi \otimes \Gamma)^{-1} \) and \( \psi^2 \) differ by an inner automorphism. By [6, corollary 2.6], there is an automorphism \( \beta \) of \( \mathcal{P} \otimes R \) such that

\[ \beta (\Phi \otimes \Gamma) \psi (\Phi \otimes \Gamma)^{-1} \beta^{-1} = \psi^2. \]

Thus by lemma 4.1, the formula

\[ \sum_{i=0}^{4} a_i z^{2i} \mapsto \sum_{i=0}^{4} \psi^{-2i} \beta (\Phi \otimes \Gamma)(a_i) z^{-2i} \]

defines an anti-automorphism of \( \mathcal{A} \) (here \( a_i \in \mathcal{B} = \mathcal{P} \otimes R \) and \( z \) is a unitary with \( z^5 = 1 \) and \( \text{Ad } z = \psi \) on \( \mathcal{B} \)).

5. \( \mathcal{A} \) possesses no involutory anti-automorphisms.

To prove the assertion of this section we shall calculate \( \chi(\mathcal{A}) \) and \( \kappa(\theta) \) for generator \( \theta \) of \( \chi(\mathcal{A}) \) using the methods of [3]. We give a brief sketch of these methods.

Given a finite subgroup \( G \) of Aut \( \mathcal{M} \) (\( \mathcal{M} \) a \( II_1 \) factor without non-trivial hypercentral sequences) with \( G \cap \text{Int } \mathcal{M} = \{ id \} \), Connes defines \( K = G \cap \text{Ct } \mathcal{M} \) and \( K^\perp \), the group of homomorphisms from \( G \) to \( T \) which vanish on \( K \). Let \( \mathcal{M}^G \) be the fixed point algebra of the group \( G \) and let Fin be the subgroup of Aut \( \mathcal{M} \) consisting of all those automorphisms of the form \( \text{Ad } x \) with \( x \in \mathcal{M}^G \). Let Fin be its closure in Aut \( \mathcal{M} \). Connes defines \( L = \varepsilon (G \text{Ct } \mathcal{M} \cap \text{Fin}) \), a subgroup of Out \( \mathcal{M} \). He shows that there is an exact sequence

\[ 0 \to K^\perp \to \chi(W^*(\mathcal{M}, G)) \to L \to 0. \]

The map \( \partial \) comes from the dual action: write elements in the crossed product in the form \( \sum_{g \in G} a_g u_g \) with \( a_g \in \mathcal{M} \) and \( \text{Ad } u_g = g \) on \( \mathcal{M} \). Then for each \( \eta \in K^\perp \), \( \eta: G \to T \), define

\[ \delta(\eta) \left( \sum_{g \in G} a_g u_g \right) = \sum_{g \in G} \eta(g) a_g u_g. \]
One checks that $\delta(\eta) \in \text{Ct } \mathcal{N} \cap \overline{\text{Int } \mathcal{N}}$ ($\mathcal{N}$ is $W^*(\mathcal{M}, G)$) and $\delta$ is defined by taking the quotient $\varepsilon \circ \delta$. The injectivity of $\delta$ comes from the fact that the dual action is outer.

The map $\Pi: \chi(\mathcal{N}) \to L$ is defined as follows. Given $x \in \text{Ct } \mathcal{N} \cap \overline{\text{Int } \mathcal{N}}$, one can show, using the detailed knowledge of central sequences in $\mathcal{N}$ coming from the hypothesis $G \cap \overline{\text{Int } \mathcal{M}} = \{\text{id}\}$, that there is a unitary $v \in \mathcal{N}$ and a sequence $u_n$ of unitaries in $\mathcal{M}_G$ such that $x = \text{Ad } v \lim_{n \to \infty} \text{Ad } u_n$. Let $\psi_x = (\text{Ad } v^* x)|_\mathcal{M}$. By construction $\psi_x \in \text{Fint}$ and one checks that $\psi_x \in G \text{Ct } \mathcal{M}$ (use Galois theory on $\mathcal{M}_\omega$). The image of $\psi_x$ in $\text{Out } \mathcal{M}$ depends only on $\varepsilon(x)$, so we may define a map $\Pi: \chi(\mathcal{N}) \to L$ by $\Pi(\theta) = \varepsilon(\psi_x)$ with $\varepsilon(x) = \theta$.

The surjectivity of $\Pi$ involves constructing a set-theoretic section for $\Pi$. This will be the most important construction for this paper. It is done as follows: given $\mu \in L$, let $x_\mu \in \overline{\text{Int } G \text{Ct } \mathcal{M}}$ be such that $\varepsilon(x_\mu) = \mu$. This $x_\mu$ commutes with $G$ since it is the limit of such automorphisms. Thus we may define an automorphism $\beta_\mu$ of $W^*(\mathcal{M}, G)$ by

$$\beta_\mu\left(\sum_{g \in G} a_g u_g\right) = \sum_{g \in G} x_\mu(a_g) u_g.$$  

One checks that $\beta_\mu \in \text{Ct } \mathcal{N} \cap \overline{\text{Int } \mathcal{N}}$ and it is clear that $\Pi(\varepsilon(\beta_\mu)) = \mu$. Thus $\mu \mapsto \varepsilon(\beta_\mu)$ provides the required section.

**Remark 5.1.** A bonus of this description is that we can easily calculate $\chi(\varepsilon(\beta_\mu))$. For if $u_\mu$ are unitaries in $\mathcal{M}_G$ with $\lim_{n \to \infty} \text{Ad } u_\mu^n = x_\mu$ in $\text{Aut } \mathcal{M}$, then $\beta_\mu = \lim_{n \to \infty} \text{Ad } u_\mu^n$ in $\text{Aut } \mathcal{M}$ so that we only need to calculate

$$\lim_{n \to \infty} (u_\mu^n)^* \beta_\mu(u_\mu^n) = \lim_{n \to \infty} (u_\mu^n)^* x_\mu(u_\mu^n)$$

and this is $\chi(\varepsilon(\beta_\mu))$.

We want to show that if $\mathcal{A}$ and $\mathcal{B}$ are as in section 3, $\chi(\mathcal{A}) = Z_{25}$ and if $\sigma = e^{2\pi i/25}$, a generator $\theta$ of $\chi(\mathcal{A})$ satisfies $\chi(\theta) = \overline{\sigma}$. To do this we shall show that $K = Z_{25}$, $L = Z_5$ and that a lifting to $\chi(\mathcal{A})$ of a generator of $L$ satisfies $\theta^5 \neq 1$ and $\chi(\theta) = \overline{\sigma}$.

To apply [3, theorem 4] we need to know that $\mathcal{B}$ has no non-trivial hypercentral sequences. Together with our lemma 3.1, [1, lemma 2.11] shows that all central sequences in $\mathcal{P} \otimes R$ come from $R$ and it is well known that $R$ has no non-trivial hypercentral sequences. We also need to know that $G \cap \overline{\text{Int } \mathcal{B}} = \{\text{id}\}$. Here $G$ is the group generated by $\psi = \text{Ad } t (r_s^x \otimes s_y)$. It suffices to show that $\psi \notin \text{Int } \mathcal{B}$. This follows immediately from [5, 3.3] and our remark 3.2. Thus we may apply [3, theorem 4] with impunity.

We need to know $K = G \cap \text{Ct } \mathcal{B}$. 

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Lemma 5.2. The group $K = G \cap \text{Ct} \mathcal{B}$ is just the identity.

Proof. It suffices to show that $\psi \notin \text{Ct} \mathcal{B}$. But $s^\mathcal{B}_5$ is not in $\text{Ct} R$ (see [6]) and if $(x_n)$ is central in $R$, $(1 \otimes x_n)$ is central in $\mathcal{P} \otimes R$ so that $r^5_3 \otimes s^\mathcal{B}_5 \notin \text{Ct} \mathcal{B}$.

Next we determine $L$ and a lifting of a generator.

Lemma 5.3. With notation as above, $\varepsilon(G \text{Ct} \mathcal{B} \cap \overline{\text{Fint}}) \cong \mathbb{Z}_5$, and a generator is

$$\mu = \varepsilon(\text{id} \otimes (\text{Ad} v s^\mathcal{B}_5))$$

where $v$ is a unitary in $R$ such that $s^\mathcal{B}_5(v) = \sigma v$.

If $\beta_\mu \in \text{Ct} \mathcal{A} \cap \text{Int} \mathcal{A}$ is a described above, then $\chi(\varepsilon(\beta_\mu)) = \bar{\sigma}$.

Proof. Note first that such a $v$ exists by [6, corollary 2.6].

We now want to show that $\text{id} \otimes (\text{Ad} v s^\mathcal{B}_5) \in G \text{Ct} \mathcal{B} \cap \overline{\text{Fint}}$. Consider first of all $G \text{Ct} \mathcal{B}$: multiplication of $\text{id} \otimes (\text{Ad} v s^\mathcal{B}_5)$ by $\psi^{-1} = (\text{Ad} t(r^3_5 \otimes s^\mathcal{B}_5))^{-1}$ yields

$$\text{Ad} t^*(1 \otimes v)((r^3_5)^{-1} \otimes \text{id})$$

which is certainly in $\text{Ct} \mathcal{B}$. Thus $\text{id} \otimes (\text{Ad} v s^\mathcal{B}_5) \in G \text{Ct} \mathcal{B}$. To show that

$$\text{id} \otimes (\text{Ad} v s^\mathcal{B}_5) \in \overline{\text{Fint}}$$

we must exhibit a sequence $\{u_n\}$ of unitaries in $\mathcal{B}$ with $\psi(u_n) = u_n$ and

$$\text{id} \otimes (\text{Ad} v s^\mathcal{B}_5) = \lim_{n \to \infty} \text{Ad} u_n.$$
By standard approximation arguments (e.g. [10, 3.5]), we may assume $s_n^2(y_n \pi_n^*) = \delta y_n \pi_n^*$. Putting $x_n = y_n \pi_n^*$ gives the required result.

Now put $u_n = v x_n$. Then $\text{Ad } vs_n^2 = \lim_{n \to \omega} \text{Ad } u_n$ with $s_n^2(u_n) = u_n$, so that

$$r_n^2 \otimes s_n^2(1 \otimes u_n) = 1 \otimes u_n,$$

and since $t$ was chosen in the centre of the fixed point algebra for $r_n^2 \otimes s_n^2$,

$$\psi(1 \otimes u_n) = \text{Ad } t(r_n^2 \otimes s_n^2)(1 \otimes u_n) = 1 \otimes u_n.$$

Thus $1 \otimes u_n$ is the desired sequence and $\text{id} \otimes (\text{Ad } vs_n^2) \in \overline{\text{Fint}}$.

Let us now calculate $\chi(\epsilon(\beta_\mu))$. Choosing a subsequence we may assume

$$\text{id} \otimes (\text{Ad } vs_n^2) = \lim_{n \to \infty} \text{Ad } 1 \otimes u_n.$$

By remark 5.1, to calculate $\chi(\epsilon(\beta_\mu))$ we need only calculate $(\text{id} \otimes (\text{Ad } vs_n^2))(1 \otimes u_n)$. But

$$\text{id} \otimes s_n^2(1 \otimes u_n) = 1 \otimes u_n,$$

so

$$\text{id} \otimes (\text{Ad } vs_n^2)(1 \otimes u_n) = 1 \otimes vu_n v^*.$$

Now $\lim_{n \to \infty} \text{Ad } u_n = s_n^2$ and $s_n^2(v) = \delta v$. Thus $\lim_{n \to \infty} u_n v^* u_n^* = \delta v^*$, and

$$\lim_{n \to \infty} \| \text{id} \otimes (\text{Ad } vs_n^2)(1 \otimes u_n) - \delta 1 \otimes u_n \|_2 = 0.$$

By definition $\chi(\epsilon(\beta_\mu)) = \delta$.

Thus far we know that $\epsilon(\text{id} \otimes (\text{Ad } vs_n^2))$ is an element (of period 5) of $L = (G \text{Ct } \mathcal{B} \cap \overline{\text{Fint}})$. We want to show that it generates $L$. This means that if $\varphi \in G \text{Ct } \mathcal{B} \cap \overline{\text{Fint}}$ then it differs by an inner automorphism from a power of $\text{id} \otimes (\text{Ad } vs_n^2)$. By the same argument as in [2, theorem 3.2 a)], if $\alpha \in \text{Ct } \mathcal{B}$, then $\alpha = \text{Ad } z(v \otimes \text{id})$ for some unitary $z \in \mathcal{B}$ and an automorphism $v$ of $\mathcal{B}$ (basically because otherwise $\alpha$ would act on central sequences coming from $R$). Thus since $\varphi \in G \text{Ct } \mathcal{B}$, there is an $n \in \{0, 1, 2, 3, 4\}$ such that $\varphi \text{Ad } t^n(r_n^2 \otimes s_n^2)^n$ is of the form $\text{Ad } z(v \otimes \text{id})$. Solving for $\varphi$ gives

$$\varphi = \text{Ad } x'(v(r_n^2)^{-n} \otimes (s_n^2)^{-n})$$

for some $x \in \mathcal{B}$. Since $\varphi \in \overline{\text{Int } \mathcal{B}}$ and $\mathcal{B}$ is full we may apply [5, corollary 3.3] to conclude that

$$\varphi = \text{Ad } x'(\text{id} \otimes (s_n^2)^{-n})$$

(or argue as in [2, 2.1]). Thus $\epsilon(\varphi) = \mu^{-n}$, and $\mu$ generates $L$.

**Lemma 5.4.** We have $\chi(\mathcal{A}) \cong \mathbb{Z}_{25}$ and $\Omega_{sd} = 0$. 

PROOF. By 5.2, 5.3 and [3, theorem 4], we have an exact sequence

$$0 \rightarrow \mathbb{Z}_5 \rightarrow \chi(\mathcal{A}) \rightarrow \mathbb{Z}_5 \rightarrow 0.$$ 

A lifting to $\chi(\mathcal{A})$ of a generator of $\mathbb{Z}_5$ is defined on $\mathcal{A} = W^*(\mathcal{B}, \mathbb{Z}_5)$ by $\varepsilon(x)$ where

$$\alpha \left( \sum_{i=0}^{4} a_i z_i^i \right) = \sum_{i=0}^{4} \text{Id} \otimes (\text{Ad } v s_j)(a_i) z_i^i.$$ 

Since 5 is a prime number, to prove that $\chi(\mathcal{A}) \cong \mathbb{Z}_{25}$, it suffices to show that $\varepsilon(x)^5 = 1$, i.e. $\alpha^5 \notin \text{Int } \mathcal{A}$.

By definition,

$$\alpha^5 \left( \sum_{i=0}^{4} a_i z_i^i \right) = \sum_{i=0}^{4} \text{Ad}(1 \otimes v^5 w)(a_i) z_i^i = \text{Ad}(1 \otimes v^5 w) \left( \sum_{i=0}^{4} \gamma^{2i} a_i z_i^i \right).$$

To see this last step, remember that $r_5^2 \otimes s_5^2 (1 \otimes v^5 w) = \sigma^5 \gamma(1 \otimes v^5 w)$, so that

$$\text{Ad } t(1 \otimes v^5 w) = 1 \otimes v^5 w$$

and

$$\text{Ad } t(r_5^2 \otimes s_5^2)(1 \otimes v^5 w) = \gamma^2 (1 \otimes v^5 w).$$

Also $\text{Ad } z = \text{Ad } t(r_5^2 \otimes s_5^2)$ on $\mathcal{B}$. Thus $\alpha^5$ is a dual action times an inner automorphism, which is outer. Hence $\chi(\mathcal{A}) \cong \mathbb{Z}_{25}$.

The $H^3$ obstruction $\Omega_{\mathcal{A}}$ is represented by $\gamma(\theta)$ for generator of $\chi(\mathcal{A})$. But from the above calculation, the period of $\theta$ is 25, and if $\theta = \varepsilon(\beta_m)$, we may suppose, by lemma 5.3, that $\kappa(\theta) = \sigma$. By lemma 2.7, $\gamma(\theta) = 1$, which means $\Omega_{\mathcal{A}} = 0$.

**Theorem 5.5.** $\mathcal{A}$ possesses no involutory antiautomorphism.

**Proof.** If $\Phi$ were such an antiautomorphism, conjugation by $\Phi$ would induce an automorphism of period 1 or 2 of $\chi(\mathcal{A})$. Since $\chi(\mathcal{A}) = \mathbb{Z}_{25}$, the only such automorphisms are the identity and the map $\theta \mapsto \theta^{-1}$. By lemmas 2.6 and 2.7, neither of these is possible since $\kappa(\theta) = \sigma$, and $\sigma \neq \sigma$.

**Remark 5.6.** Since everything happened in $\text{Out } \mathcal{A}$, it follows from the above that $\mathcal{A}$ has no antiautomorphism whose square is an inner automorphism. This may also be deduced from [15, theorem 5.5 and theorem 4.4].

**Remark 5.7.** It is clear that $\mathcal{A}$ is not the von Neumann algebra generated by the left regular representation of a discrete group. Thus a modernised version of [14, problem 4.4.30] would be: "If $\mathcal{M}$ is a $\text{II}_1$ factor with an involutory antiautomorphism, is there a discrete group $G$ such that $\mathcal{M} = U(G)$?"
6. Example of a II$_1$ factor $\mathcal{M}$ with $\Omega_{\mathcal{M}} \neq 0$.

We have seen that for an element $\theta \in \chi(\mathcal{M})$, $\gamma(\theta) = \pm 1$ (if $\mathcal{M}$ has no non-trivial hypercentral sequences). We shall give an example of such a factor with $\chi(\mathcal{M}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and an element $\theta \in \chi(\mathcal{M})$ with $\gamma(\theta) = -1$. This also implies $\Omega_{\mathcal{M}} \neq 0$.

The example is obtained by replacing 5 by 2 and 24 by 3 in the above construction of $\mathcal{A}$ and $\mathcal{B}$. (If one doesn't believe lemma 3.1 any more, just add a few dummy generators to $F_3$.) Now $\gamma$ will be $-1$ and $\sigma$ will be $i$. All the calculations work in the same way up to 5.4. Thus we obtain a $\theta \in \chi(\mathcal{A})$ lifting a generator of $e(C_t \mathcal{B} \cap \text{Fin}) \cong \mathbb{Z}_2$ with $\varkappa(\theta) = i$. But in the calculation of 5.4, we notice that

$$\alpha^2 \left( \sum_{i=0}^{2} a_i z^i \right) = \text{Ad} (1 \otimes v^2 w) \left( \sum_{i=0}^{2} \gamma^{2i} a_i z^i \right).$$

And now $\gamma^{2i} = 1$, so that $e(\alpha^2) = 1$. Thus the sequence $0 \to \mathbb{Z}_2 \to \chi(\mathcal{A}) \to \mathbb{Z}_2 \to 0$ is split, and since $\theta^2 = 1$ and $\varkappa(\theta) = i$, by 2.8. $\gamma(\theta) = -1$.

That $\mathcal{A}$ has no non-trivial hypercentral sequences follows from the fact that $\mathcal{B}$ has none and the control over central sequences given by the hypothesis $G \cap \text{Int} \mathcal{B} = \text{id}$ (see [5, theorem 3.1]).

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