HYPERBOLIC HARDY CLASS $H^1$

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1. Introduction.

A meromorphic function $f$ in $D = \{ |z| < 1 \}$ is known to be of bounded Nevanlinna characteristic if and only if the Shimizu–Ahlfors characteristic function of $f$,

$$ \int_0^r \pi^{-1} t^{-1} \left[ \iint_{|z| < t} \frac{|f'(z)|^2}{(1 + |f(z)|^2)^2} \, dx \, dy \right] dt, $$

is bounded for $0 < r < 1$, where $z = x + iy$; see [3, p. 13]. Similarly, a holomorphic function $f$ in $D$ is of Hardy class $H^2$ if and only if

$$ \int_0^r \pi^{-1} t^{-1} \left[ \iint_{|z| < t} |f'(z)|^2 \, dx \, dy \right] dt, $$

is bounded for $0 < r < 1$. This follows on integrating an obvious version of the Hardy–Stein equality [2, Theorem 3.1, p. 42] applied to $f$ and $\lambda = 2$.

Let now $B$ be the family of $f$ holomorphic and bounded, $|f| < 1$, in $D$. Set for $f \in B$,

$$ T^*(r, f) = \int_0^r \pi^{-1} t^{-1} \left[ \iint_{|z| < t} \frac{|f'(z)|^2}{(1 - |f(z)|^2)^2} \, dx \, dy \right] dt, $$

where $0 < r < 1$. The main purpose of the present paper is to study $f$ whose hyperbolic Shimizu–Ahlfors characteristic function $T^*(r, f)$ remains bounded for $0 < r < 1$.

It is well known that the disk $D$ is endowed with the non-Euclidean hyperbolic distance

$$ \sigma(z, w) = \frac{1}{2} \log \frac{|1 - \bar{z}w| + |z - w|}{|1 - \bar{z}w| - |z - w|}, \quad z, w \in D. $$

As will be shown in Lemma 1, for each $f \in B$ and for each constant $a \in D$, the function $\log \sigma(f, a)$ is subharmonic in $D$. Therefore, $\sigma(f, a)^p = \exp [p \log \sigma(f, a)]$ is also subharmonic for each $p > 0$. In particular,

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\[ M_1^*(r, f, a) = \frac{1}{2\pi} \int_0^{2\pi} \sigma(f(re^{i\theta}), a) d\theta \]

is bounded for \(0 \leq r < 1\) if and only if the subharmonic function \(\sigma(f, a)\) admits a harmonic majorant \(U\) in \(D\), that is, \(U\) is harmonic and \(\sigma(f, a) \leq U\) in \(D\). Therefore, it is reasonable to say that \(f \in B\) is of hyperbolic \(H^1\) if \(\sigma(f, 0)\) admits a harmonic majorant in \(D\). Let \(H^1_h\) be the family of \(f \in B\), being of hyperbolic \(H^1\). Our main result is

**Theorem 1.** For \(f \in B\) to be of \(H^1_h\) it is necessary and sufficient that \(T^*(r, f)\) is bounded for \(0 < r < 1\).

Theorem 1 follows from the next theorem on setting \(a = 0\).

**Theorem 2.** For each \(f \in B\), each \(a \in D\), and each \(r, 0 < r < 1\), the following inequality holds.

\[
T^*(r, f) \leq M_1^*(r, f, a) + \frac{1}{2} \log \left[ 1 - \left| (f(0) - a)/(1 - \bar{a}f(0)) \right|^2 \right] \\
< T^*(r, f) + \log 2 .
\]

The inequality (1.1) is sharp. First, \(T^*(r, 0) = M_1^*(r, 0, 0) = 0\) for each \(0 < r < 1\). Next, the constant \(\log 2\) in (1.1) cannot be replaced by a smaller positive constant. Actually, for \(f(z) = z\),

\[
M_1^*(r, f, 0) - T^*(r, f) = \log (1 + r) \to \log 2 \text{ as } r \to 1 .
\]

Let \(G\) be a subdomain of \(D\) such that the boundary of \(G\) has the only one point 1 in common with the unit circle. Assume that there exist \(r_0, 0 < r_0 < 1\), and a function \(A(r), r_0 < r < 1\), both depending on \(G\), such that the intersection of \(G\) with each circle \(|z| = r\), \(r_0 < r < 1\), is of linear measure \(rA(r)\), where

\[
0 < \inf_{r_0 < r < 1} (1 - r)^{-1} A(r) \text{ and } \sup_{r_0 < r < 1} (1 - r)^{-1} A(r) < +\infty .
\]

Let \(G\) be the family of all domains \(G\) of the described type. A typical example of \(G \in G\) is a triangular domain in \(D\) with one vertex at 1 and the other vertices in \(D\). Set \(G(\theta) = \{e^{i\theta}z \mid z \in G\}\) for \(G \in G\) and \(\theta \in [0, 2\pi]\), and set for \(f \in B\),

\[
S(f, G(\theta)) = \iint_{G(\theta)} \frac{|f'(z)|^2}{(1 - |f(z)|^2)^2} \, dx \, dy ,
\]

being the non-Euclidean area of the Riemannian image of \(G(\theta)\) by \(f\). We next propose a criterion for \(f \in B\) to belong to \(H^1_h\).
THEOREM 3. Assume that \( f \in B \), and assume that, for a certain \( G \in \mathcal{G} \),

\[
\int_0^{2\pi} S(f, G(\theta)) \, d\theta < +\infty.
\]

Then \( f \in H^1_h \). Conversely, if \( f \in H^1_h \), then (1.2) holds for each \( G \in \mathcal{G} \).

Finally, we assert that \( H^1_h \) is closed for the multiplication and that \( H^1_h \) is convex.

THEOREM 4. For each \( f \in H^1_h \), each \( g \in H^1_h \), and each constant \( t, 0 < t < 1 \),

\[
fg \in H^1_h \text{ and } tf + (1 - t)g \in H^1_h.
\]

2. Proof of Theorem 2.

The following lemma is fundamental to deduce Theorem 1 from Theorem 2.

**Lemma 1.** For each \( f \in B \) and for each constant \( a \in D \), the function \( \log \sigma(f, a) \) is subharmonic in \( D \).

Setting \( u(z) = \sigma(z, 0) \), \( z \in D \), one obtains for \( z \neq 0 \), the identity

\[
\Delta \log u(z) = (1 - |z|^2)^{-2} u(z)^{-2} [ |z|^{-1} (1 + |z|^2) u(z) - 1 ] ,
\]

which, together with the inequality

\[
\frac{1 + x^2}{2x} \log \frac{1 + x}{1 - x} - 1 \geq 0 \quad \text{for} \quad 0 < x < 1 ,
\]

shows that \( \Delta \log u \geq 0 \) in \( D - \{0\} \). Since \( \log u(0) = -\infty \), the function \( \log u \) is subharmonic in the whole \( D \). Since

\[
T_a(f) = (f - a)/(1 - \bar{a}f)
\]

is holomorphic in \( D \), it follows that the composed function

\[
\log \sigma(T_a(f), 0) = \log \sigma(f, a)
\]

is subharmonic in \( D \).

For the proof of Theorem 2 we note first that, for \( f \in B \) and \( a \in D \), the following inequality holds in \( D \).

\[
-\frac{1}{2} \log (1 - |T_a(f)|^2) \leq \sigma(f, a) < -\frac{1}{2} \log (1 - |T_a(f)|^2) + \log 2 .
\]

In effect, for \( 0 \leq x < 1 \),

\[
-\frac{1}{2} \log (1 - x^2) \leq \sigma(x, 0) < -\frac{1}{2} \log (1 - x^2) + \log 2 ,
\]
which, together with \( \sigma(f, a) = \sigma(|T_a(f)|, 0) \), proves (2.1). We next note that, in \( D \),

\[
-\Delta \log (1 - |f|^2) = 4f^*^2, \quad f \in B,
\]

where \( f^* = |f'|/(1 - |f|^2) \). Since \( (T_a(f))^* = f^* \), it follows that

(2.2) \[
-\Delta \log (1 - |T_a(f)|^2) = 4f^*^2
\]

in \( D \).

Setting

\[
I(r, f, a) = -\frac{1}{2\pi} \int_0^{2\pi} \log (1 - |T_a(f(re^{i\theta}))|^2) \, d\theta
\]

for \( f \in B, \ a \in D \), and \( 0 < r < 1 \), one observes by the Green formula, together with (2.2), that

(2.3) \[
r \frac{d}{dr} I(r, f, a) = \frac{2}{\pi} \iint_{|z| < r} f^*^2 \, dx \, dy, \quad 0 < r < 1.
\]

Since \( \lim_{r \to 0} I(r, f, a) = -\log (1 - |T_a(f(0))|^2) \), the integration of (2.3) yields

\[
\frac{1}{2} [I(r, f, a) + \log (1 - |T_a(f(0))|^2)] = T^*(r, f), \quad 0 < r < 1.
\]

On the other hand, it follows from (2.1) that

\[
\frac{1}{2} I(r, f, a) \leq M^*_1(r, f, a) < \frac{1}{2} I(r, f, a) + \log 2,
\]

whence follows (1.1) of Theorem 2.

**Remark.** For \( f \in B \), the function \( \log [-\log (1 - |f|^2)] \) is subharmonic in \( D \). In effect, except for the zeros of \( f \), one obtains in \( D \),

\[
\Delta \log [-\log (1 - |f|^2)] = 4 \frac{\partial^2}{\partial z \partial \bar{z}} \log [-\log (1 - f\bar{f})]
\]

\[
= 4[\log (1 - |f|^2)]^{-2} f^*^2 [-\log (1 - |f|^2) - |f|^2] \geq 0.
\]

By (2.1) for \( a = 0 \) and by Theorem 1 one observes that \( f \in B \) is of \( H^1_k \) if and only if \( -\log (1 - |f|^2) \) has a harmonic majorant in \( D \). Let \( f(e^{i\theta}) \) be the angular limit of \( f \in B \), whose existence at almost every \( e^{i\theta}, \ \theta \in [0, 2\pi] \), is well known. Assume that \( f \in H^1_k \). It then follows from [1, Theorem] (see also [4]) that

\[
\sigma(f, 0) = \exp [\log \sigma(f, 0)]
\]

and

\[
-\log (1 - |f|^2) = \exp [\log [-\log (1 - |f|^2)]]
\]

both have the least harmonic majorants in \( D \), being the Poisson integrals of \( \sigma(f(e^{i\theta}), 0) \) and \( -\log (1 - |f(e^{i\theta})|^2) \) on \([0, 2\pi]\), respectively.
3. Proof of Theorem 3.

We shall make use of

**Lemma 2.** Let $f \in B$. Then, $f \in H^1$ if and only if

$$E_1 = \iint_D (1 - |z|) f^*(z)^2 \, dx \, dy < +\infty.$$ 

First of all, by Theorem 1, $f$ is a member of $H^1$ if and only if

$$E_2 = \int_0^1 \left[ \iint_{|z| < r} f^*(z)^2 \, dx \, dy \right] \, dr < +\infty.$$ 

Let $X_r$ be the function in $D$, being one on $\{|z| < r\}$, and zero otherwise. Then

$$E_2 = \int_0^1 \left[ \iint_D X_r(z) f^*(z)^2 \, dx \, dy \right] \, dr$$

$$= \iint_D \left[ \int_0^1 X_r(z) \, dr \right] f^*(z)^2 \, dx \, dy = E_1,$$

which proves lemma 2.

For the proof of Theorem 3, let $r_0$ and $A$ be as described in the definition of $G \in \mathcal{G}$. Let $X(z, \theta)$ be the function of $z$ in $D$, being one on $G(\theta)$ and zero otherwise, $\theta$ ranging on $[0, 2\pi]$. Then

$$A(|z|) = \int_0^{2\pi} X(z, \theta) \, d\theta, \quad r_0 < |z| < 1.$$ 

On the other hand,

$$\int_0^{2\pi} S(f, G(\theta)) \, d\theta = \iint_D \left[ \int_0^{2\pi} X(z, \theta) \, d\theta \right] f^*(z)^2 \, dx \, dy.$$ 

It then follows from (3.1) that (1.2) holds if and only if

$$\iint_{r_0 < |z| < 1} (1 - |z|) f^*(z)^2 \, dx \, dy < +\infty.$$ 

Our Theorem 3 now follows from Lemma 2.


According to the remark at the end of Section 2, $F \in H^1$ if and only if $-\log (1 - |F|^2)$ has a harmonic majorant in $D$. For $0 \leq P < 1$ and $0 \leq Q < 1$, 

\[(4.1) \quad -\log (1 - P) - \log (1 - Q) \geq -\log (1 - PQ),\]

being a consequence of \(2PQ \leq (2\sqrt{PQ} \leq) P + Q\). Now, to prove that \(fg \in H^1_h\), we have only to apply (4.1) to \(P = |f|^2\) and \(Q = |g|^2\).

Since the function \(-\log (1 - x^2)\) is convex for \(0 \leq x < 1\), it follows that

\[(4.2) \quad -\log [1 - (tP + (1-t)Q)^2] \leq -t \log (1 - P^2) - (1-t) \log (1 - Q^2)\]

for \(0 \leq P < 1\) and \(0 \leq Q < 1\). Therefore,

\[-\log (1 - |tf + (1-t)g|^2) \leq -\log [1 - (t|f| + (1-t)|g|)^2],\]

together with (4.2), where \(P = |f|, Q = |g|\), proves that \(tf + (1-t)g \in H^1_h\).

REFERENCES