THE LEVI PROBLEM IN STEIN SPACES

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1. Introduction.

Let $X$ denote a Stein space and let $\Omega$ be an open set in $X$. Assume that for every $p \in \partial \Omega$ there exists an open neighborhood $U(p)$ such that $\Omega \cap U(p)$ is Stein.

**The Levi Problem. Is $\Omega$ necessarily Stein?**

In case $X$ is a complex manifold this was solved affirmatively by Docquier and Grauert [4], and in case $X$ has at most isolated singularities it was solved affirmatively by Andreotti and Narasimhan [1].

**The Union Problem. If $\Omega^\text{open} \subset X^{\text{Stein}}$ and $\Omega_1 \subset \Omega_2 \subset \ldots \subset \bigcup \Omega^\text{open}_n = \Omega$ with each $\Omega_n$ Stein, is $\Omega$ Stein?**

This was proved to be true if $X = \mathbb{C}^4$ by Behnke and Stein [2]. The case when $X$ is a Stein manifold follows from the work of Docquier and Grauert [4] via the embedding of $X$ as a closed complex submanifold of some $\mathbb{C}^l$, Remmert [13], Bishop [3] and Narasimhan [9].

If one drops the assumption that $X$ is Stein, the result is not true, Fornæss [6].

Suppose next that $\{\Omega_t\}_{t \in \mathbb{R}}$ is a family of Stein open subsets of $X$ and that $\bigcup_{t < r} \Omega_t$ is a union of connected components of $\Omega_t$ and that $\Omega_t$ is a union of connected components of $\text{int} \cap_{t > s} \Omega_t$ for each $t \in \mathbb{R}$.

**The Runge Problem. Is $\Omega_t$ Runge in $\Omega_s$ whenever $r < s$.**

When $X$ is complex manifold this was answered affirmatively by Docquier and Grauert [4].

In this short note we will solve affirmatively the above problems in the Stein space

$$X = Z \times \mathbb{C}, \quad Z = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 ; z_1z_2 = z_3z_4\}.$$

Received April 4, 1978.
The reason that we find the space $Z$ interesting is the following observation by Grauert and Remmert [7]. The map $\Phi: \mathbb{C} \times (\mathbb{C}^2 - \{0\}) \to Z$ by

$$\Phi(t, w, \eta) = (w, tw, \eta, tw)$$

is biholomorphic onto the open set $\Omega = Z - \{z_1 = z_3 = 0\}$. For every $p \in \partial \Omega$, $p \neq 0$, there exists an open neighborhood $U(p)$ in $Z$ such that $U(p)$ is Stein. However $\Omega$ is obviously not Stein. This can be compared to the following theorem by Grauert and Remmert [7].

**THEOREM.** If $\Omega^\text{open} \subset \mathbb{C}^n$ and for every $p \in \partial \Omega$, $p \neq 0$, there exists an open neighborhood $U(p)$ such that $U(p) \cap \Omega$ is Stein, then $\Omega$ is Stein, unless $\Omega \cup \{0\}$ is open (in which case $\Omega \cup \{0\}$ is Stein).

A function $f: X \to \mathbb{R} \cup \{-\infty\}$ where $X$ is a (reduced) complex space will be said to be plurisubharmonic if for every $x \in X$ there is an open neighborhood $U(x)$ which can be realized as a closed complex subvariety $Y \subset V^\text{open} \subset \mathbb{C}^n$, $\Phi: U(x) \cong Y$ such that $f \circ \Phi^{-1}$ is the restriction to $Y$ of a plurisubharmonic function on $V$. The function $f$ is continuous (smooth) and plurisubharmonic if in addition $f \circ \Phi^{-1}$ can be chosen to be continuous (smooth). Also $f$ is said to be (continuous/smooth) strongly plurisubharmonic if $f \circ \Phi^{-1} + \varepsilon \tau$ is (continuous/smooth) plurisubharmonic for all $\varepsilon \geq 0$ sufficiently small whenever $\tau \in C^\infty_0(V)$, $\tau: V \to \mathbb{R}$.

It is a theorem by Richberg [14] that strongly plurisubharmonic functions which are continuous are continuous strongly plurisubharmonic.

The results and proofs in this paper are equally valid in Stein spaces $X' = Z' \times M$ where $M$ is any Stein manifold and

$$Z' = \{(z_1, \ldots, z_n, w_1, \ldots, w_n) \in \mathbb{C}^{2n} : z_iw_j = z_jw_i \text{ for all } i, j\}.$$

2. Preliminary remarks.

We would like here to briefly recall a few results which we will need.

**THEOREM 1.** (Narasimhan [10, 11]). Let $X$ be a complex space. Then $X$ is Stein if and only if there exists a continuous strongly plurisubharmonic function $\varphi: X \to \mathbb{R}$ such that $X_\alpha = \{x \in X : \varphi(x) < \alpha\}$ is relatively compact in $X$ for all $\alpha \in \mathbb{R}$.

**THEOREM 2.** (Narasimhan [10, 11]). Let $X$ be a Stein space and let $\varphi: X \to \mathbb{R}$ be a continuous plurisubharmonic function. Then $X_\alpha = \{x \in X : \varphi(x) < \alpha\}$ is Stein and Runge in $X$ for all $\alpha \in \mathbb{R}$.
A particularly useful consequence of the two above theorems is the following well known result:

**Corollary 3.** If $X$ is a Stein space and $K = \hat{K}$ is a compact set in $X$, then $K$ has a neighborhood basis of Stein open sets which are Runge in $X$.

We also need the following theorem due to Richberg [14].

**Theorem 4.** If $\varphi$ is a continuous strongly plurisubharmonic function on a countably compact complex manifold $M$ and $\tau : M \to \mathbb{R}^+$ is a strictly positive continuous function, then there exists a smooth strongly plurisubharmonic function $\varphi^*$ on $M$ such that $\varphi < \varphi^* < \varphi + \tau$. If $\sigma$ is a continuous nonnegative plurisubharmonic function on a countably compact complex space $X$, $\sigma \equiv 0$ in a neighborhood of the singular locus of $X$ and there exists a bounded continuous strongly plurisubharmonic function on $X$, then for every $\varepsilon > 0$ there exists a smooth plurisubharmonic function $\sigma^*$ on $X$ with $\sigma < \sigma^* < \sigma + \varepsilon$.

Let us consider the Stein space $Z \times \mathbb{C}$ where $Z = \{z \in \mathbb{C}^4 : z_1z_2 = z_3z_4\}$. If $\Omega^{\text{open}} \subset Z \times \mathbb{C}$, we can define a distance function $\delta : \Omega \to \mathbb{R} \cup \{\infty\}$ as follows. For any $\varphi = (p, c) \in \Omega$, we let

$$\delta(\varphi) = \sup \{r : (p, c+z) \in \Omega \text{ for all } z \in \mathbb{C}, |z|<r\}.$$

**Proposition 5.** The function $-\log \delta : \Omega \to \mathbb{R} \cup \{-\infty\}$ is plurisubharmonic if $\Omega$ is Stein, except on those connected components of $\Omega$ where $-\log \delta \equiv -\infty$.

**Proof.** By the theorem of Siu [15] there exists a domain of holomorphy, $\hat{\Omega}$, in $\mathbb{C}^4$ such that $\hat{\Omega} \cap (Z \times \mathbb{C}) = \Omega$. If we define $\hat{\delta} : \hat{\Omega} \to \mathbb{R} \cup \{\infty\}$ in the same way as $\delta$, we obtain a plurisubharmonic function $-\log \hat{\delta} : \hat{\Omega} \to \mathbb{R} \cup \{-\infty\}$ such that $-\log \hat{\delta} \mid \Omega = -\log \delta$.

3. **$Z$ as a branched Riemann domain.**

In the paper of Andreotti and Narasimhan [1] they make fundamental use of the fact that a pure $n$-dimensional Stein space $X$ may be realized as a branched Riemann domain over $\mathbb{C}^n$ in many different ways. Although the singular points of $X$ necessarily are branch points, one can always make the branch locus avoid any given regular point.

Let $Z = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : z_1z_2 = z_3z_4\}$. We consider two holomorphic maps $\Phi_1, \Phi_2 : \mathbb{C}^3 \to Z$, by

$$\Phi_1(t, w, \eta) = (w, t\eta, \eta, tw) \quad \text{and} \quad \Phi_2(t, w, \eta) = (tw, \eta, t\eta, w).$$

The following lemma is easily verified.
Lemma 6. $\Phi_1(t, 0, 0) = \Phi_2(t, 0, 0) = 0$ and if

$$U = \{ (t, w, \eta) \in \mathbb{C}^3 ; \ (w, \eta) \neq (0, 0) \}$$

then $\Phi_i|_U$ is biholomorphic onto the open set $\Phi_i(U)$, $i = 1, 2$. Moreover $Z - (0) = \Phi_1(U) \cup \Phi_2(U)$.

We will now define four holomorphic maps $\pi_i : Z \to \mathbb{C}^3$, $i = 1, 2, 3, 4$. More precisely $\pi_1(z_1, z_2, z_3, z_4) = (z_1, z_2, z_3 + 2z_4)$, $\pi_2(z) = (z_1, z_2, 2z_3 + z_4)$, $\pi_3(z) = (z_1 + 2z_2, z_3, z_4)$ and $\pi_4(x) = (2z_1 + z_3, z_3, z_4)$. Also define the holomorphic functions $f_j : Z \to \mathbb{C}$ by $f_1(z) = z_3 - 2z_4$, $f_2(z) = 2z_3 - z_4$, $f_3(z) = z_1 - 2z_2$ and $f_4(z) = 2z_1 - z_2$.

Lemma 7. The holomorphic map $\pi_j : Z \to \mathbb{C}^3$ makes $Z$ into a doubly sheeted branched covering of $\mathbb{C}^3$. The set of branch points is precisely $S_j = \{ f_j = 0 \}$. If $p \in S_j - \{ 0 \}$ one can find local holomorphic coordinates $w = (w_1, w_2, w_3, p = 0$ on $Z$ and local holomorphic coordinates $\eta = (\eta_1, \eta_2, \eta_3)$. $\pi_j(p) = 0$ on $\mathbb{C}^3$ such that

$$\pi_j(w_1, w_2, w_3) = (w_1, w_2, w_3^2) = (\eta_1, \eta_2, \eta_3).$$

The proof of this is a straightforward computation and will be omitted. We should also point out that in the coordinates system of the lemma, $f_j/w_3$ is a nonzero holomorphic function.

Following the argument of Andreotti and Narasimhan [1] we obtain plurisubharmonic functions on open subsets $\Omega$ of $Z \times \mathbb{C}$ which are locally Stein away from $(0) \times \mathbb{C}$. First we define $\tilde{\pi}_j : Z \times \mathbb{C} \to \mathbb{C}^4 = \mathbb{C}^3 \times \mathbb{C}$ by $\tilde{\pi}_j(p, c) = (\pi_j(p), c)$. Clearly this defines $Z \times \mathbb{C}$ as a branched Riemann domain over $\mathbb{C}^4$ with branch locus $\tilde{S}_j = \{ \tilde{f}_j = 0 \}$ where $\tilde{f}_j(p, c) = f_j(p)$.

For any $j \in \{ 1, 2, 3, 4 \}$, $\tilde{\pi}_j : \Omega - \tilde{S}_j \to \mathbb{C}^4$ realizes this as an unbranched Riemann domain. From the classical theory on the Levi problem one now has that $-\log d_j : \Omega - \tilde{S}_j \to \mathbb{R}$ is continuous and plurisubharmonic. Here $d_j(q)$ is the radius of the largest ball centered at $\tilde{\pi}_j(q)$ onto which $\tilde{\pi}_j$ maps a neighborhood of $q$ biholomorphically. Let us define $\varphi_j : \Omega \to \mathbb{R} \cup \{-\infty\}$ by

$$\hat{\varphi}_j = -\log d_j + 3 \log |\tilde{f}_j| \text{ on } \Omega - \tilde{S}_j, \quad \hat{\varphi}_j \equiv -\infty \text{ on } \Omega \cap \tilde{S}_j.$$

Proposition 8. The function $\varphi_j = \max \{ \hat{\varphi}_j, 0 \}$ is continuous and plurisubharmonic on $\Omega$. Moreover, if $q \in \partial \Omega - \tilde{S}_j$, $\varphi_j(p) \to \infty$ when $p \in \Omega$ and $p \to q$.

Proof. It is clear that if $q \in \partial \Omega - \tilde{S}_j$, then $\varphi_j(p) \to \infty$ when $p \in \Omega$ and $p \to q$. In fact $\varphi_j(p)$ grows like $-\log \text{dist} (p, \partial \Omega)$ measured in any smooth Hermitian metric defined on $Z \times \mathbb{C}$ near $q$.

Near a point $q \in \Omega \cap \tilde{S}_j$, $q \neq 0$, $-\log d_j$ grows like $2 \log |\tilde{f}_j|$. Hence $\varphi_j$ is
plurisubharmonic across $\tilde{S}_j$ away from $(0) \times C$. To complete the proof it suffices to show that if \( q = (0, c) \in \Omega \), then $\varphi_j \equiv 0$ in a neighborhood of $q$, because then $\varphi_j$ is locally on $\Omega$ the restriction to $\Omega$ of a plurisubharmonic function defined on an open set in $C^4$.

Let us consider $j = 1$. Then one computes that the image of the branch locus is

$$\{ \tau = (\tau_1, \tau_2, \tau_3, \tau_4) ; \quad \tau_1 \tau_2 = \tau_3^2/8 \} = S'_1.$$

Moreover $|\tilde{f}_1|^2(\tilde{q}) = |\tau_3^2 - 8\tau_1\tau_2|$ if $\tilde{\pi}_1(\tilde{q}) = \tau$. Hence already $-\log d_1 + 2 \log |\tilde{f}_1|$ approaches $-\infty$ when $\tilde{q} \to q$. The same argument applies to $j = 2, 3, 4$.

We remark that we could have defined $\varphi_j = \max \{-\log d_j + 2 \log |\tilde{f}_j|, 0\}$ without altering the conclusion of Proposition 8.

4. Another distance function.

We have in the preceding sections described two sorts of plurisubharmonic functions on a Stein open set $\Omega$ of $Z \times C$. One is the $\varphi_j$'s which blow up at nonsingular boundary points and the other measures boundary distance in the $C$-direction. In this section we want to construct plurisubharmonic functions which blow up the $Z$-direction when we approach a point $(0, c) \in \partial \Omega$.

Let us first define a holomorphic map $\Gamma : C^4 \to Z$ by

$$\Gamma(w) = (w_1w_2, w_3w_4, w_1w_3, w_2w_4).$$

**Lemma 9.** The holomorphic map $\Gamma : C^4 \to Z$ is onto. Furthermore

$$\Gamma^{-1}(0) = \{w_1 = w_4 = 0\} \cup \{w_2 = w_3 = 0\}$$

while for every $w \in C^4 \setminus \Gamma^{-1}(0)$, we have

$$\Gamma^{-1}(\Gamma(w)) = \{(w_1\tau, w_2/\tau, w_3/\tau, w_4\tau) ; \quad \tau \in C \setminus \{0\}\}. $$

The proof is straightforward and may be omitted. Let us now consider the map $\tilde{\Gamma} : C^5 \to Z \times C$ by $\tilde{\Gamma}(p, c) = (\Gamma(p), c)$. For any open set $\Omega \subset Z \times C$ we can define the distance functions $A_1, A_2 : \tilde{\Omega} = \tilde{\Gamma}^{-1}(\Omega) \to R \cup \{\infty\}$ as follows:

Let

$$A_1(w, c) = \sup \{r ; \quad (w_1 + \tau_1, w_2, w_3, w_4 + \tau_2, c) \in \tilde{\Omega} \quad \text{for all } (\tau_1, \tau_2) \in C^2, |\tau_1|^2 + |\tau_2|^2 < r^2 \}$$

and let

$$A_2(w, c) = \sup \{r ; \quad (w_1, w_2 + \tau_1, w_3 + \tau_2, w_4, c) \in \tilde{\Omega} \quad \text{for all } (\tau_1, \tau_2) \in C^2, |\tau_1|^2 + |\tau_2|^2 < r^2 \}. $$
Lemma 10. $A_1 \cdot A_2$ is constant on each fibre of $\hat{\Gamma}$. Moreover, if $\hat{\Gamma}(q) = (p, c)$ and if $(0, c) \notin \Omega$, then $A_1 \cdot A_2(q) \leq 2\|p\|.$

Proof. We easily check that $A_1(w_1 \tau, w_2/\tau, w_3/\tau, w_4 \tau, c) = |\tau|A_1(w, c)$ and

$$A_2(w_1 \tau, w_2/\tau, w_3/\tau, w_4 \tau, c) = \frac{1}{|\tau|} A_2(w, c)$$

from which the first statement follows.

Next assume that $q = (w, c) \in \hat{\Omega}$ and that $(0, c) \notin \Omega$. Let $(z, c) = \hat{\Gamma}(w, c)$, and assume $|z_1| = \max_{j=1,\ldots,4} \{|z_j|\}$. The argument is similar for the other possibilities. By the first statement, we may suppose that we have chosen $w_1 = \sqrt{|z_1|}$. Thus $w_2 = \sqrt{|z_1|}, w_3 = z_3/\sqrt{|z_1|}$ and $w_4 = z_4/\sqrt{|z_1|}$ as one deduces from the definition of $\hat{\Gamma}$. In particular, this implies that $|w_3|, |w_4| \leq |w_1| = |w_2| = \sqrt{|z_1|}$. Since $\hat{\Gamma}^{-1}(0, c) \subset \mathbb{C}^3 - \hat{\Omega}$, one obtains that

$$A_1(w, c) \leq (|w_1|^2 + |w_4|^2)^{1/2} \leq \sqrt{2}\sqrt{|z_1|}$$

and similarly $A_2(w, c) \leq \sqrt{2}\sqrt{|z_1|}$. Hence $A_1 \cdot A_2(w, c) \leq 2|z_1| \leq 2\|z\|$ as desired.

Definition 11. Assume $\Omega \subset \mathbb{Z} \times \mathbb{C}$ is an open subset. Then $\Delta(q): \Omega \to \mathbb{R} \cup \{\infty\}$ is defined as $\Delta(q) = A_1(q) \cdot A_2(q)$ for any $q \in \hat{\Gamma}^{-1}(q)$.

Lemma 12. Assume $\Omega \subset \mathbb{Z} \times \mathbb{C}$ is Stein or an increasing union of Stein open sets or is locally Stein. Then $\Delta^* = \max \{-\log \Delta, 0\}: \Omega \to \mathbb{R}$ is plurisubharmonic. Moreover $\Delta^* \equiv 0$ in an open neighborhood of $(0, c)$ if $(0, c) \in \Omega$. Also if $(0, c) \notin \Omega$, $\Delta^*(p, c) \geq -\log \Delta \geq -\log \|p\| - \log 2$ whenever $(p, c) \in \Omega$.

Proof. The set $\hat{\Omega} = \hat{\Gamma}^{-1}(\Omega)$ is a domain of holomorphy. This implies that $-\log \Delta_1$ and $-\log \Delta_2$ are plurisubharmonic on $\hat{\Omega}$. Therefore $-\log \Delta_1 \Delta_2$ is also plurisubharmonic. Clearly $-\log \Delta_1 \Delta_2 \equiv -\infty$ on $\hat{\Gamma}^{-1}(0, c)$ if $(0, c) \in \Omega$. The Lemma now follows from Lemma 10 and the observation that for every $(z^0, c) \in (\mathbb{Z} - (0)) \times \mathbb{C}$ there exists a holomorphic map $T$ defined in an open neighborhood of $(z^0, c)$ in $\mathbb{Z} \times \mathbb{C}$ to $\mathbb{C}^3$ such that $\hat{\Gamma} \circ T = \text{Id}$. For example, if $z^0_4 \neq 0$, we can define

$$T(z, c) = (1, z_1, z_3, z_4/z_1, c).$$

Lemma 13. If $\Omega \subset \mathbb{Z} \times \mathbb{C}$ is locally Stein or an increasing union of Stein open sets and if $\partial \Omega$ is smooth away from $0 \times \mathbb{C}$, then $\Delta^*$ is continuous.
This is proved using the result by Kerzman [8] that smoothly bounded domains of holomorphy are taut. Let us just point out that if \((w^0, c) \in \mathbb{C}^5, \tilde{F}(w^0, c) = (z^0, c) \neq (0, c)\), and if say

\[ A_1(w^0, c) = \sqrt{w_1^0 \bar{w}_1^0 + w_4^0 \bar{w}_4^0} = \delta, \]

then we can show that \(A_1\) is upper semicontinuous at \((w^0, c)\) by contradiction as follows. Assume for some \(\delta' > 0\) that there exists a sequence \(\{(w^n, c_n)\}_{n=1}^{\infty} \in \tilde{F}^{-1}(\Omega) = \tilde{\Omega}\) such that \((w^n, c_n) \to (w^0, c)\) and

\[ \{(w_1, w_2^0, w_3^0, w_4, c_n) ; \sqrt{|w_1 - w_1^n|^2 + |w_4 - w_4^n|^2} < \delta + \delta'\} \]

is contained in \(\tilde{\Omega}\) for all \(n\). By tautness away from \(\tilde{F}^{-1}(0 \times \mathbb{C})\) it follows that \(\tilde{\Omega}\) contains

\[ \{(w_1, w_2^0, w_3^0, w_4, c) ; \sqrt{|w_1 - w_1^0|^2 + |w_4 - w_4^0|^2} < \delta + \delta' \text{ and } (w_1, w_4) \neq (0, 0)\} \]

Since \(\tilde{\Omega}\) is a domain of holomorphy, \((0, w_2^0, w_3^0, 0, c) \in \tilde{\Omega}\) as well and hence \(A_1(w^0, c) \geq \delta + \delta'\) which gives a contradiction.

5. The Levi problem.

Assume \(\Omega\) is an open subset of \(X = Z \times \mathbb{C}\), \(Z = \{z \in \mathbb{C}^4 ; z_1 z_2 = z_3 z_4\}\).

**Theorem 14.** If \(\Omega\) is locally Stein, i.e., for every point \(p \in \partial \Omega\) there is an open neighborhood \(U(p)\) such that \(U(p) \cap \Omega\) is Stein, then \(\Omega\) is Stein.

**Proof.** The function \(\|z\|^2 + c\bar{z}, z \in Z, c \in \mathbb{C}\) is a continuous plurisubharmonic function on \(X\). Hence it follows from Theorem 2 that we may assume that \(\Omega \subset \subset Z \times \mathbb{C}\). The maps \(\pi_j : \Omega - \tilde{S}_j \to \mathbb{C}^4\) realize \(\Omega - \tilde{S}_j\) as a locally Stein unbranched Riemann domain over \(\mathbb{C}^4, j = 1, 2, 3, 4\). By Okà's [12] solution of the Levi Problem it follows that \(\Omega - \tilde{S}_j\) is Stein. Therefore the functions \(\varphi_j\) constructed in Proposition 8 are continuous plurisubharmonic functions on \(\Omega\) which are identically zero in a neighborhood of \(\Omega \cap \{0 \times \mathbb{C}\}\).

Hence by Theorem 4 there is a smooth plurisubharmonic function \(\varphi : \Omega \to \mathbb{R}\) such that \(|\varphi - \sup \varphi_j| < 1\) on \(\Omega\). In particular \(\varphi(p) \to \infty\) if \(p \in \Omega\) approaches any point \(q \in \partial \Omega - (0 \times \mathbb{C})\).

This implies, by Sard's theorem, that there exists arbitrarily large \(\alpha \in \mathbb{R}\) such that \(\Omega_\alpha = \{\varphi < \alpha\}\) has smooth boundary away from \((0) \times \mathbb{C}\).

By Theorem 2 it suffices to prove that any such \(\Omega_\alpha\) is Stein. So we fix an \(\Omega_\alpha\) with the above boundary property in the rest of the proof.

From Lemma 13, applied to \(\Omega_\alpha\), it follows that \(A^*\) is a continuous
nonnegative plurisubharmonic function on $\Omega_z$ which is identically zero in an open set containing $\Omega_z \cap (0 \times C)$. Hence using Theorem 4 again, we find a smooth plurisubharmonic function $\hat{A} : \Omega_z \to R$ such that $|A - A^*| < 1$ on $\Omega_z$.

Let $\Omega^{(\beta)} = \{ q \in \Omega_z ; \hat{A}(q) < \beta \}$ for $\beta \in R$. From Sard's theorem it follows that $\partial \Omega^{(\beta)}$ is smooth away from $\partial \Omega_z$ and $\Omega_z \cap (0 \times C)$ for arbitrarily large $\beta$. We fix such an $\Omega^{(\beta)}$ in the rest of the proof and observe that by Theorem 2 it suffices to show that $\Omega^{(\beta)}$ is Stein.

We will construct a continuous strongly plurisubharmonic exhaustion function on $\Omega^{(\beta)}$. Since, if $\varphi_j : \Omega^{(\beta)} \to R$ is as in Proposition 8, max$_{j=1,2,3,4} \{ \varphi_j \} + \|z\|^2 + c\bar{c}$ is a nonnegative strongly plurisubharmonic function which blows up at every boundary point of $\Omega^{(\beta)}$, except along $(0) \times C$, it suffices to find a continuous nonnegative plurisubharmonic function on $\Omega^{(\beta)}$ which blows up at every boundary point of $\Omega^{(\beta)}$ on $(0) \times C$. In fact, we will prove that max $\{ -\log \delta, \gamma \} = \delta^*$ is such a function if $\delta$ is as in Proposition 5, and if $\gamma$ is sufficiently large.

The local Stein-ness of $\Omega^{(\beta)}$ follows from Theorem 2 and implies via Proposition 5 that $\delta^*$ is plurisubharmonic if $\gamma$ is sufficiently large. It remains to prove that $\delta$ is continuous and that $\delta \to 0$ when we approach $\partial \Omega^{(\beta)} \cap (0 \times C)$.

Let $U = \Omega^{(\beta)} \cap (0 \times C)$ and consider a point $(0, c) \notin U$. First of all, we observe that $(0, c) \notin \Omega_z$ since we may assume $\beta >> 1$. If $(p, c) \in \Omega^{(\beta)}$, then $\hat{A} (p, c) < \beta$ and hence $A^*(p, c) < \beta + 1$. From Lemma 12 it now follows that $-\log \|p\| - \log 2 < \beta + 1$, and so $\|p\| > e^{-\beta - 2}$. Therefore, if $(p, c) \in \Omega^{(\beta)}$ and $\|p\| < e^{-\beta - 2}$, we must necessarily have $(0, c) \in U$. This implies that $\delta \to 0$ when we approach $\partial \Omega^{(\beta)} \cap (0 \times C)$. Also, this implies that $\delta$ is continuous at every point in $U$.

Fix a point $(p, c) \in \Omega^{(\beta)}$, $p \neq 0$. We will show that $\delta$ is continuous at this point. Since $\Omega^{(\beta)}$ is open, $\delta$ is lower semicontinuous. Assume $\delta$ is not upper semicontinuous. Let $(p, c')$ be a point on $\partial \Omega^{(\beta)}$ with $|c' - c| = \delta(p, c)$. There exists an $\varepsilon > 0$ and a sequence $\{p^n\}_{n=1}^\infty$ such that $p^n \to p$ and

$$A^n = \{ (p^n, c'') ; |c'' - c| < \delta(p^n, c) + \varepsilon \} \subset \Omega^{(\beta)}$$

for all $n$. Let $A = \{ (p, c') ; |c' - c| < \delta(p, c) + \varepsilon \}$, and observe that since $A^n \subset \Omega_z$ and $\Omega_z$ is taut at smooth boundary points, Kerzman [8], it follows that $A \subset \Omega_z$. This implies that $(p, c') \in \partial \Omega^{(\beta)} \cap \Omega_z$, which contradicts the same result of Kerzman since $\partial \Omega^{(\beta)}$ is smooth away from $\partial \Omega_z$ and $(0) \times C$. Hence $\delta$ is upper semicontinuous at $(p, c)$ as well.

6. The union problem.

Let $\{\Omega_z\}$ be a sequence of Stein open subsets of

$$X = Z \times C, \quad Z = \{ z \in C^4 ; z_1 z_2 = z_3 z_4 \}.$$
THEOREM 15. If $\Omega_1 \subset \Omega_2 \subset \ldots$ and $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$, then $\Omega$ is Stein.

We will first prove a standard Lemma which reduces the proof to an estimate of the hulls of compact subsets of $\Omega$.

LEMMA 16. If for every compact set $K \subset \Omega$ there exists a compact set $F \subset \Omega$ such that $\hat{K}_{\Omega_n} \subset F$ for all $\Omega_n \ni K$, then $\Omega$ is Stein.

PROOF OF THE LEMMA. Choose compact sets $\{K_n\}_{n=1}^{\infty}$ such that $K_n \subset \text{int} K_{n+1}$ for all $n$ and $\Omega = \bigcup K_n$. Let $\{F_n\}_{n=1}^{\infty}$ be the corresponding compact sets given by the hypothesis of the Lemma. We may assume that $F_n \subset F_{n+1}$ for all $n$. To show that $\Omega$ is Stein, it suffices to prove that for every sequence $\{p_n\} \subset \Omega$ without cluster point in $\Omega$ there exists a holomorphic function $f: \Omega \to \mathbb{C}$ such that $\sup_n |f(p_n)| = \infty$.

Taking suitable subsequences, we may assume that $p_n \in K_{n+1} \subset F_{n+1} \subset \Omega_n$ and that $p_n \notin F_n$. Choose inductively a sequence $\{f_n\}$ of holomorphic functions, $f_n: \Omega_n \to \mathbb{C}$ with the property that

(i) $|f_{n+1} - f_n| < 1/2^n$ on $K_{n+1}$
(ii) $f_n(p_k) = k$, $k = 1, \ldots, n$.

This clearly is possible. If $f = \lim_{n \to \infty} f_n$, then $f$ has the desired properties.

PROOF OF THE THEOREM. Let us fix a compact set $K \subset \Omega$ and show that there exists a compact set $F \subset \Omega$ such that $\hat{K}_n := \hat{K}_{\Omega_n} \subset F$ for all $n$. By Lemma 16 this will complete the proof.

Let $\varphi = \max_{j=1,2,3,4} \varphi_j$ be the function constructed in Proposition 8. Since the Union Problem has been solved on unbranched Riemann domains, the function $\varphi$ is plurisubharmonic on $\Omega$.

By Theorem 2 we may assume that $\Omega$ is bounded. Using Theorem 4, we find a smooth plurisubharmonic function $\varphi^*$ on $\Omega$ such that $|\varphi - \varphi^*| < 1$.

If $m = \max_K \varphi^*$, it follows that $\hat{K}_n \subset \{q \in \Omega \ ; \ \varphi^*(q) < m + 1\} = \Omega^{m+1}$ for all $n$ such that $\Omega_n \ni K$. This is clear because $\{\varphi^* < \alpha\} \cap \Omega_n$ is Runge in $\Omega_n$ for all $\alpha$ by Theorem 2.

We fix an $\alpha > m + 1$ such that $\partial \Omega^\alpha$ is smooth away from $(0) \times \mathbb{C}$. Next we consider the function $\Delta^* : \Omega^\alpha \to \mathbb{R}$ constructed in Lemma 12. By Lemma 13, $\Delta^*$ is continuous and plurisubharmonic on $\Omega^\alpha$.

Let $m'$ be the maximum value of $\Delta^*$ on $K$. We fix a $\beta > m'$. Let us denote by $\Omega_n^\beta$ the set $\Omega^\alpha \cap \Omega_n$ and by $\Omega_n, \beta$ the set $\{q \in \Omega_n^\beta : \Delta^* < \beta\}$. Then $\Omega_n, \beta \ni K$ and is Runge in $\Omega_n$. Therefore $\hat{K}_{\Omega_n} = \hat{K}_{\Omega_n, \beta} \subset \Omega_n, \beta$. In particular

$$\hat{K}_n \subset \{q \in \Omega^\alpha : \Delta^* < \beta\} = \Omega^\beta_{\beta} \quad \text{for all large } n.$$
The sets $U = \Omega \cap (0 \times \mathbb{C})$ and $\Omega^x \cap (0 \times \mathbb{C})$ and $\Omega^x_\beta \cap (0 \times \mathbb{C})$ are all equal since $\varphi$ and $A^*$ are 0 in an open set containing $U$.

We obtain from Lemma 12 that if $(p, c) \in \Omega^x_\beta$ and $(0, c) \notin U$, then $\|p\| > e^{-\beta - 1}$. Now the sets $\Omega^x_{\beta, n} = \Omega^x_\beta \cap \Omega_n$ are Stein and

$$\Omega^x_{\beta, 1} \subset \Omega^x_{\beta, 2} \subset \ldots \subset \bigcup_{n=1}^{\infty} \Omega^x_{\beta, n} = \Omega^x_\beta.$$

Let $n_0$ be some index such that $K \subset \Omega^x_{\beta, n_0}$. If $n \geq n_0$ and $f$ is a holomorphic function on $\Omega^x_{\beta, n}$, then $\partial f/\partial c'$ is also holomorphic on $\Omega^x_{\beta, n}$. Moreover, choose a positive number $\varepsilon > 0$ such that $(p, c) \in K$ and $c' \in \mathbb{C}$, $|c'| < \varepsilon$ implies that $(p, c + c') \in \Omega^x_{\beta, n_0}$. It follows that if $(p, c) \in \hat{K}_{\Omega^x_\beta, n}$, $n \geq n_0$ then $(p, c + c') \in \Omega^x_{\beta, n}$ for all $c' \in \mathbb{C}$, $|c'| < \varepsilon$. In particular, if $\|p\| \leq e^{-\beta - 1}$, then $(0, c) \in U$ and $(0, c + c') \in U$ for all $c' \in \mathbb{C}$, $|c'| < \varepsilon$.

In conclusion, we have shown that if $K$ is a compact subset of $\Omega$, then there exists a compact subset $F$ of $\Omega$ such that $\hat{K}_{\Omega^x_\beta} \subset F$ whenever $K \subset \Omega_n$. This completes the proof of the Theorem.


As always, let $X = \mathbb{C} \times \mathbb{C}$ with $Z = \{z \in \mathbb{C}^4 : z_1z_2 = z_3z_4\}$. Let $\{\Omega_t\}_{t \in \mathbb{R}}$ be a family of Stein open subsets of $X$ such that $\Omega_t$ is a union of connected components of the interior of $\cap_{t > t} \Omega_t$ and such that $\cup_{t < t} \Omega_t$ is a union of connected components of $\Omega_t$ for each $t \in \mathbb{R}$.

**Theorem 17.** If $t_1 < t_2$ are real numbers, then $\Omega_{t_1}$ is Runge in $\Omega_{t_2}$.

**Proof.** We fix a $t \in \mathbb{R}$ and a compact set $K \subset \Omega_t$. To arrive at a contradiction, let us assume that for some $t > t$ the set $\hat{K}_t = \hat{K}_{\Omega_t}$ is not contained in $\Omega_t$. From Corollary 3 it follows that $\hat{K}_t$ is contained in the union $\cup_{\lambda < t} \Omega_{\lambda}$. Hence there exists a number $t'$, $t \leq t' < t$ such that $\hat{K}_t \subset \Omega_{\lambda}$ when $\lambda > t'$ and $\hat{K}_t \subset \Omega_{\lambda}$ when $\lambda < t'$. Let us assume that $\hat{K}_t \cap \partial \Omega_{t'} = \emptyset$. It would then follow from Corollary 3 that $\hat{K}_t \subset \Omega_{t'}$. This implies that $t' > t$ since $\hat{K}_t \subset \Omega_{t'}$. Therefore $\hat{K}_t \subset \cup_{\lambda < t'} \Omega_{\lambda}$, again by Corollary 3. Hence $\hat{K}_t \subset \Omega_{t'}$ for some $\lambda < t'$ contradicting the choice of $t'$.

We may assume therefore that $\hat{K}_t \cap \partial \Omega_{t'} \neq \emptyset$, and hence we may also assume that $t = t'$.

Summarizing, we assume that $t < t$ and that $K$ is a compact subset of $\Omega_t$ such that $\hat{K}_t \cap \partial \Omega_{t'} \neq \emptyset$ while $\hat{K}_t \subset \Omega_{\lambda}$ for all $\lambda > t$. We denote $\hat{K}_t \cap \partial \Omega_{t'}$ by $F$. Let us first prove that $F \subset (0) \times \mathbb{C}$.

Pick a point $(p, c) \in F$ with $p \neq 0$. There exists a $j \in \{1, 2, 3, 4\}$ such that $(p, c) \notin \hat{S}_j$. Hence the map $\hat{n}_j : \mathbb{C} \times \mathbb{C} \to \mathbb{C}^4$ is regular at $(p, c)$. Let $d^*_j$ be the
distance function on $\Omega_\lambda - \overline{S}_J$ obtained from viewing this set as an unbranched Riemann domain over $C^4$.

If $\lambda > t$, the functions $\varphi_j^\lambda = \max \{ -\log d_j^\lambda + 3 \log |f_j|, 0 \}$ are continuous and plurisubharmonic on $\Omega_\lambda$ as was established in Proposition 8. Moreover they are uniformly bounded on $K$. By Corollary 3, $\tilde{K}_\lambda = \tilde{K}_r$ for all $\lambda \in (t, \tau)$. Therefore, by Theorem 2, the functions $\varphi_j^\lambda$ are uniformly bounded at $(p, c)$, $\lambda \in (t, \tau)$. Hence $(p, c)$ is an interior point of $\bigcap_{\lambda > t} \Omega_\lambda$. This contradicts that $(p, c) \in \partial \Omega_t$ and that $\Omega_t$ consists of connected components of the interior of $\bigcap_{\lambda > t} \Omega_\lambda$. This shows that we must have $F \subset (0) \times C$.

Let now $U = \Omega_t \cap ((0) \times C)$ and let $\epsilon > 0$ be a number such that $(p, c + c') \in \Omega_t$ whenever $(p, c) \in K$ and $|c'| < 4\epsilon$.

We will show that there exists a positive number $\delta > 0$ such that if $(p, c) \in \Omega_t$ and $p \neq 0$ and $\|p\| < \delta$ and moreover $(0, c) \notin U$ or has distance from $\partial U$, $U$ thought of as an open set in $C$, less than $\epsilon$, then $(p, c) \notin \tilde{K}_r$.

Let $\varphi_j^\lambda : \Omega_\lambda \to \mathbb{R}$ be the continuous plurisubharmonic function constructed in Proposition 8, $\lambda \in (t, \tau)$. From the inclusions we have the obvious estimate that $\varphi_j^\lambda \leq \varphi_j^\lambda$ on $\Omega_t$.

Using Theorem 4 we find smooth plurisubharmonic functions $\varphi^\lambda : \Omega_\lambda \to \mathbb{R}$ such that

$$\left| \varphi^\lambda - \max_{j=1, \ldots, 4} \{ \varphi_j^\lambda \} \right| < 1 \quad \text{on} \quad \Omega_\lambda.$$

Here we use again the observation that by Theorem 2 we may assume that the sets $\Omega_\lambda \subset \subset X$ for all $\lambda$.

We choose a number $m$ such that $\Omega'_\lambda = \{ q \in \Omega_t : \varphi^\lambda(q) < m \}$ has smooth boundary away from $0 \times C$ and such that if $(p, c) \in K$ and $|c'| < 3\epsilon$, then $(p, c + c') \in \Omega'_\lambda$. Next we define $\Omega'_\lambda$ for $\lambda \in (t, \tau)$ as $\{ q \in \Omega_\lambda : \varphi^\lambda(q) < m_\lambda \}$ where $m_\lambda \in (m + 2, m + 3)$ is chosen such that $\Omega'_\lambda$ has smooth boundary away from $\{0\} \times C$. Then each $\Omega'_\lambda$ is Stein and we have the estimates

(i) if $(p, c) \in K$ and $|c'| < 3\epsilon$, then $(p, c + c') \in \Omega'_\lambda$;

(ii) $\tilde{K}_\lambda \subset \subset \Omega'_\lambda$ and

(iii) For any positive number $\eta > 0$ there exists a $\lambda(\eta) > t$ such that if $\lambda \in (t, \lambda(\eta))$ and $(p, c) \in \partial \Omega_t$, $\|p\| > \eta$, then $(p, c) \notin \Omega'_\lambda$.

In fact (i) follows since $\Omega'_\lambda \supset \Omega'_\tau$, (ii) follows from Corollary 3 and (iii) follows since $(p, c) \in \partial \Omega_t$ cannot be interior points of $\bigcap_{\lambda > t} \Omega_\lambda$.

Now let $A^\lambda : \Omega'_\lambda \to \mathbb{R}$ be the functions constructed in Lemma 12. From Lemma 13 it follows that $A^\lambda$ is continuous and plurisubharmonic. We have the obvious estimate $A^\lambda \geq A^\lambda$ on $\Omega'_\lambda$ for all $\lambda \in (t, \tau)$.
We choose a $k \in \mathbb{R}$ such that if $(p, c) \in K$ and $|c'|<2\varepsilon$, then $(p, c + c') \in \Omega'_t$ and $A^*_x(p, c) < k$. If we let

$$\Omega'_{\lambda} = \{ q \in \Omega'_x ; \ A^*_x < k \}, \quad \lambda \in [t, \tau),$$

then if $(p, c) \in K$ and $|c'|<2\varepsilon$, then $(p, c + c') \in \Omega'_{\lambda}$. Furthermore, by Corollary 3, $\hat{K}_r \subset \Omega'_{\lambda}$ for all $\lambda \in (t, \tau)$. We let $\delta = \frac{\lambda}{4}e^{-k}$ and choose a point $(p, c) \in \Omega_t$ with $p \neq 0$ and $\|p\|<\delta$. To arrive at a contradiction, let us assume that $(p, c) \in \hat{K}_r$ and that $(0, c) \notin U$ or has distance from $\partial U$ less than $\varepsilon$. Since each $\Omega'_x$ is Stein it follows that if $|c'|<2\varepsilon$, then $(p, c + c') \in \Omega'_{\lambda}$, $\lambda \in (t, \tau)$. Hence we may find a possibly different point $(p, c) \in \Omega_t$ with $p \neq 0$ and $\|p\|<\delta$ such that $(0, c) \notin U$ and $(p, c) \in \Omega'_{\lambda}$ for all $\lambda \in (t, \tau)$. We consider a point $(w^0, c) \in \mathbb{C}^\delta$ such that $\hat{\Gamma}(w^0, c) = (p, c)$ in the notation of section 4. Let $\Sigma_1$ be the two dimensional complex plane

$$\Sigma_1 = \{ (w, c) ; \ w_1 = w^0_1, w_4 = w^0_4 \}$$

and similarly let

$$\Sigma_2 = \{ (w, c) ; \ w_2 = w^0_2, w_3 = w^0_3 \}.$$

It is possible to choose $w^0$ so that $\max|w^0_i|<\sqrt{\delta}$, as is seen from the definition of $\hat{\Gamma}$. Let $B_\delta$ be the open ball in $\Sigma_1$ centered at $(w^0, c)$ with radius $\sqrt{\frac{\delta}{2}}\sqrt{\delta}$. Then the point $q = (w^0_0, 0, 0, w^0_1, c) \in B_\delta$. We let $\tilde{\Omega}_t$ be the pull back to $\mathbb{C}^\delta$ of $\Omega_t$. Since $(0, c) \notin \Omega_t$, it follows that $q \notin \tilde{\Omega}_t$. Since $\tilde{\Omega}_t$ is a domain of holomorphy, it follows that $\tilde{\Omega}_t \cap \Sigma_1$ is also a domain of holomorphy in $\Sigma_1 \cong \mathbb{C}^2$. This implies that there must exist a point $q' = q$ in $B_\delta$ such that $q' \in \partial\tilde{\Omega}_t$, and hence $q' = \hat{\Gamma}(q') \in \partial\Omega_t$. Therefore, (iii) gives that $q' \notin \Omega'_x$ for all sufficiently small $\lambda > t$. In particular we get that $q' \notin \hat{\Gamma}^{-1}(\Omega'_x)$ for all such $\lambda$. Let $A^{*x}_x, A^*_x$ be the distance functions on $\hat{\Gamma}^{-1}(\Omega'_x)$ used in Lemma 10. We have now the estimate

$$A^*_x(w^0, c) < \sqrt{\frac{2}{\delta}}\sqrt{\delta},$$

and by the same argument applied to $\Sigma_2$, $A^*_x(w^0, c) < \sqrt{\frac{2}{\delta}}\sqrt{\delta}$ also. Since

$$A^*_x(p, c) = \max \{ -\log A^*_{x}A'^*_{x}(w^0, c), 0 \}$$

we get

$$A^*_x(p, c) > -\log 2 - \log \delta = \log 2 + k > k.$$

This contradicts that $(p, c) \in \Omega'_x$ because $\Omega'_x = \{ q \in \Omega'_x ; \ A^*_x < k \}$. Let us fix a number $\delta > 0$ such that if $(p, c) \in \Omega_t$, $p \neq 0$ and $\|p\|<\delta$ and moreover $(0, c) \notin U$ or has distance from $\partial U$ less than $\varepsilon$, then $(p, c) \notin \hat{K}_r$. Denote by $F_0$ the set $\hat{K}_r \cap \partial U$. We know now that if $F_0$ is empty, then $\hat{K}_r \subset U$ by Corollary 3. So we assume that there exists at least one point $(0, c') \in \hat{K}_r \cap \partial U$. Let $H$ be those points in $U \cap \hat{K}_r$ with distance from $\partial U$ in $[\varepsilon/3, \varepsilon/2]$. Then $H$ is a compact set, and we consider this as a subset of $\mathbb{C} \cong \{0\} \times \mathbb{C}$. Hence $c'$ is in a connected component $V$ of $\mathbb{C} \setminus H$. 


Let us first show that $V \supset \Omega_{\lambda}$ if $\lambda > t$ is small enough. The set $\partial V \subset H \subset \Omega_t$, and so there exists a $v > 0$ such that if $(q, c) \in \mathbb{Z}$, $\|q\| \leq v$ and $c \in \partial V$, then $(q, c) \in \Omega_t$. If $V \in \Omega_{\lambda}$, it therefore follows that

$$\{(q, c) ; \|q\| \leq v \text{ and } c \in V\} \subset \Omega_{\lambda}.$$ 

If this holds for all $\lambda > t$, $(0, c')$ must be an interior point of $\cap_{\lambda > t} \Omega_{\lambda}$ which contradicts that $(p, 0) \in \partial \Omega_t$. Hence there exists a point $(0, c'') \notin \Omega_{\lambda}$ whenever $\lambda > t$ is small enough, $c'' \in V$.

By the well known Runge theorem in one complex variable there exists a rational function $P(c) : \mathbb{C} \to \mathbb{C}$ with poles at $c''$ only such that $|P(c')| > 1 > \max_{\Omega_t} |P|$. Since $\Omega_\lambda \cap (0 \times \mathbb{C})$ is a closed subvariety of a Stein space, we may find a holomorphic extension $\bar{P} : \Omega_\lambda \to \mathbb{C}$. Since $F_0$ is compact, we may find a $\lambda_0 > t$ and a finite collection of holomorphic function $\bar{P}_1, \ldots, \bar{P}_l : \Omega_{\lambda_0} \to \mathbb{C}$ such that

$$\max_{j=1, \ldots, l} |\bar{P}_j|(0, c) > 1 \quad \text{for all } c \in F_0$$

and each $|\bar{P}_j| < 1$ on $H$.

Let $W$ be a Stein open set in $\Omega_{\lambda_0}$ containing $\hat{K}_t$ which is Runge in $\Omega_{\lambda_0}$. We can by Corollary 3 assume that $W$ is contained in any given neighborhood of $\hat{K}_t$. In what follows we will assume $W$ is sufficiently small.

Let $\chi : \mathbb{R} \to \mathbb{R}$ be a continuous convex function such that $\chi(x) > 0$ when $x > 0$, $\chi(x) \equiv 0$ when $x \leq 0$. We define a continuous plurisubharmonic function, $\varrho$, on $W$ by

(i) $\varrho \equiv 0$ for those points in $W$ which lie near points in $\hat{K}_t \cap \Omega_t$ except near those in $(0) \times U$ whose distance to $\partial U$ is less than $\varepsilon/2$.

(ii) $\varrho = \chi \cdot \max_j |\bar{P}_j|$ otherwise.

Then by Theorem 2 the set $F_0$ must be empty since $W$ is Runge in $\Omega_{\lambda_0}$ and $\varrho | F_0 > 0$ while $\varrho | K \equiv 0$.

8. Some remarks.

We will list a few other problems than the ones mentioned in the introduction, but which are suggested by the preceding proofs.

**Problem 1.** Assume $\varrho : X \to \mathbb{R}$ is a continuous function on a complex space $X$ such that $\varrho \circ \psi : \Delta \to \mathbb{R}$ is subharmonic whenever $\psi : \Delta \to X$ is a holomorphic map of the open unit disc into $X$. Is $\varrho$ necessarily plurisubharmonic?

This problem was posed in Narasimhan [10]. Clearly there are other similar problems with other regularity conditions on the functions.
Problem 2. If \( \{ \varphi_n \}_{n=1}^{\infty} : X \to \mathbb{R} \) is a sequence of continuous plurisubharmonic functions on \( X \) converging uniformly to \( \varphi : X \to \mathbb{R} \) on compact subsets of the complex space \( X \). Is \( \varphi \) plurisubharmonic?

This problem was posed in Richberg [14]. Again similar problems arise with other regularity conditions on the functions. Theorem 4 of Richberg suggests the following type of problem:

Problem 3. If \( \varphi : X \to \mathbb{R} \) is a plurisubharmonic function on a complex space \( X \), does there exist a sequence of smooth plurisubharmonic functions \( \{ \varphi_n \} : X \to \mathbb{R} \) such that \( \varphi_n \searrow \varphi \) when \( n \to \infty \).

This is of course true if \( X \) is a Stein manifold.

Problem 4. Assume \( \varphi : X \to \mathbb{R} \) is a strongly plurisubharmonic function on a complex space \( X \) such that \( \{ \varphi < \alpha \} \subset \subset X \) for all \( \alpha \in \mathbb{R} \). Is \( X \) Stein?

If we assume in addition that \( \varphi \) is continuous, this is Theorem 2 by Narasimhan [10]. Problem 4 is still open if \( X \) is a complex manifold. If \( \varphi : X \to \mathbb{R} \) is a plurisubharmonic exhaustion function and there exists a continuous strongly plurisubharmonic function \( \psi : X \to \mathbb{R} \), then \( X \) is Stein if it is a complex manifold, Richberg [14], Suzuki [16] and Elenewaig [5].

Problem 5. Assume \( \varphi : X \to \mathbb{R} \) is a plurisubharmonic function on a Stein space \( X \). Is \( X_\alpha = \{ \varphi < \alpha \} \) Runge in \( X \) and/or Stein for any \( \alpha \in \mathbb{R} \)?

This is true if \( X \) is a Stein manifold. Also if \( \varphi \) is continuous it reduces via Richberg's theorem to Theorem 2 by Narasimhan [10].

References

8. N. Kerzman, Personal communication.