A REMARK ON
SINGULAR SUPPORTS OF CONVOLUTIONS

LARS HÖRMANDER

For distributions \( f, g \in \mathcal{E}'(\mathbb{R}^n) \) it was proved in [2] that

\[
(1) \quad \text{ch} \, \text{sing supp} \, (\varphi f) \ast (\psi g) \subset \text{ch} \, \text{sing supp} \, f \ast g
\]

when \( \varphi \) and \( \psi \) are polynomials. Here \( \text{ch} \) denotes convex hull. The question of the validity of (1) for all \( \varphi, \psi \in C^\infty \) was also raised in [2], and in [1] an extension to entire analytic functions \( \varphi, \psi \) of exponential type was stated. (Dr. Dostal has informed the author that the published proof is not correct.) The purpose of this note is to show that the methods of [3] give fairly complete information concerning the validity of (1):

**Theorem 1.** Let \( f, g \in \mathcal{E}'(\mathbb{R}^n) \) and assume that \( \varphi, \psi \in C^\infty(\mathbb{R}^n) \) are real analytic near \( \text{sing supp} \, f \) and \( \text{sing supp} \, g \) respectively. Then it follows that (1) is valid.

**Theorem 2.** In any Denjoy–Carleman class of \( C^\infty \) functions which is strictly larger than the analytic class it is possible to find a function \( \varphi \) such that for some \( f, g \in \mathcal{E}' \), with \( \text{sing supp} \, f = \text{sing supp} \, g = \{0\} \)

\[
\text{sing supp} \, f \ast g = \emptyset, \quad \text{ch} \, \text{sing supp} \, (\varphi f) \ast g = \{0\}.
\]

In the proof we shall use the notations of [3]. In particular, we write

\[
v_f(z, \xi) = (\log |\hat{f}(\xi + z \log |\xi|)|)/\log |\xi|.
\]

Recall that every sequence \( \xi_j \to \infty \) in \( \mathbb{R}^n \) has a subsequence for which \( v_f(z, \xi_{j_k}) \) converges to a plurisubharmonic function (possibly \( -\infty \)) having a supporting function \( H \) in the sense of [3, section 3]. The set of such supporting functions is denoted by \( \mathcal{H}(f) \). We write \( \mathcal{H}(f, g) \) for the set of pairs of supporting functions corresponding to simultaneous limits of \( v_f \) and \( v_g \).

**Theorem 3.** Let \( f \in \mathcal{E}' \) and assume that \( \varphi \in C^\infty \) is real analytic near \( \text{sing supp} \, f \). Then \( (h_1, h) \in \mathcal{H}(\varphi f, f) \) implies \( h_1 \leq h \).

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Proof that Theorem 3 implies Theorem 1. By Theorem 5.1 and Lemma 5.2 in [3] the supporting function of the left hand side of (1) is the supremum of all sums $h_1 + h_2$ with $(h_1, h_2) \in \mathcal{H}(\varphi f, \psi g)$. By Lemma 5.1 in [3] one can find $h_3, h_4$ so that $(h_1, h_2, h_3, h_4) \in \mathcal{H}(\varphi f, \psi g, f, g)$, thus $(h_1, h_3) \in \mathcal{H}(\varphi f, f)$ and $(h_2, h_4) \in \mathcal{H}(\psi g, g)$. Hence Theorem 3 gives $h_1 \leq h_3$ and $h_2 \leq h_4$ so $h_1 + h_2 \leq h_3 + h_4 \in \mathcal{H}(f \ast g)$, by Theorem 5.1 in [3], so Lemma 5.2 in [3] proves Theorem 1.

Proof of Theorem 3. Let $\xi_j$ be a sequence such that $v_f(z, \xi_j)$ and $v_{\psi f}(z, \xi_j)$ converge to plurisubharmonic functions with supporting functions $h$ and $h_1$ respectively. Choose $C$ so that the limit of $v_f(z, \xi_j)$ is $\leq C - 1 + h(\text{Im} z)$. For every $R > 0$ it follows that for $j > j(R)$

$$v_f(z, \xi_j) \leq C + h(\text{Im} z), \quad |z| < R,$$

that is,

$$|\hat{f}(z + \xi_j)| \leq |\xi_j|^{C e^{h(\text{Im} z)}}, \quad |z| < R \log |\xi_j|.$$  \hspace{1cm} (2)

In addition

$$|\hat{f}(z)| \leq C_1 (1 + |z|)^C z e^{C_1 |\text{Im} z|}, \quad z \in \mathbb{C}^n,$$  \hspace{1cm} (3)

for some positive constants $C_1, C_2, C_3$. It is no restriction to assume that $\varphi$ is analytic in a neighborhood of supp $f$, for $f$ may be replaced by $\chi f$ where $\chi \in C^\infty_0$ is equal to 1 near sing supp $f$ and $\varphi$ is analytic near supp $\chi$. We can then use [4, Proposition 2.4] to choose a sequence of functions $\varphi_k \in C^\infty_0$ equal to $\varphi$ near supp $f$ so that for every $k$

$$|\hat{\varphi}_k(\xi)| \leq C, \quad |\varphi_k(z)| \leq C^{k+1} (k/(k + |\xi|))^k, \quad \xi \in \mathbb{R}^n.$$  \hspace{1cm} (4)

If $F = \varphi f$ then

$$\hat{F}(\xi_j + z) = (2\pi)^{-n} \int \hat{f}(\xi_j + z - \theta) \hat{\varphi}_k(\theta) d\theta,$$

and we shall estimate this when $|z| < \gamma R \log |\xi_j|$ where $\gamma \in (0, 1/2)$ will be chosen later on. By (2) the integral over the set where $|\theta| < (R/2) \log |\xi_j|$ can be estimated by

$$|\xi_j|^{C + 1} e^{h(\text{Im} z)}$$  \hspace{1cm} (5)

and the remaining part of the integral can be estimated by

$$C_1 (1 + |\xi_j| + |z|)^C z e^{C_1 |\text{Im} z|} \int_{|\theta| > R/2 \log |\xi_j|} (1 + |\theta|)^C |\varphi_k(\theta)| d\theta.$$  

If $a = Ce$ where $C$ is the constant in (4), we have for large $k$

$$\int_{|\theta| > ka} (1 + |\theta|)^C |\varphi_k(\theta)| d\theta \leq C' C^k (ak)^{C_1 + n} a^{-k} \leq C'' e^{-k} k^{C_2 + n}.$$
We choose \( k \) equal to the integral part of \((R/2a)\log|\xi_j|\) and obtain
\[
e^{-k} < e^{l|\xi_j|^{R/2a}}.
\]
Without restriction we may assume that \( h \geq 0 \). If \( \gamma < 1/2aC_3 \) and \( C_4 > \max(C+1, C_2+1) \) we then obtain for large \( j \)
\[
|\hat{F}(\xi_j + z)| \leq |\xi_j|^{C_4 e^{h(\text{Im}z)}} |z| < \gamma R \log|\xi_j|.
\]
This implies that \( h_1 \leq h \) which completes the proof of Theorem 3.

Before passing to the proof of Theorem 2 we recall the basic definitions involved. By a Denjoy–Carleman class \( C^M \) where \( M = (M_0, M_1, \ldots) \) is an increasing logarithmically convex sequence with \( M_0 = M_1 = 1 \) we mean the space of \( C^\infty \) functions \( \varphi \) such that for every compact set \( K \) there is a constant \( C_K \) such that for all multiindices \( \alpha \)
\[
|D^\alpha \varphi(x)| \leq C_K |\alpha|^{1+M_{|\alpha|}}, \quad x \in K.
\]
We assume \( M_k \geq k! \) so that \( C^M \) contains the real analytic class. Set
\[
M(t) = \sum_{k=0}^{\infty} t^k/M_k
\]
which is then convergent. It is obvious that for all \( \alpha \)
\[
|D^\alpha \varphi| \leq (2\pi)^{-n} M_{|\alpha|} \int |\hat{\varphi}(\xi)|M(|\xi|) d\xi
\]
so \( C^M \) contains the Banach space \( B \) of all \( \varphi \in \mathcal{S}' \) with \( \hat{\varphi} \in L^1 \) and the norm
\[
\|\varphi\|_B = (2\pi)^{-n} \int |\hat{\varphi}(\xi)|M(|\xi|) d\xi
\]
finite. It is well known that \( C^M \) is the analytic class if and only if
\[
e^{ct} \leq CM(t)
\]
for some \( c > 0 \) and \( C \). If this is not the case we therefore have
\[
\lim_{t \to \infty} M(t)e^{-t/j} = 0
\]
for every positive integer \( j \). Hence we can choose a sequence \( a_j \to \infty \), increasing as rapidly as we please, so that
\[
jM(a_j) < \exp(a_j/j).
\]
Choose a sequence \( \xi_j \in \mathbb{R}^n \) with
\[
\log|\xi_j| = a_j/j.
\]
If $a_j$ increases sufficiently rapidly then the balls
\[ \{ \xi \in \mathbb{R}^n ; \; |\xi - \xi_j| \leq a_j \} \]
are disjoint and $a_j/|\xi_j| \to 0$. Set
\[ E = \{ \xi ; \; |\xi - \xi_j| \geq a_j/2 \text{ for all } j \} . \]
Then we have
\[ |\xi - \xi_j| \geq a_j/2 = (j/2) \log |\xi_j| , \quad \xi \in E , \]
so it follows from [3, Theorem 5.2] that we can find $f \in \mathcal{E}'$ with $\text{sing supp } f = \{0\}$ so that
\[ v_f(z; \xi) \to -\infty \quad \text{when } E \ni \xi \to \infty \]
but $v_f(z; \xi_j)$ does not converge to $-\infty$. Choose $\eta_j \in E$ with $|\eta_j - \xi_j| = a_j$.

**Proposition 4.** If $M$ is not the analytic class and $f$ is chosen as just described then $v_{\phi f}(z, \eta_j)$ does not converge to $-\infty$ for all $\phi \in B$.

**Proof.** If $v_{\phi f}(z, \eta_j)$ converges to $-\infty$ then
\[ \sup_j |\eta_j|^N |(\phi f) \hat{\gamma}_j(\eta_j)| \]
is finite for every $N$, and if this is true for every $\phi \in B$, then
\[ \sup_j |\eta_j|^N |(\phi f) \hat{\gamma}_j(\eta_j)| \leq C_N \| \phi \|_B \]
by the closed graph theorem. Thus
\[ |\eta_j|^N \left| \int \hat{\phi}(\xi) \hat{f}(\eta_j - \xi) d\xi \right| \leq C_N \int |\hat{\phi}(\xi)| M(|\xi|) d\xi \]
which means that
\[ |\eta_j|^N \sup_\xi |\hat{f}(\eta_j - \xi)/M(|\xi|) | \leq C_N . \]

(7)

Now there is a subsequence $\xi_{j_k}$ such that
\[ v_f(z, \xi_{j_k}) \]
converges to a plurisubharmonic function which is not $-\infty$ identically and therefore constant since the supporting function is 0. (See [3, Lemma 3.6].) Hence
\[ v_f(z, \xi_{j_k}) \to C \]
in $L^1_{\text{loc}}(C^n)$ and also in $L^1_{\text{loc}}(R^n)$. It follows that we can find $\theta_{j_k} \in R^n$ so that for large $k$

$$|\theta_{j_k} - \eta_{j_k}| \leq a_{j_k}, |\hat{f}(\theta_{j_k})| > |\xi_{j_k}|^{C-1}. $$

With $\zeta = \eta_{j_k} - \theta_{j_k}$ and $j = j_k$ in (7) we obtain

$$|\xi_{j_k}|^{N+C-1} \leq C_N M(a_{j_k}).$$

Choose $N$ so that $N + C > 2$. Then we obtain

$$\exp(a_{j_k}/j_k) \leq C_N M(a_{j_k})$$

which contradicts (6), so the proof is complete.

PROOF OF THEOREM 2. Assuming that $M$ is not the analytic class we use Proposition 4 to choose $\varphi \in B_M$ so that $v_{\varphi f}(z, \eta_j)$ does not tend to $-\infty$. Let $\eta_{j_k}$ be a subsequence for which there is a finite limit. With

$$E_1 = \{\xi ; |\xi - \eta_{j_k}| \geq a_{j_k}/2 \text{ for all } k\}$$

we choose according to [3, Theorem 5.2] a distribution $g \in \mathcal{E}'$ with sing supp $g = \{0\}$ so that $v_g(z, \eta_{j_k})$ does not tend to $-\infty$ but $v_g(z, \xi) \to -\infty$ when $\xi \to \infty$ in $E_1$. Then $f \ast g \in C^\infty$ by [3, Lemma 5.2 and Theorem 5.1], for every sequence $\to \infty$ in $R^n$ contains a subsequence in $E$ or one in $E_1$, so for $(h_f, h_g) \in \mathcal{E}'(f, g)$ we always have $h_f = -\infty$ or $h_g = -\infty$. On the other hand, for a subsequence of $\eta_{j_k}$ we know that both $v_{\varphi f}(z, \eta_{j_k})$ and $v_g(z, \eta_{j_k})$ have finite limits, so $(0, 0) \in \mathcal{E}'(\varphi f, g)$. Hence $(\varphi f) \ast g$ is not in $C^\infty$ so the singular support is $\{0\}$.

It is clear that by a slight modification of the preceding construction one can modify the statement of Theorem 2 so that sing supp $f \ast g$ and sing supp $(\varphi f) \ast g$ are two arbitrary convex compact sets.

REFERENCES