WHITNEY (b)-REGULARITY IS WEAKER THAN KUO'S RATIO TEST FOR REAL ALGEBRAIC STRATIFICATIONS

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We give examples of real algebraic hypersurfaces such that the full partition by dimension gives a stratification which is Whitney (b)-regular, but which fails to satisfy Kuo's ratio test (r), and hence also fails to satisfy the (w)-regularity of Verdier. Such a hypersurface can be a $C^1$ submanifold, so that the stratification is $C^1$ trivial, showing that (r) and (w) are not invariant under $C^1$ changes of coordinates, although they are $C^2$ invariant. We show that (w)-regularity is characterised by the possibility of extending rugose vector fields defined on some strata to rugose vector fields tangent to the remaining strata.

1. On regularity.

Let $X$ be a $C^1$ submanifold of $\mathbb{R}^n$, and a subanalytic set (defined in [2]). Let $Y$ be an analytic submanifold of $\mathbb{R}^n$ such that $0 \in Y \subset X \setminus X$. Verdier [8] defines $X$ to be (w)-regular over $Y$ at 0 if,

(w) There is a constant $C > 0$ and a neighborhood $U$ of 0 in $\mathbb{R}^n$ such that if $x \in U \cap X$ and $y \in U \cap Y$, then $d(T_x Y, T_x X) \leq C|x - y|$. 

Here $d(\cdot, \cdot)$ is defined as follows.

**Definition.** Let $A, B$, be vector subspaces of $\mathbb{R}^n$.

$$d(A, B) = \sup_{a \in A \atop |a| = 1} |a - \pi_B(a)|,$$

where $\pi_B$ is orthogonal projection onto $B$.

This is not symmetric in $A$ and $B$. Clearly $d(A, B) = 0$ if and only if $A \subseteq B$. It is clear from the definition of (w) that it is a $C^2$ invariant, or more precisely

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that it is invariant under a $C^1$ diffeomorphism with Lipschitz derivative. We shall see below that it is not a $C^1$ invariant.

Kuo's ratio test.

We suppose that $Y$ is linear (apply a local analytic isomorphism at 0 to $\mathbb{R}^n$). Let $\pi_Y$ denote orthogonal projection onto $Y$.

Reformulate (w) by the condition that $d(T_y Y, T_x X)/|x - y|$ is bounded near 0. Then in particular $d(T_0 Y, T_x X)/|x - \pi_Y(x)|$ is bounded for $x$ near 0 (recall $Y$ is linear). Then it is clear that if $X$ is (w)-regular over $Y$ at 0, then $(X, Y)_0$ satisfies the ratio test of Kuo [3]:

(r) Given any vector $v \in T_0 Y$,

$$\lim_{x \to 0, x \in X} \frac{|\pi_x(v)| \cdot |x|}{|x - \pi_Y(x)|} = 0.$$  

Here $\pi_x$ denotes orthogonal projection onto the normal space to $X$ at $x$, so that for unit vectors $v$, $|\pi_x(v)| = d(\langle v \rangle, T_x X)$. In [3] Kuo proved that (r) implies Whitney (b)-regularity (defined in [9]) and that (b) implies (r) when $Y$ is 1-dimensional. In [6] a fairly complicated semialgebraic example was given with $Y$ 2-dimensional showing that (b) is weaker than (r). We give a simple algebraic example below.

First observe that if (b) (respectively (w)) holds for a pair of strata $(X, Y)$ at 0 in $\mathbb{R}^n$, then (b) (respectively (w)) holds for $(X \times \mathbb{R}, Y \times \mathbb{R})$ along $0 \times \mathbb{R}$ in $\mathbb{R}^n \times \mathbb{R}$. However (r) does not have this property.

**Proposition 1.** Let $(X, Y)$ be a pair of strata in $\mathbb{R}^n$ not (w)-regular at 0 (but possibly satisfying (r)) and let $Y$ be linear. Then $(X \times \mathbb{R}, Y \times \mathbb{R})$ fails to satisfy (r) at any point of $0 \times \mathbb{R}$ in $\mathbb{R}^n \times \mathbb{R}$.

**Proof.** Let $X$, $Y$ have dimensions $m$, $p$ respectively and identify the set of one dimensional subspaces of $T_0 Y$ with the Grassmannian $G^n_p$.

Define three subsets of $\mathbb{R}^n \times \mathbb{R}^n \times G^m_n \times G_p^n \times \mathbb{R}$:

$$V_1 = \{(x, \pi_Y(x), T_x X) : x \in X\} \times G^n_p \times \mathbb{R}$$

$$V_2 = \{(x, y, T, \langle v \rangle, \epsilon) : |x - y| < \epsilon d(\langle v \rangle, T)\}$$

$$V_3 = \mathbb{R}^n \times \mathbb{R}^n \times \{(T, \langle v \rangle) : d(\langle v \rangle, T) = d(T_0 Y, T)\} \times \mathbb{R}$$

$V_1$ is subanalytic using Verdier [8, Lemma 1.6] (by restricting to a compact neighbourhood of 0 in $\mathbb{R}^n$ if necessary), $V_2$ is semialgebraic, and $V_3$ is algebraic. Hence $V = V_1 \cap V_2 \cap V_3$ is a subanalytic set.
We have that \((w)\) fails for the pair \((X, Y)\) at 0, which is equivalent to the existence of \(\tau \in G^\sigma_m\) and \(v \in T_0 Y\) with \(\|v\| = 1\) such that

\[
(0, 0, \tau, \langle v \rangle, 0) \in \bar{V} \subseteq \mathbb{R}^n \times \mathbb{R}^n \times G^\sigma_m \times G^\sigma_l \times \mathbb{R}.
\]

By curve selection [2] we can find an analytic arc

\[
\alpha: [0, 1] \to \mathbb{R}^n \times \mathbb{R}^n \times G^\sigma_m \times G^\sigma_l \times \mathbb{R},
\]

such that \(\alpha(0) = (0, 0, \tau, \langle v \rangle, 0)\) and such that \(\alpha(t) \in V\) if \(t \neq 0\). Write

\[
\alpha(t) = (x_t, \pi_y(x_t), T_{x_t}X, \langle v_t \rangle, e_t)
\]

where \(v_t \in T_0 Y, \|v_t\| = 1\) and \(v_t \to v\) as \(t \to 0\). Then

\[
\frac{d(\langle v_t \rangle, T_{x_t}X)}{|x_t - \pi_y(x_t)|}
\]

is unbounded as \(t\) tends to 0. We assert that

\[
d(\langle v \rangle, T_{x_t}X) \geq \frac{1}{2}d(\langle v_t \rangle, T_{x_t}X)
\]

for \(t\) sufficiently small. This is a consequence of the definition of \(V_3\), as follows:

Let \(v = v_t \cos \varphi_t + u_t \sin \varphi_t\) where \(\|u_t\| = 1, v_t \perp u_t\) and \(\varphi_t\) is the positive angle between \(v\) and \(v_t\), we can assume \(0 \leq \varphi_t < \pi/2\). Let \(\pi_t\) denote the orthogonal projection onto \(T_{x_t}X\). Then

\[
d(\langle v \rangle, T_{x_t}X) = |v - \pi_t(v)| = |(v_t - \pi_t(v_t))\cos \varphi_t + (u_t - \pi_t(u_t))\sin \varphi_t|
\]

\[
\geq |v_t - \pi_t(v_t)| \cos \varphi_t - |u_t - \pi_t(u_t)| \sin \varphi_t
\]

(by the triangle inequality)

\[
\geq |v_t - \pi_t(v_t)| (\cos \varphi_t - \sin \varphi_t)
\]

(Using definition of \(V_3, |v_t - \pi_t(v_t)| \geq |u_t - \pi_t(u_t)|\))

\[
d(\langle v_t \rangle, T_{x_t}X) (\cos \varphi_t - \sin \varphi_t)
\]

Since \(\varphi_t\) tends to 0 as \(t\) tends to 0, it follows that, for \(t\) sufficiently small,

\[
d(\langle v \rangle, T_{x_t}X) \geq \frac{1}{2}d(\langle v_t \rangle, T_{x_t}X).
\]

We deduce that \(d(\langle v \rangle, T_{x_t}X)/|x_t - \pi_y(x_t)|\) is also unbounded as \(t\) tends to 0. After reparametrisation we can suppose that

\[
\frac{d(\langle v \rangle, T_{x_t}X)}{|x_t - \pi_y(x_t)|} \sim t^{-k} \quad \text{for some } k \geq 1
\]

In \(\mathbb{R}^n \times \mathbb{R}\) consider the curve \(q(t) = (x_t, t_0 + t)\). Using the canonical inclusion \(T_0 Y \subseteq T_{(0, t_0)}(Y \times \mathbb{R})\), we can consider \(v\) as a unit vector of \(T_{(0, t_0)}(Y \times \mathbb{R})\). Then
\[ d(\langle v \rangle, T_{q(t)}(X \times \mathbb{R})) |q(t) - (0, t_0)| \]
\[ |q(t) - \pi_{Y \times \mathbb{R}}(q(t))| \]
\[ = \frac{d(\langle v \rangle, T_x X) \cdot |(x_t, t)|}{|x_t - \pi_y(x_t)|} \]
\[ \geq \frac{d(v, T_{x_t} X) \cdot t}{|x_t - \pi_y(x_t)|} \sim t^{-(k-1)}, \]

which does not tend to zero as \( t \) approaches zero since \( k \geq 1 \). Hence the ratio test \((r)\) fails for the pair \((X \times \mathbb{R}, Y \times \mathbb{R})\) at every point \((0, t_0)\) of \( 0 \times \mathbb{R} \) in \( \mathbb{R}^n \times \mathbb{R} \), completing the proof of Proposition 1.

**Example 1.** Let \( V = \{ y^3 = z^2x^3 + x^5 \} \subset \mathbb{R}^3 \), and let \( Y \) be the \( z \)-axis and \( X = V - Y \).

\( (z^2x^3 + x^5)^{1/3} \) is a \( C^1 \) function of \( x \) and \( z \), and so \( V \), as the graph of a \( C^1 \) map, is a \( C^1 \) submanifold of \( \mathbb{R}^3 \). Hence \( X \) is \((\beta)\)-regular over \( Y \). By Theorem 2 of [3] we deduce that \((X, Y)\) satisfies \((r)\) at 0, since \( \dim Y = 1 \).

Consider the curve \( p(t) = (t^3, \sqrt[3]{2} \cdot t^5, t^3) \) from the origin into \( X \). The normal direction to \( X \) at \((x, y, z)\) is \((3x^2z^2 + 5x^4, -3(z^2x^3 + x^5)^{2/3}, 2zx^3)\). At \( p(t) \) this becomes

\[ (8t^2, -3 \cdot 2^{2/3} \cdot 2t^2) \cdot \]

So

\[ d(T_0 Y, T_{p(t)} X) = \frac{2t^2}{(68t^4 + 18\sqrt[3]{2})^{1/2}} \]

and

\[ \frac{d(T_0 Y, T_{p(t)} X)}{|p(t) - \pi_Y(p(t))|} \sim \frac{t^2}{t^3} \sim \frac{1}{t}, \]

which is unbounded as \( t \) approaches zero, so that \((w)\) fails for \((X, Y)\) at 0.

Now let

\[ V' = V \times \mathbb{R} = \{ y^3 = z^2x^3 + x^5 \} \subset \mathbb{R}^4 = \{(x, y, z, u)\} \]

Let

\[ Y' = Y \times \mathbb{R} = \{ y = x = 0 \} \subset \mathbb{R}^4 \quad \text{and} \quad X' = V' - Y' \]

By Proposition 1, \((X', Y')\) fails to satisfy \((r)\) at any point of \( 0 \times \mathbb{R} \) (for example consider the curve \( q(t) = (p(t), t) \) from 0 into \( X' \)). But since \( V' \) is a \( C^1 \) submanifold, \((X', Y')\) is \((b)\)-regular.
Example 1 describes the first example of a pair \((X, Y)\) satisfying (b) but not (r) where \(X\) is the regular part of an algebraic variety and \(Y\) the singular locus. Contrast this with the complex hypersurface case where (b)-regularity, the ratio test, and (w)-regularity are equivalent. This is a consequence of the equivalence of (b)-regularity with Teissier's (c)-cosecanceness [5] (references for the implications giving this equivalence may be found in [1]); (c)-cosecanceness trivially implies (w)-regularity, and hence also the ratio test. It remains to be seen whether (b), (r) and (w) are distinct when \(V\) is a complex analytic variety of codimension greater than 1.

Example 2 (from [7]). \(V \equiv \{ y^4 = z^4 x + x^3 \} \subset \mathbb{R}^3, Y = \{ z \text{-axis} \}, X = V \setminus Y\). Here \(y\) is not a \(C^1\) function of \(x\) and \(z\), but \(V\) is still a \(C^1\) submanifold of \(\mathbb{R}^3\), so that (b) holds for \((X, Y)\). (w) fails along the curve \(p(t) = (t^4, \sqrt[4]{2} \cdot t^3, t^2)\). As with Example 1 we can apply Proposition 1 to show that \((X \times \mathbb{R}, Y \times \mathbb{R})\) fails to satisfy (r) on \(0 \times \mathbb{R} \subset \mathbb{R}^4\), but (b) clearly holds.

Example 3 (due to Kuo [4]). \(V \equiv \{ y^4 = z^2 x^5 + x^7 \} \subset \mathbb{R}^3, Y\) the z-axis, \(X = V \setminus Y\). \(V\) is no longer a \(C^1\) submanifold— for each \(z\), \(y^4 = z^2 x^5 + x^7\) defines a plane curve of "cusp type” near 0. However (b) does hold and (w) fails. We can apply Proposition 1 as before.

Examples 1 and 2, and indeed the second discordant horn of [6], show that (r) and (w) are not invariant under \(C^1\) diffeomorphisms. So (b) is more natural in differential topology; it is a \(C^1\) invariant.

Looking closely at the proofs in [3] we see why it is not surprising that (r) is strictly stronger than (b) when \(\dim Y \geq 2\). It is proved in [3] that (b) is equivalent to the conjunction of (a) and (r') defined as follows.

\((r')\) If \(p(t), t \in [0, 1]\) is an analytic arc in \(\mathbb{R}^n\) with \(p(0) = 0\) and \(p(t) \in X\) for \(t \neq 0\), then

\[
\lim_{t \to 0} \frac{|\pi_t(v)||p(t)|}{|p(t) - \pi_Y(p(t))|} = 0,
\]

where \(v\) is the tangent at 0 to the arc \(\pi_Y \circ p([0, 1])\) on \(Y\), and \(\pi_t\) is projection onto the normal space to \(X\) at \(p(t)\).

It is obvious that (r) implies (a) + (r') and that (a) + (r') implies (r) when \(Y\) has dimension one. Being able to choose a vector \(v\) in \(T_0 Y\) and a curve whose tangent at 0 is orthogonal to \(v\) suggested the counterexample in [6], and gives rise to the examples here too.
Rugose vector fields.

Given a (b)-regular stratification, one might hope to be able to find rugose vector fields tangent to the strata. Verdier shows that these exist on (w)-regular stratifications [8] and derives rugose trivialisations. However it can be impossible to extend a constant vector field on a base stratum $Y$ to a rugose vector field on an attaching stratum $X$ when $(X, Y)$ is (b)-regular. This is a consequence of our next proposition and the existence of (b)-regular examples which do not satisfy (w).

We refer to [8] for the definition of rugose vector field. (Note the misprint in the definition of rugose function on page 307 of [8], as described below).

**Proposition 2.** Let $X$ be a $C^2$ submanifold of $\mathbb{R}^n$ and let $Y = \mathbb{R}^m \times 0 \subset \mathbb{R}^n$. Suppose that each of the constant vector fields \{\frac{\partial}{\partial y_i}\}, $i = 1, \ldots, m$, on $Y$ extends to a rugose vector field on $X \cup Y$. Then $X$ is (w)-regular over $Y$.

**Proof.** Let $\tilde{v}_i$ denote the extension of $\frac{\partial}{\partial y_i}$. For each $i$ there exists a constant $C$ and a neighbourhood $U$ of 0 such that

$$\left| \tilde{v}_i(x) - \frac{\partial}{\partial y_i} \right| \leq C|x - y|$$

for all $x \in U \cap X, y \in U \cap Y$. We can assume that $C$ and $U$ are the same for all $i$. Let $x \in U$. Then

$$d\left( \frac{\partial}{\partial y_i}, T_x X \right) \leq \left| \frac{\partial}{\partial y_i} - \tilde{v}_i(x) \right|,$$

hence

(*) $$d\left( \frac{\partial}{\partial y_i}, T_x X \right) \leq C|x - y| \quad \text{for all } x \in X \cap U, y \in Y \cap U.$$

Take $v \in T_x Y$ with $|v| = 1$.

$$v = \sum_{i=1}^{m} a_i \frac{\partial}{\partial y_i}, \quad \text{with } \sum_{i=1}^{m} a_i^2 = 1.$$

Let $N_x X$ denote the orthogonal complement of $T_x X$ in $\mathbb{R}^n$ and $\pi_x : \mathbb{R}^n \to N_x X$ the orthogonal projection.

$$d(v, T_x X) = |\pi_x(v)| = \left| \sum_{i=1}^{m} a_i \pi_x \left( \frac{\partial}{\partial y_i} \right) \right|.$$

$$\leq \sum_{i=1}^{m} \left| \pi_x \left( \frac{\partial}{\partial y_i} \right) \right| = \sum_{i=1}^{m} d\left( \frac{\partial}{\partial y_i}, T_x X \right).$$

$$\leq mC|x - y| \quad \text{by (*)}.$$
Hence
\[ d(T_y Y, T_x X) = \sup_{\{v: v \in T_y Y\}} d(v, T_x X) \leq mC|x - y| \quad \text{for all } x \in X \cap U, y \in Y \cap U, \]
i.e. \( X \) is (w)-regular over \( Y \) at 0. Repeating the above argument for each \( y \in Y \), we obtain that \( X \) is (w)-regular over \( Y \), completing the proof of Proposition 2.

**Corollary.** Let \( A = X \cup B \) be a closed subset of \( \mathbb{R}^n \), \( B \cap X = \emptyset \), \( X \) a \( C^2 \) submanifold, \( B \) a closed subset, and let \((B, \Sigma)\) be a (w)-regular stratification, with each stratum a \( C^2 \) submanifold. Then the stratification \( \Sigma' \) of \( A \) given by adding \( X \) to \( \Sigma \) is (w)-regular if and only if every rugose vector field on \( B \) tangent to \( \Sigma \) can be extended to a rugose vector field on \( A \) tangent to \( \Sigma' \).

**Proof.** "Only if" is proved by Verdier [8]. "If" follows from Proposition 2 above by making the stratum containing a given point \( y \), affine near \( y \), by a \( C^2 \) change of local coordinates.

**Warning.** The definition of rugosity in [8] should read "for all \( x \in S_x \), there is a constant \( C \) and a neighbourhood \( V \) of \( x \) such that for all \( x' \in V \cap S_x \) and all \( y \in V \cap A \),

\[ |f(x') - f(y)| \leq C|x' - y| \]

and not

\[ |f(x') - f(y)| \leq C|x - y| \]

To see that these are effectively distinct notions in the case of vector fields we can use Example 2. (w) fails, so by Proposition 2 no lift of \( \partial / \partial z \) satisfies (**). However the canonical lift of \( \partial / \partial z \) (namely the vector field \( v(x, y, z) \) on \( V \) defined by projecting \( \partial / \partial z \) onto the tangent space to \( X \) at each point of \( X \)) satisfies (***) as follows.

Let \( f(x, y, z) = -y^4 + z^4 x + x^3 \). Then
\[
v(x, y, z) = (0, 0, 1) - \frac{(f_x, f_y, f_z)}{|\text{grad } f|} \cdot \frac{f_z}{|\text{grad } f|}.
\]

Hence
\[ |v(x, y, z) - (0, 0, 1)| = \frac{|f_z|}{|\text{grad } f|} \]

We must check that \( |v(x, y, z) - (0, 0, 1)|/|(x, y, z)| \) is bounded as \((x, y, z)\) tends to 0 on \( X \).
\[
\frac{|v(x,y,z) - (0,0,1)|}{|(x,y,z)|} = \frac{|f_2|}{|\text{grad } f| \cdot |(x,y,z)|}
= \frac{|4z^3 x|}{|(z^4 + 3x^2, -4(z^4 x + x^3)^{3/4}, 4z^3 x)| \cdot |(x, (z^4 x + x^3)^{1/4}, z)|}
\]

CASE 1. \(|x/z^2| \leq 1\). Dividing through by \(z^5\), gives
\[
\frac{|4x/z^2|}{|(1 + (3x^2/z^4), \ldots)| \cdot |(x/z, \ldots, 1)|}
\]
which is at most 4.

CASE 2. \(|z^2/x| \leq 1\). Dividing through by \(x^2 z\), gives
\[
\frac{|4z^2/x|}{|(z^4/x^2 + 3, \ldots, 4z^3/x)| \cdot |(x/z, \ldots, 1)|}
\]
which is at most \(4/3\).

We have shown that (***) is satisfied.

BIBLIOGRAPHY