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DUALITY IN THE THEORY OF CROSSED PRODUCTS

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Abstract.

If A is a von Neumann algebra, G a locally compact abelian group and α a continuous action of G on A, it was shown by Takesaki that it is possible to define a continuous action $\hat{\alpha}$ of the dual group $\hat{G}$ on the crossed product $A \otimes_\alpha G$ such that the repeated crossed product $(A \otimes_\alpha G) \otimes_{\hat{\alpha}} \hat{G}$ is isomorphic to $A \otimes \mathscr{B}(L_2(G))$, where $\mathscr{B}(L_2(G))$ denotes the von Neumann algebra of all bounded linear operators on $L_2(G)$. The duality theorem was used by Takesaki to obtain an important structure theorem for type III von Neumann algebras. Takesaki's duality theorem has been extended to the non-abelian case by Landstad and Nakagami. Because of the lack of symmetry between the group and its dual object in the non-commutative case, the extended theorem in fact consists of two versions, a direct result and a dual one.

In this paper we will show that both the direct theorem and the dual one are essentially two forms of one result. Moreover it turns out that all together our approach is both simpler and shorter.

1. Introduction.

We shall give a new proof of the duality theorems for locally compact groups, which one finds in the papers of Landstad [2] and Nakagami [3]. First we remark that the covariance algebra, (respectively the dual covariance algebra) defined by Landstad in [2] describes the same notion as the crossed product (respectively the crossed dual product) of Nakagami in [3]. In this paper we will use the terminology of Nakagami. Let us now formulate the most important definitions and results of the duality theorem of Landstad and Nakagami.

If G is a locally compact group, equipped with a left invariant Haar measure, we consider $L_\infty(G)$ as the von Neumann algebra on $L_2(G)$ generated by the multiplication operators, while $L(G)$ is the von Neumann algebra generated by the left regular representation on $L_2(G)$, i.e. $\{\lambda_t, t \in G\}''$.

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DEFINITION 1.1. Let A_1 be a von Neumann algebra acting on $\mathscr{H}_1$. Then π_1 is an action of G on A_1, if π_1 is a normal injective *-homomorphism of A_1 into $A_1 \otimes L_\infty(G)$ such that the following diagram is commutative:

$$\begin{array}{ccc} A_1 & \xrightarrow{\pi_1} & A_1 \otimes L_\infty(G) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_1 \otimes 1} \\ A_1 \otimes L_\infty(G) & \xrightarrow{1 \otimes \delta_1} & A_1 \otimes L_\infty(G) \otimes L_\infty(G) \end{array}$$

Here δ_1 is defined on $L_\infty(G)$ by $\delta_1(x) = U_1^*(x \otimes 1)U_1$ with U_1 the unitary operator on $L_2(G) \otimes L_2(G)$ such that $(U_1 f)(s,t) = f(t^{-1}s, t)$ for f a continuous function with compact support from $G \times G$ to $\mathbf{C}$ (i.e. $f \in K(G \times G)$). We remark that we use the fact that $L_2(G) \otimes L_2(G)$ and $L_2(G \times G)$ are isomorphic. We denote by 1 the identity operator both on $\mathscr{H}_1$ and $L_2(G)$.

The von Neumann algebra generated by $\pi_1(A_1)$ and $1 \otimes L(G)$ is denoted by $A_1 \otimes_{\pi_1} L(G)$ and is called the crossed product of A_1 and G by the action π_1.

DEFINITION 1.2. Let A_2 be a von Neumann algebra on $\mathscr{H}_2$. Then π_2 is a dual action of G on A_2, if π_2 is a normal injective *-homomorphism of A_2 into $A_2 \otimes L(G)$ such that the following diagram is commutative:

$$\begin{array}{ccc} A_2 & \xrightarrow{\pi_2} & A_2 \otimes L(G) \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \pi_2 \otimes 1} \\ A_2 \otimes L(G) & \xrightarrow{1 \otimes \delta_2} & A_2 \otimes L(G) \otimes L(G) \end{array}$$

Here δ_2 is defined on $L(G)$ by $\delta_2(x) = U_2^*(x \otimes 1)U_2$ with U_2 the operator on $L_2(G) \otimes L_2(G)$ such that $(U_2 f)(s,t) = f(s, st)$ for $f \in K(G \times G)$.

The von Neumann algebra generated by $\pi_2(A_2)$ and $1 \otimes L_\infty(G)$ is denoted by $A_2 \otimes_{\pi_2} L_\infty(G)$ and is called the crossed dual product of A_2 and G by the action π_2.

Let us now consider the von Neumann algebra $A_1 \otimes_{\pi_1} L(G)$. Then it is possible to find a dual action $\hat{\pi}_1$ on $A_1 \otimes_{\pi_1} L(G)$ such that the crossed dual product $(A_1 \otimes_{\pi_1} L(G)) \otimes_{\hat{\pi}_1} L_\infty(G)$ is isomorphic to $A_1 \otimes \mathscr{B}(L_2(G))$ (see [2] and [3]). This is the duality theorem for crossed products and can be formulated in the following way:

THEOREM 1.3. *If π_1 is an action of a locally compact group G on a von Neumann algebra A_1, then*

$$(A_1 \otimes_{\pi_1} L(G)) \otimes_{\hat{\pi}_1} L_\infty(G) \cong A_1 \otimes \mathscr{B}(L_2(G))$$

where $\hat{\pi}_1$ is the mapping of $A_1 \otimes_{\pi_1} L(G)$ into $(A_1 \otimes_{\pi_1} L(G)) \otimes L(G)$ defined by

$$\hat{\pi}_1(x) = (1 \otimes U_2^*)(x \otimes 1)(1 \otimes U_2) \quad \textit{for all } x \in A_1 \otimes_{\pi_1} L(G)\,.$$

Besides this direct theorem there exists also a duality theorem for crossed dual products. Indeed, Landstad and Nakagami have shown in [2] and [3] that there exists an action $\hat{\pi}_2$ on $A_2 \otimes_{\pi_2} L_\infty(G)$ such that the crossed product $(A_2 \otimes_{\pi_2} L_\infty(G)) \otimes_{\hat{\pi}_2} L(G)$ is isomorphic to $A_2 \otimes \mathscr{B}(L_2(G))$.

In the commutative case there is no difference between the two duality theorems because of the isomorphism between $L_\infty(G)$ and $L(\hat{G})$, where $\hat{G}$ is the dual group of G. So the two duality theorems reduce to one theorem namely that of Takesaki [5]. Both the direct theorem and the dual one in the non-commutative case have many aspects in common. Indeed, we can prove Theorem 1.3 by using some special properties of $L_\infty(G)$, $L(G)$ and U_1, while we need analogous properties of $L(G)$, $L_\infty(G)$ and U_2 for the dual theorem. So, in order to obtain a general duality theorem, we shall study two von Neumann algebras M_1 and M_2, acting on the same Hilbert space $\mathscr{K}$ and a unitary operator W_1 on $\mathscr{K} \otimes \mathscr{K}$, which are related to each other in a particular way. Furthermore the most important part of the proof of the duality theorem is to show the equation

$$A_1 \otimes_{\pi_1} \mathscr{B}(L_2(G)) = A_1 \otimes \mathscr{B}(L_2(G))$$

and the equation

$$A_2 \otimes_{\pi_2} \mathscr{B}(L_2(G)) = A_2 \otimes \mathscr{B}(L_2(G))$$

in the dual case.

Especially for the second theorem this is a difficult problem (see for example [2] and [3]). We shall treat this situation by introducing the concept of a non-degenerate action. This means that we define an action in such a way that it is a generalization of the action π_1 of Definition 1.1 and also of the dual action π_2 of Definition 1.2. The non-degeneracy property refers to the two equations

$$A_1 \otimes_{\pi_1} \mathscr{B}(L_2(G)) = A_1 \otimes \mathscr{B}(L_2(G))$$

and

$$A_2 \otimes_{\pi_2} \mathscr{B}(L_2(G)) = A_2 \otimes \mathscr{B}(L_2(G)) .$$

In this way we are able to formulate and prove a general duality theorem at the end of section 2. At last we shall show, in section 3, that the direct and dual theorem for locally compact groups are indeed special cases of the general one.

We also remark that the proofs given in this paper are much simpler than those of Nakagami and Landstad in [2] and [3]. We mention that Enock and Schwarz have also generalized the duality theory by introducing the concept of Katz-algebras (see [1]). In the preprint [4] of Stratila, Voiculescu and Zsido one can also find a different treatment of this subject.

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2. The general duality theorem.

In this section we shall prove a duality theorem having both the direct theorem and the dual theorem of Landstad and Nakagami as a special case.

NOTATIONS 2.1. Let G be a locally compact group, and let $L_\infty(G)$, $L(G)$, U_1 and U_2 be as in the introduction. Then we define J_1 on $L_2(G)$ by $J_1 f(s)=\overline{f(s)}$ and J_2 on $L_2(G)$ by $J_2 f(s)=\Delta(s)^{-\frac{1}{2}}\overline{f(s^{-1})}$.

S is an operator on $L_2(G)\otimes L_2(G)$ such that $S(f\otimes g)=g\otimes f$ for g and f belonging to $L_2(G)$.

With these definitions we have the following relation between U_1 and U_2:

$$U_2 = S(1\otimes J_1J_2)U_1(1\otimes J_1J_2)S .$$

This gives a relation between an action π_1 of G on a von Neumann algebra A_1 and its dual action $\hat{\pi}_1$. Indeed, by Definition 1.1

$$(\pi_1\otimes 1)\pi_1 = (1\otimes\delta_1)\pi_1$$

with

$$\delta_1(x) = U_1^*(x\otimes 1)U_1$$

while $\hat{\pi}_1$ is defined by

$$\hat{\pi}_1(x) = (1\otimes U_2^*)(x\otimes 1)(1\otimes U_2) \qquad \text{for } x\in A_1\otimes_{\pi_1} L(G)$$

(see Theorem 1.3). Considering now the 5-tuple $(L_\infty(G), L(G), U_1, J_1, J_2)$, we can prove the following interesting properties:

PROPOSITION 2.2. i) $L(G)\cap L_\infty(G)=\mathbf{C}1$, $\quad J_1^2=J_2^2=1$ *and* $J_1J_2=J_2J_1$

ii) $J_1L_\infty(G)J_1=L_\infty(G)'=J_2L_\infty(G)'J_2$ *and* $J_2L(G)J_2=L(G)'=J_1L(G)'J_1$

iii) $(1\otimes S)(U_1\otimes 1)(1\otimes S)(U_1\otimes 1)=(1\otimes U_1^*)(U_1\otimes 1)(1\otimes U_1)$

iv) $U_1^*=(J_1\otimes J_2)U_1(J_1\otimes J_2)$

v) $U_1\in L(G)\otimes L_\infty(G)$

vi) $SU_1^*(L_\infty(G)\otimes 1)U_1S=(J_2\otimes J_2)U_1^*(L_\infty(G)\otimes 1)U_1(J_2\otimes J_2)$

vii) $SU_1^*(1\otimes L(G)')U_1S=(J_2\otimes J_2)U_1^*(1\otimes L(G)')U_1(J_2\otimes J_2)$

PROOF. Since these equations can be found by straightforward calculations, we omit the proof (for the relations i) to v) see for example [6]. Property vi) and vii) will be proved by similar computations).

In this way by studying the duality theorem for crossed products, we can derive Proposition 2.2 for the 5-tuple $(L_\infty(G), L(G), U_1, J_1, J_2)$ and the important formula

$$U_2 = S(1 \otimes J_1 J_2) U_1 (1 \otimes J_1 J_2) S ,$$

which gives a relation between $\hat{\pi}_1$ and π_1. Now, we want to prove a general duality theorem, which means also a generalization of the duality theorem for crossed dual products. Therefore we shall look for a 5-tuple with properties analogous to those of Proposition 2.2 and we shall find a formula, which gives a relation between $\hat{\pi}'_2$ and π_2. For this let us recall the formulation of the dual duality theorem as given by Landstad and Nakagami in [2] and [3]:

If π_2 is a dual action of a locally compact group G on a von Neumann algebra A_2, then there exists an action $\hat{\pi}'_2$ of G on the crossed dual product $A_2 \otimes_{\pi_2} L_\infty(G)$ such that

$$(A_2 \otimes_{\pi_2} L_\infty(G)) \otimes_{\hat{\pi}'_2} L(G)) \cong A_2 \otimes \mathscr{B}(L_2(G)) .$$

Here $\hat{\pi}'_2$ is the mapping of $A_2 \otimes_{\pi_2} L_\infty(G)$ into $(A_2 \otimes_{\pi_2} L_\infty(G)) \otimes L_\infty(G)$ defined by

$$\hat{\pi}'_2(x) = (1 \otimes V^*)(x \otimes 1)(1 \otimes V) \quad \text{for all } x \in A_2 \otimes_{\pi_2} L_\infty(G) ,$$

with V an operator on $L_2(G) \otimes L_2(G)$ such that $Vf(s,t) = \Delta(t)^{\frac{1}{2}} f(st,t)$ for $f \in K(G \times G)$. As π_2 is a dual action,

$$(\pi_2 \otimes 1)\pi_2 = (1 \otimes \delta_2)\pi_2$$

with $\delta_2(x) = U_2^*(x \otimes 1)U_2$ and U_2 as in Definition 1.2.

Computing $S(1 \otimes J_1 J_2) U_2 (1 \otimes J_1 J_2) S$, we do not get V, but an operator $U_3 = (1 \otimes J_1 J_2) V (1 \otimes J_1 J_2)$. Furthermore we have that $J_1 J_2 L(G) J_2 J_1 = L(G)'$ by Proposition 2.2. Hence

$$\begin{aligned}(1 \otimes 1 \otimes J_1 J_2)\big((A_2 \otimes_{\pi_2} L_\infty(G)) \otimes_{\hat{\pi}'_2} L(G)\big)(1 \otimes 1 \otimes J_1 J_2) \\ = (A_2 \otimes_{\pi_2} L_\infty(G)) \otimes_{\hat{\pi}_2} L(G)'\end{aligned}$$

where

$$\hat{\pi}_2(x) = (1 \otimes U_3^*)(x \otimes 1)(1 \otimes U_3) \quad \text{for all } x \in A_2 \otimes_{\pi_2} L_\infty(G) .$$

Because of the symmetry with the direct Duality Theorem 1.3 we will formulate the dual one as follows:

THEOREM 2.3. *If π_2 is a dual action of a locally compact group G on a von Neumann algebra A_2, then there exists an injective normal *-homomorphism $\hat{\pi}_2$ of the crossed dual product $A_2 \otimes_{\pi_2} L_\infty(G)$ into $(A_2 \otimes_{\pi_2} L_\infty(G)) \otimes L_\infty(G)$ such that*

$$(A_2 \otimes_{\pi_2} L_\infty(G)) \otimes_{\hat{\pi}_2} L(G)' \cong A_2 \otimes \mathscr{B}(L_2(G)) .$$

Here $\hat{\pi}_2$ is defined by $\hat{\pi}_2(x) = (1 \otimes U_3^*)(x \otimes 1)(1 \otimes U_3)$.

Considering the 5-tuple $(L(G), L_\infty(G)', U_2, J_2, J_1)$ we get the following properties.

PROPOSITION 2.4. i) $L_\infty(G)' \cap L(G) = \mathbf{C}1, \quad J_2^2 = J_1^2 = 1$ *and* $J_2 J_1 = J_1 J_2$

ii) $J_2 L(G) J_2 = L(G)' = J_1 L(G)' J_1$ *and* $J_1 L_\infty(G)' J_1 = L_\infty(G) = J_2 L_\infty(G) J_2$

iii) $(1 \otimes S)(U_2 \otimes 1)(1 \otimes S)(U_2 \otimes 1) = (1 \otimes U_2^*)(U_2 \otimes 1)(1 \otimes U_2)$

iv) $U_2^* = (J_2 \otimes J_1) U_2 (J_2 \otimes J_1)$

v) $U_2 \in L_\infty(G)' \otimes L(G)$

vi) $S U_2^* (L(G) \otimes 1) U_2 S = (J_1 \otimes J_1) U_2^* (L(G) \otimes 1) U_2 (J_1 \otimes J_1)$

vii) $S U_2^* (1 \otimes L_\infty(G)) U_2 S = (J_1 \otimes J_1) U_2^* (1 \otimes L_\infty(G)) U_2 (J_1 \otimes J_1)$.

The proof is similar to the one of Proposition 2.2.

Comparing Proposition 2.2 and 2.4, we see that we get the same results for the 5-tuple $(L_\infty(G), L(G), U_1, J_1, J_2)$ and $(L(G), L_\infty(G)', U_2, J_2, J_1)$. Moreover it turns out that the results for $(L(G), L_\infty(G)', U_2, J_2, J_1)$ can be derived from those of $(L_\infty(G), L(G), U_1, J_1, J_2)$, only by using the relation

$$U_2 = S(1 \otimes J_1 J_2) U_1 (1 \otimes J_1 J_2) S .$$

We will generalize this situation in the following theorem.

THEOREM 2.5. *Let M_1 and M_2 be von Neumann algebras on a Hilbertspace $\mathscr{K}$, W_1 a unitary operator on $\mathscr{K} \otimes \mathscr{K}$ and K_1 and K_2 self-adjoint conjugate linear operators on $\mathscr{K}$, satisfying the following properties*:

i) $M_2 \cap M_1 = \mathbf{C}1,\quad K_1^2 = K_2^2 = 1$ *and* $K_1 K_2 = K_2 K_1$

ii) $K_1 M_1 K_1 = M_1' = K_2 M_1' K_2$ *and* $K_2 M_2 K_2 = M_2' = K_1 M_2' K_1$

iii) $(1 \otimes S)(W_1 \otimes 1)(1 \otimes S)(W_1 \otimes 1) = (1 \otimes W_1^*)(W_1 \otimes 1)(1 \otimes W_1)$

iv) $W_1^* = (K_1 \otimes K_2) W_1 (K_1 \otimes K_2)$

v) $W_1 \in M_2 \otimes M_1$

vi) $S W_1^* (M_1 \otimes 1) W_1 S = (K_2 \otimes K_2) W_1^* (M_1 \otimes 1) W_1 (K_2 \otimes K_2)$

vii) $S W_1^* (1 \otimes M_2') W_1 S = (K_2 \otimes K_2) W_1^* (1 \otimes M_2') W_1 (K_2 \otimes K_2)$.

Then the same properties hold if we replace M_1, M_2, W_1, K_1, K_2 by M_2, M_1', W_2, K_2, K_1 respectively where W_2 is defined by

$$W_2 = S(1 \otimes K_1 K_2) W_1 (1 \otimes K_1 K_2) S . \tag{viii}$$

PROOF. The properties i) and ii) and also iv) to vii) for the 5-tuple $(M_2, M_1', W_2, K_2, K_1)$ can be proved as a consequence of the corresponding relations for the 5-tuple $(M_1, M_2, W_1, K_1, K_2)$. Since this can be done by an easy calculation, we shall only show the difficult relation iii). From iv) and (viii) we have

(ix) $$W_2 = S(K_1 \otimes K_1)W_1^*(K_1 \otimes K_1)S\,.$$

But

(x) $$(1 \otimes S)(W_1^* \otimes 1)(1 \otimes S) = (S \otimes 1)(1 \otimes W_1^*)(S \otimes 1)\,.$$

From (ix) and (x) we get that

(xi) $$\begin{aligned}&(1 \otimes S)(W_2 \otimes 1)(1 \otimes S)(W_2 \otimes 1)\\ &= (K_1 \otimes K_1 \otimes K_1)(1 \otimes S)(S \otimes 1)(1 \otimes S)(S \otimes 1)(1 \otimes W_1^*)(S \otimes 1)\\ &\quad (1 \otimes W_1^*)(S \otimes 1)(1 \otimes S)(S \otimes 1)(K_1 \otimes K_1 \otimes K_1)\,.\end{aligned}$$

From iii) and (x)

(xii) $$(S \otimes 1)(1 \otimes W_1^*)(S \otimes 1)(1 \otimes W_1^*) = (W_1 \otimes 1)(1 \otimes W_1^*)(W_1^* \otimes 1)$$

and (xi) and (xii) give

$$\begin{aligned}&(1 \otimes S)(W_2 \otimes 1)(1 \otimes S)(W_2 \otimes 1)\\ &= (K_1 \otimes K_1 \otimes K_1)(1 \otimes S)(S \otimes 1)(1 \otimes S)(W_1 \otimes 1)(1 \otimes W_1^*)(W_1^* \otimes 1)\\ &\quad (S \otimes 1)(1 \otimes S)(S \otimes 1)(K_1 \otimes K_1 \otimes K_1)\,.\end{aligned}$$

Using (x) again, we get

$$\begin{aligned}&(1 \otimes S)(W_2 \otimes 1)(1 \otimes S)(W_2 \otimes 1)\\ &= (K_1 \otimes K_1 \otimes K_1)(1 \otimes S)(1 \otimes W_1)(1 \otimes S)(S \otimes 1)(W_1^* \otimes 1)\\ &\quad (S \otimes 1)(1 \otimes S)(1 \otimes W_1^*)(1 \otimes S)(K_1 \otimes K_1 \otimes K_1)\,,\end{aligned}$$

and so, using (ix),

$$(1 \otimes S)(W_2 \otimes 1)(1 \otimes S)(W_2 \otimes 1) = (1 \otimes W_2^*)(W_2 \otimes 1)(1 \otimes W_2)\,.$$

Before we can state the duality theorem, we have to generalize the concepts of action, dual action, crossed product and crossed dual product:

DEFINITION 2.6. If $\mathscr{M} = (M_1, M_2, W_1, K_1, K_2)$ is the 5-tuple of Theorem 2.5 and A is a von Neumann algebra acting on $\mathscr{H}$, then an action of $\mathscr{M}$ on A is an injective normal *-homomorphism $\pi\colon A \to A \otimes M_1$ such that the following diagram is commutative:

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[d, "\pi"'] & A \otimes M_1 \arrow[d, "\pi \otimes 1"] \\ A \otimes M_1 \arrow[r, "1 \otimes \delta"] & A \otimes M_1 \otimes M_1 \end{tikzcd}$$

Here δ is defined on M_1 by $\delta(x) = W_1^*(x \otimes 1)W_1$.

Let us first show that δ maps M_1 into $M_1 \otimes M_1$ so that this definition makes sense. Indeed, $W_1 \in M_2 \otimes M_1$ implies that

$$W_1^*(M_1 \otimes 1)W_1 \subset \mathscr{B}(L_2(G)) \otimes M_1 ,$$

which, together with

$$SW_1^*(M_1 \otimes 1)W_1 S = (K_2 \otimes K_2)W_1^*(M_1 \otimes 1)W_1(K_2 \otimes K_2)$$

and $K_2 M_1 K_2 = M_1$, gives $W_1^*(M_1 \otimes 1)W_1 \subset M_1 \otimes M_1$.

We can see immediately that an action of a locally compact group on a von Neumann algebra A as defined in 1.1 is the same as an action of $\mathscr{M} = (L_\infty(G), L(G), U_1, J_1, J_2)$ on A as defined above. A dual action of a locally compact group G (see Definition 1.2) agrees with an action of $\mathscr{M} = (L(G), L_\infty(G)', U_2, J_2, J_1)$.

DEFINITION 2.7. If π is an action of $\mathscr{M} = (M_1, M_2, W_1, K_1, K_2)$ on A, then π is called non-degenerate if the von Neumann algebra $A \otimes_\pi \mathscr{B}(\mathscr{K})$ generated by $\pi(A)$ and $1 \otimes \mathscr{B}(\mathscr{K})$ is equal to $A \otimes \mathscr{B}(\mathscr{K})$.

In section 3 we will show that both an action of a locally compact group and a dual action are non-degenerate.

DEFINITION 2.8. If π is an action of $\mathscr{M} = (M_1, M_2, W_1, K_1, K_2)$ on A, then the von Neumann algebra generated by $\pi(A)$ and $1 \otimes M_2$ is called the δ-product of A and $\mathscr{M}$ by π and is denoted by $A \otimes_\pi M_2$.

Comparing the definitions of crossed product and crossed dual product with 2.8, we see that a crossed product is a δ-product where $\mathscr{M} = (L_\infty(G)), L(G), U_1, J_1, J_2)$ and a crossed dual product is a δ-product where $\mathscr{M} = (L(G)), L_\infty(G)', U_2, J_2, J_1)$.

DEFINITION 2.9. If $A \otimes_\pi M_2$ is a δ-product of A and $\mathscr{M} = (M_1, M_2, W_1, J_1, J_2)$ by π, then $\hat{\pi}$ is called the dual action of π if

$$\hat{\pi}(x) = (1 \otimes W_2^*)(x \otimes 1)(1 \otimes W_2) \quad \text{for all } x \in A \otimes_\pi M_2$$

with

$$W_2 = S(1 \otimes K_1 K_2)W_1(1 \otimes K_1 K_2)S .$$

Then, the dual δ-product is the von Neumann algebra generated by $(1 \otimes W_2^*)(A \otimes_\pi M_2 \otimes 1)(1 \otimes W_2)$ and $1 \otimes 1 \otimes M_1'$ and will be denoted by $(A \otimes_\pi M_2) \otimes_{\hat{\pi}} M_1'$.

Now we come to the main theorem of this paper, namely the general duality theorem.

THEOREM 2.10. *If π is a non-degenerate action of $\mathscr{M}=(M_1,M_2,W_1,J_1,J_2)$ on the von Neumann algebra A, then the dual δ-product $(A\otimes_\pi M_2)\otimes_{\hat\pi} M_1'$ is isomorphic to $A\otimes\mathscr{B}(\mathscr{H})$ and the isomorphism is given by*

$$(A\otimes_\pi M_2)\otimes_{\hat\pi} M_1' \;=\; (1\otimes W_1)(\pi\otimes 1)(A\otimes\mathscr{B}(\mathscr{H}))(1\otimes W_1^*)\,.$$

PROOF. $(A\otimes_\pi M_2)\otimes_{\hat\pi} M_1'$ is the von Neumann algebra generated by $\hat\pi(A\otimes_\pi M_2)$ and $1\otimes 1\otimes M_1'$.

But $\hat\pi(A\otimes_\pi M_2)$ is generated by $(1\otimes W_2^*)(\pi(A)\otimes 1)(1\otimes W_2)$ and $\hat\pi(1\otimes M_2)$. Because of the definition of an action of $\mathscr{M}$ on A, $\pi(A)\subset A\otimes M_1$. But $W_2\in M_1'\otimes M_2$ because of Theorem 2.5. Hence

$$(1\otimes W_2^*)(\pi(A)\otimes 1)(1\otimes W_2) \;=\; \pi(A)\otimes 1\,.$$

By the same theorem $W_1\in M_2\otimes M_1$. Hence

$$1\otimes 1\otimes M_1' \;=\; (1\otimes W_1)(1\otimes 1\otimes M_1')(1\otimes W_1^*)\,.$$

By the definition of W_2 we have

$$\begin{aligned}&W_2^*(M_2\otimes 1)W_2\\ &\quad= S(1\otimes K_1K_2)W_1^*(1\otimes K_1K_2)S(M_2\otimes 1)S(1\otimes K_1K_2)W_1(1\otimes K_1K_2)S\end{aligned}$$

and by Theorem 2.5

$$K_1K_2M_2K_2K_1 \;=\; K_1M_2'K_1 \;=\; M_2'$$

and also

$$W_1^*(1\otimes M_2')W_1 \;=\; (K_2\otimes K_2)SW_1^*(1\otimes M_2')W_1S(K_2\otimes K_2)\,.$$

So we get

$$\begin{aligned}W_2^*(M_2\otimes 1)W_2 &= S(K_2\otimes K_1K_2K_2)SW_1^*(1\otimes M_2')W_1S(K_2\otimes K_2K_2K_1)S\\ &= (K_1\otimes K_2)W_1^*(1\otimes M_2')W_1(K_1\otimes K_2)\end{aligned}$$

or

$$W_2^*(M_2\otimes 1)W_2=W_1(1\otimes M_2)W_1^*$$

because of the fact that

$$W_1^* \;=\; (K_1\otimes K_2)W_1(K_1\otimes K_2) \quad\text{ and }\quad K_2M_2'K_2 \;=\; M_2\,.$$

Hence

$$\begin{aligned}(A\otimes_\pi M_2)\otimes_{\hat\pi} M_1' &= \{\pi(A)\otimes 1,\ (1\otimes W_1)(1\otimes 1\otimes M_2)(1\otimes W_1^*),\\ &\qquad (1\otimes W_1)(1\otimes 1\otimes M_1')(1\otimes W_1^*)\}''\\ &= (1\otimes W_1)\{(1\otimes W_1^*)(\pi(A)\otimes 1)(1\otimes W_1),\\ &\qquad 1\otimes 1\otimes M_2,\ 1\otimes 1\otimes M_1'\}''(1\otimes W_1^*)\,.\end{aligned}$$

By Theorem 2.5, $M_2\cup M_1'=\mathscr{B}(\mathscr{K})$ and by the definition of an action

$$(1\otimes W_1^*)(\pi(A)\otimes 1)(1\otimes W_1) = (\pi\otimes 1)\pi(A)\,.$$

Hence

$$(A\otimes_\pi M_2)\otimes_{\hat\pi} M_1' = (1\otimes W_1)(\pi\otimes 1)\{\pi(A),\ 1\otimes\mathscr{B}(\mathscr{K})\}''(1\otimes W_1^*)\,.$$

But π is non-degenerate, so

$$\{\pi(A), 1\otimes\mathscr{B}(\mathscr{K})\}'' = A\otimes\mathscr{B}(\mathscr{K})$$

and

$$(A\otimes_\pi M_2)\otimes_{\hat\pi} M_1' = (1\otimes W_1)(\pi\otimes 1)(A\otimes\mathscr{B}(\mathscr{K}))(1\otimes W_1^*)\,.$$

3. Non-degeneracy for actions and dual actions of locally compact groups.

In this section we shall show that the duality theorem of Landstad and Nakagami are special cases of the general one of section 2. In the first part we prove the duality theorem for crossed products, where we use the fact that a crossed product is a δ-product and that the action defined in this case is non-degenerate.

THEOREM 3.1. *If π_1 is an action of $\mathscr{M}=(L_\infty(G), L(G), U_1, J_1, J_2)$ on a von Neumann algebra A_1, then π_1 is non-degenerate.*

PROOF. We have to prove that

$$A_1\otimes_{\pi_1}\mathscr{B}(L_2(G)) = A_1\otimes\mathscr{B}(L_2(G))\,.$$

The inclusion $A_1\otimes_{\pi_1}\mathscr{B}(L_2(G))\subset A_1\otimes\mathscr{B}(L_2(G))$ is a consequence of the definition of an action, because $\pi_1(A_1)\subset A_1\otimes L_\infty(G)$. For the other inclusion must show that $A_1\otimes 1\subset A_1\otimes_{\pi_1}\mathscr{B}(L_2(G))$. But by definition $(\pi_1\otimes 1)\pi_1 =(1\otimes\delta_1)\pi_1$ with δ_1 as in Definition 1.1. So this is equivalent to

$$\pi_1(A_1)\otimes 1 \subset (\pi_1\otimes 1)(A_1\otimes_{\pi_1}\mathscr{B}(L_2(G))) = \pi_1(A_1)\otimes_{1\otimes\delta_1}\mathscr{B}(L_2(G))\,.$$

For this it is sufficient to prove that

$$x\otimes 1 \in \{(1\otimes U_1^*)(x\otimes 1)(1\otimes U_1),\ 1\otimes 1\otimes \mathscr{B}(L_2(G))\}'' \quad \text{if } x \in \mathscr{B}(\mathscr{H}_1\otimes L_2(G))\,.$$

Let f_K be an element of $K(G)$, the set of the functions with compact support, such that $\int |f_K|^2\,ds=1$, where K denotes the support of f_K. Let A_1 act on $\mathscr{H}_1$, then we have for $\tilde{\xi} \in \mathscr{H}_1\otimes L_2(G)$

$$\|(1\otimes U_1)(\tilde{\xi}\otimes f_K)-\tilde{\xi}\otimes f_K\|^2 = \int |f_K(t)|^2\,\|(\lambda_t-1)\tilde{\xi}\|^2\,dt$$

because

$$(U_1 f)(s,t) = f(t^{-1}s,t) \quad \text{if } f \in K(G\times G)\,.$$

But λ is a strongly continuous representation of G, so $\forall\,\varepsilon>0$ there exists a neighbourhood V of e such that $\|(\lambda_t-1)\tilde{\xi}\|<\varepsilon$ if $t \in V$. Take $K\subset V$. Then

$$\|(1\otimes U_1)(\tilde{\xi}\otimes f_K)-\tilde{\xi}\otimes f_K\|^2 < \varepsilon^2 \int |f_K(t)|^2\,dt = \varepsilon^2\,.$$

Furthermore, if $\tilde{\xi}$ and $\tilde{\eta} \in \mathscr{H}_1\otimes L_2(G)$, we get

$$\begin{aligned}
&|\langle(1\otimes U_1^*)(x\otimes 1)(1\otimes U_1)(\tilde{\xi}\otimes f_K),\tilde{\eta}\otimes f_K\rangle-\langle x\tilde{\xi},\tilde{\eta}\rangle|\\
&= |\langle(x\otimes 1)(1\otimes U_1)(\tilde{\xi}\otimes f_K),\ (1\otimes U_1)(\tilde{\eta}\otimes f_K)\rangle-\langle x\tilde{\xi},\tilde{\eta}\rangle|\\
&\leqq |\langle(x\otimes 1)(1\otimes U_1)(\tilde{\xi}\otimes f_K),\ (1\otimes U_1)(\tilde{\eta}\otimes f_K)\rangle\\
&\quad -\langle(x\otimes 1)(1\otimes U_1)(\tilde{\xi}\otimes f_K),\ \tilde{\eta}\otimes f_K\rangle|\\
&\quad +|\langle(x\otimes 1)(1\otimes U_1)(\tilde{\xi}\otimes f_K),\ \tilde{\eta}\otimes f_K\rangle-\langle(x\otimes 1)(\tilde{\xi}\otimes f_K),\ \tilde{\eta}\otimes f_K\rangle|\\
&\leqq \|x\|\ \|\tilde{\xi}\|\ \|f_K\|\ \|(1\otimes U_1)(\tilde{\eta}\otimes f_K)-(\tilde{\eta}\otimes f_K)\|\\
&\quad +\|x\|\ \|(1\otimes U_1)(\tilde{\xi}\otimes f_K)-(\tilde{\xi}\otimes f_K)\|\ \|\tilde{\eta}\|\ \|f_K\|\,.
\end{aligned}$$

But we know that $\|(1\otimes U_1)(\tilde{\eta}\otimes f_K)-(\tilde{\eta}\otimes f_K)\| \to 0$ if K decreases and also $\|(1\otimes U_1)(\tilde{\xi}\otimes f_K)-(\tilde{\xi}\otimes f_K)\| \to 0$ if K decreases.

Therefore if we put $\omega_K(y)=\langle y f_K, f_K\rangle$ for $y \in L_\infty(G)$, we get

$$(1\otimes 1\otimes\omega_K)((1\otimes U_1^*)(x\otimes 1)(1\otimes U_1)) \to x \quad \text{in the weak sense}\,.$$

Let

$$y\otimes 1 \in \mathscr{B}(\mathscr{H}_1\otimes L_2(G))\otimes 1 \cap (1\otimes U_1^*)(x\otimes 1)'(1\otimes U_1)\,.$$

Then since

$$(1\otimes 1\otimes\omega_K)(y\otimes 1) = y$$

we have

$$y(1\otimes 1\otimes \omega_K)((1\otimes U_1^*)(x\otimes 1)(1\otimes U_1))$$
$$= (1\otimes 1\otimes \omega_K)((1\otimes U_1^*)(x\otimes 1)(1\otimes U_1))y .$$

But

$$(1\otimes 1\otimes \omega_K)((1\otimes U_1^*)(x\otimes 1)(1\otimes U_1)) \to x \quad \text{in the weak sense} .$$

Hence

$$yx = xy \quad \text{or} \quad x\otimes 1 \in \{(1\otimes U_1^*)(x\otimes 1)(1\otimes U_1),\ 1\otimes 1\otimes \mathscr{B}(L_2(G))\}'' .$$

We shall now prove the duality theorem for crossed products:

THEOREM 3.2. *If A_1 is a von Neumann algebra on $\mathscr{H}_1$, G a locally compact group and π_1 an action of G on A_1, then*

$$(A_1 \otimes_{\pi_1} L(G)) \otimes_{\hat{\pi}_1} L_\infty(G) = (1\otimes U_1)(\pi_1\otimes 1)(A_1\otimes \mathscr{B}(L_2(G))(1\otimes U_1^*) .$$

PROOF. This theorem is an immediate consequence of the general duality Theorem 2.10. Indeed, because of the definition $\hat{\pi}_1$ (see Theorem 1.3) and the fact that $L_\infty(G) = L_\infty(G)'$, $(A_1 \otimes_{\pi_1} L(G)) \otimes_{\hat{\pi}_1} L_\infty(G)$ is a dual δ-product. Furthermore π_1 is a non-degenerate action by Theorem 3.1.

Now we shall show that also the duality theorem for crossed dual products is a consequence of the general one. For this we must prove that a dual action of a locally compact group is non-degenerate.

Let us first consider the case where G is a compact group. We define ω on $L(G)$ by $\omega(x) = \langle x1, 1\rangle$ where 1 denotes the L_2-function such that $1(t) = 1$ for all $t \in G$.

If f and g are elements of $L_2(G)$ we have that

$$\begin{aligned}\langle (1\otimes\omega)(U_2)f, g\rangle &= \langle U_2(f\otimes 1), g\otimes 1\rangle \\ &= \iint f(s)1(st)g(s)1(t)\,ds\,dt \\ &= \iint f(s)g(s)\,ds\,dt \\ &= \langle f\otimes 1, g\otimes 1\rangle = \langle f,g\rangle\langle 1,1\rangle \\ &= \langle (1\otimes\omega)(1)f, g\rangle\end{aligned}$$

So, if $x \in \mathscr{B}(\mathscr{H}_2 \otimes L_2(G))$, because ω is multiplicative on $L(G)$, we have

$$(1 \otimes 1 \otimes \omega)((1 \otimes U_2^*)(x \otimes 1)(1 \otimes U_2)) = x$$

and we get that

$$\pi_2(A_2) \otimes_{1 \otimes \delta_2} \mathscr{B}(L_2(G)) = \pi_2(A_2) \otimes \mathscr{B}(L_2(G))$$

with δ_2 as in Definition 1.2. which means that the action π_2 on A_2 is non-degenerate.

It is clear that we can not do the same thing for a non-compact group. But we will do something similar in the following lemmas.

LEMMA 3.3. *There exists a net $\{\omega_K\}$ in $L(G)_*$ such that*

$$\omega_K(\lambda_p) = \omega_K^*(\lambda_p) \quad \textit{and} \quad \omega_K(\lambda_p) \to 1 \textit{ for all } p \textit{ in } G, \quad \textit{when } K \to G .$$

PROOF. Let f_0 be an element of $K(G)$ such that $f_0 \geqq 0$ and $\int f_0(s)\,ds = 1$. Put $g_K(t) = \int_K f_0(ts)\,ds$ for any compact subset K of G. So

$$g_K = \left(\int_K \varrho_s \Delta(s)^{-\frac{1}{2}}\,ds\right) f .$$

If $K \to G$, we have

$$g_K(t) \to \int f_0(ts)\,ds = \int f_0(s)\,ds = 1$$

so by Dini's theorem g_K converges to 1 uniformly on compact sets. Then define ω_K by

$$\omega_K(x) = \langle x f_0, g_K \rangle \quad \text{for } x \in L(G) .$$

Then

$$\omega_K(\lambda_p) = \int f_0(p^{-1}s) g_K(s)\,ds \to \int f_0(p^{-1}s)\,ds = \int f_0(s)\,ds = 1 .$$

Furthermore,

$$\begin{aligned}
\omega_K^*(\lambda_p) = \overline{\omega_K(\lambda_p^*)} &= \overline{\langle \lambda_p^* f_0, g_K \rangle} = \langle g_K, \lambda_p^* f_0 \rangle \\
&= \int \langle \varrho_s \Delta(s)^{-\frac{1}{2}} f_0, \lambda_p^* f_0 \rangle\,ds \\
&= \int \langle \lambda_p f_0, \varrho_{s^{-1}} \Delta(s)^{-\frac{1}{2}} f_0 \rangle\,ds \\
&= \langle \lambda_p f_0, g_K \rangle = \omega_K(\lambda_p) .
\end{aligned}$$

In the next lemma we identify $\mathscr{H}_2 \otimes L_2(G)$ with the completion of $K(G, \mathscr{H}_2)$, i.e. the continuous functions of G to $\mathscr{H}_2$ with compact support.

LEMMA 3.4. *Let $x \in \mathscr{B}(\mathscr{H}_2 \otimes L_2(G))$ and $\xi \in K(G, \mathscr{H}_2)$. Put*

$$\varphi_K(x) = (1 \otimes 1 \otimes \omega_K)((1 \otimes U_2^*)(x \otimes 1)(1 \otimes U_2))$$

with ω_K as in Lemma 3.3.

Then there is a number M such that

$$|\langle \varphi_K(x)\xi, \eta \rangle| \leqq \|(x \otimes 1)(1 \otimes U_2)(\xi \otimes f_0)\| \, \|\eta\| M$$

for all K, $\eta \in \mathscr{H}_2 \otimes L_2(G)$ and with f_0 as in Lemma 3.3.

PROOF. By the definition of U_2 we have

$$((1 \otimes U_2)(\xi \otimes f_0))(p, q) = \xi(p) f_0(pq) .$$

As ξ and f_0 have compact support, q must belong to a compact set. Choose $h \in K(G)$ such that $h \geqq 0$ and $h = 1$ on this compact set. Then

$$(1 \otimes 1 \otimes m_h)(1 \otimes U_2)(\xi \otimes f_0) = (1 \otimes U_2)(\xi \otimes f_0)$$

and

$$\begin{aligned} |\langle \varphi_K(x)\xi, \eta \rangle| &= |\langle (1 \otimes 1 \otimes m_h)(x \otimes 1)(1 \otimes U_2)(\xi \otimes f_0), (1 \otimes U_2)(\eta \otimes g_K) \rangle| \\ &\leqq \|(x \otimes 1)(1 \otimes U_2)(\xi \otimes f_0)\| \, \|(1 \otimes 1 \otimes m_h)(1 \otimes U_2)(\eta \otimes g_K)\| . \end{aligned}$$

But

$$\begin{aligned} &\|(1 \otimes 1 \otimes m_h)(1 \otimes U_2)(\eta \otimes g_K)\|^2 \\ &\qquad = \iint h(q)^2 \|\eta(p)\|^2 |g_K(pq)|^2 \, dp \, dq \leqq \int h(q)^2 \, dq \|\eta\|^2 \end{aligned}$$

for $g_K \leqq 1$. So put $M = \int h(q)^2 \, dq$ and we get the desired result.

LEMMA 3.5. *If $x \in \mathscr{B}(\mathscr{H}_2 \otimes L_2(G))$ and $\xi \in \mathscr{H}_2 \otimes L_2(G)$ with compact support, then*

$$\langle \varphi_K(x)\xi, \eta \rangle \to \langle x\xi, \eta \rangle \quad \textit{for all } \eta \in \mathscr{H}_2 \otimes L_2(G) .$$

In particular, if $x \in \mathscr{B}(\mathscr{H}_2 \otimes L_2(G))$, then

$$x \in \{\varphi_K(x) \; ; \; K = K^{-1} \textit{ and } K \textit{ is compact}\}'' .$$

PROOF. Let

$$C = \{y \otimes m_f \lambda_p, \; y \in \mathscr{B}(\mathscr{H}_2), \; f \in K(G), \; p \in G\} .$$

By a simple computation we see that

$$U_2^*(\lambda_p\otimes 1)U_2 = \lambda_p\otimes\lambda_p \quad \text{and} \quad U_2^*(m_f\otimes 1)U_2 = m_f\otimes 1 .$$

Therefore, with ω_K defined as in Lemma 3.3

$$\begin{aligned} &\langle((1\otimes 1\otimes\omega_K)((1\otimes U_2^*)(y\otimes\lambda_p m_f\otimes 1)(1\otimes U_2)))\xi,\eta\rangle \\ &\quad = \langle((1\otimes 1\otimes\omega_K)(y\otimes\lambda_p m_f\otimes\lambda_p))\xi,\eta\rangle \\ &\quad = \langle(y\otimes\lambda_p m_f)\xi,\eta\rangle\langle\lambda_p f_0, g_K\rangle \to \langle(y\otimes\lambda_p m_f)\xi,\eta\rangle \end{aligned}$$

because of Lemma 3.3. Hence

$$\langle\varphi_K(y)\xi,\eta\rangle \to \langle y\xi,\eta\rangle \quad \text{if } y\in C .$$

Here φ_K is defined as in Lemma 3.4.

We also know that C spans a σ-weakly dense subset in $\mathscr{B}(\mathscr{H}_2\otimes L_2(G))$. So $\forall\,\varepsilon>0$ there exists a $x_0\in C$ such that

$$|\langle x_0\xi,\eta\rangle-\langle x\xi,\eta\rangle| < \varepsilon \quad \text{and} \quad \|((x-x_0)\otimes 1)(1\otimes U_2)(\xi\otimes f)\| < \varepsilon .$$

But by Lemma 3.4 we have

$$\begin{aligned} |\langle\varphi_K(x)\xi,\eta\rangle-\langle x\xi,\eta\rangle| &\leqq \|((x-x_0)\otimes 1)(1\otimes U_2)(\xi\otimes f)\|\,\|\eta\|M+ \\ &\quad +|\langle\varphi_K(x_0)\xi,\eta\rangle-\langle x_0\xi,\eta\rangle|+|\langle x_0\xi,\eta\rangle-\langle x\xi,\eta\rangle| . \end{aligned}$$

Therefore for $K\to G$

$$|\langle\varphi_K(x_0)\xi,\eta\rangle-\langle x_0\xi,\eta\rangle| \to 0$$

and consequently

$$|\langle\varphi_K(x)\xi,\eta\rangle-\langle x\xi,\eta\rangle| \to 0$$

since we can find a $x_0\in C$ such that

$$\|((x-x_0)\otimes 1)(1\otimes U_2)(\xi\otimes f)\|\,\|\eta\|M+|\langle x_0\xi,\eta\rangle-\langle x\xi,\eta\rangle|$$

becomes arbitrarily small.

Now let y commute with $\varphi_K(x)$ for all compact sets K such that $K=K^{-1}$. Take ξ and η with compact support. Then

$$\begin{aligned} \langle yx\xi,\eta\rangle &= \lim_{K\to G} \langle\varphi_K(x)\xi,y^*\eta\rangle \\ &= \lim_{K\to G} \langle y\xi,\varphi_K(x^*)\eta\rangle \\ &= \langle xy\xi,\eta\rangle . \end{aligned}$$

Here we used that $\langle \varphi_K(x)\xi, \eta \rangle \to \langle x\xi, \eta \rangle$ and the fact that $\varphi_K(x)^* = \varphi_K(x^*)$ which is a consequence of Lemma 3.3. Hence

$$xy = yx \quad \text{for all } y \in \{\varphi_K(x)\ ;\ K = K^{-1} \text{ and } K \text{ is compact}\}' .$$

Before we come to the duality theorem for crossed dual products, we shall show that a dual action of a locally compact group G on a von Neumann algebra A_2 is non-degenerate.

THEOREM 3.6. *If π_2 is an action of $\mathscr{M} = (L(G), L_\infty(G)', U_2, J_2, J_1)$ on a von Neumann algebra A_2 acting on $\mathscr{H}_2$, then*

$$A_2 \otimes_{\pi_2} \mathscr{B}(L_2(G)) = A_2 \otimes \mathscr{B}(L_2(G)) .$$

PROOF. As π_2 is an injective mapping it is sufficient to prove that

$$(\pi_2 \otimes 1)(A_2 \otimes_{\pi_2} \mathscr{B}(L_2(G))) = \pi_2(A_2) \otimes \mathscr{B}(L_2(G)) .$$

But

$$(\pi_2 \otimes 1)\pi_2(A_2) = (1 \otimes \delta_2)\pi_2(A_2)$$

with δ_2 as in Definition 1.2. So we have to show that

$$\pi_2(A_2) \otimes_{1 \otimes \delta_2} \mathscr{B}(L_2(G)) = \pi_2(A_2) \otimes \mathscr{B}(L_2(G)) .$$

As

$$\begin{aligned}
&(\pi_2(A_2) \otimes_{1 \otimes \delta_2} \mathscr{B}(L_2(G)))' \\
&\quad = \{(1 \otimes U_2^*)(\pi_2(A_2) \otimes 1)(1 \otimes U_2), 1 \otimes 1 \otimes \mathscr{B}(L_2(G))\}' \\
&\quad = (1 \otimes U_2^*)(\pi_2(A_2)' \otimes \mathscr{B}(L_2(G)))(1 \otimes U_2) \cap (\mathscr{B}(\mathscr{H}_2 \otimes L_2(G)) \otimes 1)
\end{aligned}$$

there exists a von Neumann algebra $A \subset \mathscr{B}(\mathscr{H}_2 \otimes L_2(G))$ such that

$$\pi_2(A_2) \otimes_{1 \otimes \delta_2} \mathscr{B}(L_2(G)) = A \otimes \mathscr{B}(L_2(G)) .$$

Because

$$\begin{aligned}
(1 \otimes \delta_2)\pi_2(A_2) = (\pi_2 \otimes 1)\pi_2(A_2) &\subset (\pi_2 \otimes 1)(A_2 \otimes L(G)) \\
&\subset \pi_2(A_2) \otimes L(G) ,
\end{aligned}$$

we have that

$$A \subset \pi_2(A_2) .$$

The other inclusion follows from Lemma 3.5. Indeed, let $y \in A'$, then

$$y \otimes 1 \in \mathscr{B}(\mathscr{H}_2 \otimes L_2(G)) \otimes 1 \cap (1 \otimes U_2^*)(\pi_2(A_2) \otimes 1)'(1 \otimes U_2) .$$

Since $(1\otimes 1\otimes\omega_K)(y\otimes 1)=y$ we have for $x\in\pi_2(A_2)$

$$y(1\otimes 1\otimes\omega_K)((1\otimes U_2^*)(x\otimes 1)(1\otimes U_2)) = (1\otimes 1\otimes\omega_K)((1\otimes U_2^*)(x\otimes 1)(1\otimes U_2))y\ .$$

But by Lemma 3.5 we have that

$$x\in\{(1\otimes 1\otimes\omega_K)((1\otimes U_2^*)(x\otimes 1)(1\otimes U_2))\ ;\ K=K^{-1} \text{ and } K \text{ is compact}\}''\ .$$

Hence $xy=yx$ and $A'\subset\pi_2(A_2)'$, and so $\pi_2(A_2)=A$.

THEOREM 3.7. *If A_2 is a von Neumann algebra on $\mathscr{H}_2$, G is a locally compact group and π_2 a dual action of G on A_2, then*

$$(A_2\otimes_{\pi_2}L_\infty(G))\otimes_{\hat{\pi}_2}L(G)' = (1\otimes U_2)(\pi_2\otimes 1)(A_2\otimes\mathscr{B}(L_2(G))(1\otimes U_2^*)\ .$$

PROOF. This theorem is also a consequence of the general duality Theorem 2.10. Indeed, because of the definition of $\hat{\pi}_2$ (see Theorem 2.3), it follows that $A_2\otimes_{\pi_2}L_\infty(G)$ is a δ-product and $A_2\otimes_{\pi_2}L_\infty(G)\otimes_{\hat{\pi}_2}L(G)'$ is a dual δ-product, where $\mathscr{M}=(L(G),L_\infty(G)',U_2,J_2,J_1)$.

Furthermore π_2 is non-degenerate because of Theorem 3.7.

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