ENLARGING A SUBSPACE OF $C(X)$ WITHOUT CHANGING THE CHOQUET BOUNDARY

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1.

Let $X$ be a compact metric space, let $B$ be a uniformly closed subspace of $C(X)$, the Banach space of all continuous real-valued functions on $X$, and suppose that $B$ separates the points of $X$ and contains the constant functions.

For each $x \in X$ let $\nu_x$ be a maximal measure on $X$ representing $x$ with respect to $B$. We are going to study conditions on the subspace $B$ and the selection $x \to \nu_x$ of maximal representing measures which ensure that the space

$$A = \left\{ f \in C(X) : \int f \, d\nu_x = f(x) \text{ all } x \in X \right\}$$

is a simplicial space i.e. the state space for $A$ is a simplex. In [3] it is shown that it is always possible to find a selection $x \to \nu_x$ which is measurable but this is in general not a sufficient condition. The case when the Choquet boundary $\partial_B X$ for $B$ is closed is treated in [2]. There it is shown that if $\partial_B X$ is closed, then the selection $x \to \nu_x$ is continuous if and only if $A$ is simplicial.

In this note conditions on $B$ and on the selection $x \to \nu_x$ are given which imply that $A$ is a simplicial space. These conditions involve separation of maximal representing measures and upper semi-continuity of the selection $x \to \nu_x$ with respect to the cone of all finite suprema of functions from $B$.

2.

Let $X$ and $B$ be as above. By $M(X)$ we denote the space of all regular Borel measures on $X$, by $\partial_B X$ the Choquet boundary for $B$ and by $M(\partial_B X)$ the set of all boundary (or maximal) measures on $X$; since $X$ is a metric space a measure $\mu$ is in $M(\partial_B X)$ if and only if $|\mu|(X \setminus \partial_B X) = 0$ ([1, (4.11), p. 35 and II. §2]).

A measurable selection of maximal representing measures is a map $x \to \nu_x$ of $X$ into $M(\partial_B X)$ such that $\nu_x$ is a probability measure in $M(\partial_B X)$ for which

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\[ b(x) = \int b \, dv_x \quad \text{for each } b \in B \]

and such that for each \( f \in C(X) \) the function

\[ x \rightarrow \int f \, dv_x \]

is Borel measurable on \( X \). Such a selection gives rise to a map \( T \) from \( M(X) \) into \( M(\partial_B X) \) defined by (cf. [3])

\[ \int f \, dT\mu = \int \left( \int f(y) \, dv_x(y) \right) \, d\mu(x), \quad \text{for all } f \in C(X). \]

From the definition of \( T \) it is clear that \( T \) is a linear map, and that \( T\mu = \mu \) if \( \mu \) is a boundary measure. We are interested in deciding when the space

\[ A = \left\{ f \in C(X) : f(x) = \int f \, dv_x, \text{ for all } x \in X \right\} \]

is a simplicial space, that is when the state space for \( A \) is a simplex. In terms of the map \( T \) we have a condition which implies that \( A \) is simplicial:

**Proposition 1.** The space \( A \) is simplicial, if the null-space for \( T \),

\[ N(T) = \{ \mu \in M(X) : T\mu = 0 \}, \]

is a \( w^* \)-closed subspace of \( M(X) \) (when \( M(X) \) is equipped with the \( w^* \)-topology defined by \( C(X) \)).

**Proof.** We look at the space \( M(X)/N(T) \) equipped with the quotient topology which is a Hausdorff topological vector space. The set \( M_1^+(X)/N(T) \) is a compact convex subset of \( M(X)/N(T) \) where \( M_1^+(X) \) denotes the set of probability measures on \( X \). Now, if \( \varphi \) denotes the canonical projection of \( M(X) \) onto \( M(X)/N(T) \) and if \( \tilde{a} \) is a continuous affine function on \( M_1^+(X)/N(T) \) then the restriction of \( \tilde{a} \circ \varphi \) to \( M_1^+(X) \) is a continuous affine function. Since \( T\nu = \nu \) if \( \nu \in M_1^+(\partial_B X) = M_1^+(X) \cap M(\partial_B X) \), this shows that the continuous affine functions on \( M_1^+(X) \), which are constant on each of the sets \( \{ \mu \in M_1^+(X) : T\mu = \nu \} \), where \( \nu \in M_1^+(X) \), separate the points of \( M_1^+(\partial_B X) \). But these functions are just the functions in \( A \) because the continuous affine functions on \( M_1^+(X) \) are of the form

\[ \mu \rightarrow \int f \, d\mu \]

where \( f \in C(X) \). If \( x \in X \setminus \partial_B X \) then \( \delta_x \), the point mass at \( x \), and \( \nu_x \) represent \( x \) with respect to \( A \), which shows that the Choquet boundary for \( A \) is contained
in that for $B$. The reverse inclusion is clear since $A$ contains $B$. Thus $A$ and $B$
have the same Choquet boundary. But then $M_1^+(\partial_B X) = M_1^+(\partial_A X)$ and we
conclude that $A$ separates the points of $M_1^+(\partial_A X)$, which shows that each point
in the state space for $A$ is represented by a unique measure in $M_1^+(\partial_A X)$. But
then the state space for $A$ is a simplex ([1, Thm. II 3.6]).

We now turn to look at conditions given in terms of the space $B$ on
selections of representing measures which ensure that $N(T)$ is $w^*$-closed.

Let $P$ denote the cone of all pointwise suprema of finitely many functions
from $B$. The following condition is necessary for the existence of a simplicial
space $A$ containing $B$ and having the same Choquet boundary as $B$:

(*) There is a measurable selection $x \to v_x$ of maximal measures such that for
each $f \in P$ the function $x \to \int f \, dv_x$ is upper semi-continuous on $X$.

The necessity of this condition follows from [1, Thm. II 3.7]. If $\partial_B X$ is closed,
then the restriction of $P$ to $\partial_B X$ is dense in $C(\partial_B X)$ so that (*) is in that case
equivalent to the existence of a measurable selection which is upper semi-
continuous with respect to $C(X)$ which is the same thing as saying that the
selection is continuous. This is, however known to be a sufficient condition
when $\partial_B X$ is closed ([2, Thm. (2.4)]).

The second condition is concerned with separation of measures in $M_1^+(\partial_B X)$
by functions in $P$:

(**) If $\mu, v \in M_1^+(\partial_B X)$ with $\mu \neq v$ then there exist functions $f, -g \in P$ with
$f \geq g$ on $\partial_B X$ such that

\[ \int f \, d\mu < \int g \, dv . \]

We observe that condition (**) is satisfied if $\partial_B X$ is closed, because in that
case the restriction of $P$ to $\partial_B X$ is dense in $C(\partial_B X)$. We also note that if (**) is satisfied
then each point of $\partial_B X$ has a unique boundary representing measure
with respect to $B$. This may be seen as follows: Suppose $x \in \partial_B X$ has to
representing measures $\mu, v \in M_1^+(\partial_B X)$. Let $f, -g$ be as in condition (**)
relative to $\mu$ and $v$. Then

\[ f(x) \leq \int f \, d\mu < \int g \, dv \leq g(x) . \]

But $f \geq g$ on $\partial_B X$ and hence $f(x) \geq g(x)$.

The following theorem shows how conditions (*) and (**) are related to the
existence of a simplicial subspace containing $B$ and having the same Choquet
boundary as $B$. 

Theorem 2. Let \( B \) be a closed subspace of \( C(X) \) and suppose that each point in \( \partial_B X \) has a unique boundary representing measure with respect to \( B \). Then there exists a simplicial space \( A \) containing \( B \) and having the same Choquet boundary as \( B \) if and only if conditions (*) and (**) are satisfied.

Proof. Let \( A \) be a simplicial subspace of \( C(X) \) containing \( B \) and suppose that \( \partial_A X = \partial_B X \). As noted earlier condition (*) is then satisfied. To see that condition (**) is satisfied let \( \mu \) and \( \nu \) be two different elements of \( M_1^+ (\partial_B X) \) and let \( a \) be a function in \( A \) such that

\[
\int a \, d\mu < \int a \, d\nu.
\]

Let \( \hat{a} \) be the function defined for each \( x \in X \) as follows:

\[
\hat{a}(x) = \inf \{ b(x) : b \in B \, b > a \}.
\]

It follows from [1, Cor. I. 3.6], that there is a measure \( \eta \) representing \( x \) with respect to \( B \) such that

\[
\hat{a}(x) = \int a \, d\eta.
\]

Let \( \xi \in M_1^+ (\partial_B X) \) be such that \( \xi - \eta \in A^\perp \). Such a measure exists by Choquet's theorem. Then also \( \xi - \eta \in B^\perp \) which shows that \( \xi \) is a boundary representing measure (with respect to \( B \)) for \( x \). Since each point of \( \partial_B X \) has a unique boundary representing measure with respect to \( B \), we conclude that \( \xi \) is a representing measure for \( x \) with respect to \( A \), if \( x \in \partial_B X \). Thus

\[
\hat{a}(x) = \int a \, d\eta = \int a \, d\xi = a(x)
\]

if \( x \in \partial_B X \). Therefore the function \( a \) can be approximated uniformly on \( \partial_B X \) by functions in \( P \) and applying this result to \( -a \in A \) instead of \( a \), we see that \( a \) can also be approximated uniformly by functions in \( -P \). Taking (1) into account we conclude that condition (**) is satisfied.

Suppose now that conditions (*) and (**) are satisfied and let \( T \) be the linear map from \( M(X) \) into \( M(\partial_B X) \) which the selection \( x \mapsto \nu_x \) gives rise to. By Proposition 1 it suffices to show that \( N(T) \) is \( w^* \)-closed in \( M(X) \). By the Krein–Smulian Theorem it suffices to show that the set

\[
N_1 = \{ \mu \in M(X) : \|\mu\| \leq 1 \text{ and } T\mu = 0 \}
\]

is \( w^* \)-closed in \( M(X) \).

First we observe that each \( \mu \in N_1 \) can be written as \( \mu = t\mu^+ - t\mu^- \) where \( \mu^+, \mu^- \in M_1^+ (X) \) and \( 0 \leq t \leq 1/2 \).
Let $\mu$ be in the $w^*$-closure of $N_1$ and let $\{\mu_n\}$ be a sequence in $N_1$ converging to $\mu$. Each $\mu_n$ can be decomposed as $\mu_n = t_n \mu^+_n - t_n \mu^-_n$ where $\mu^+_n, \mu^-_n \in M^1_1(X)$ and where $0 \leq t_n \leq 1/2$. Passing to a subsequence if necessary we may suppose that there are $\xi, \eta \in M^1_1(X)$ and a number $t$ with $0 \leq t \leq 1/2$ such that $\{\mu^+_n\}$ and $\{\mu^-_n\}$ converge to $\xi$ and $\eta$ respectively in the $w^*$-topology and such that $\{t_n\}$ converges to $t$. Then $\mu = t\xi - t\eta$. We want to show that $T\xi = T\eta$. Suppose this is not the case. Then there are functions $f, -g \in P$ such that $f \geq g$ on $\partial P X$ and such that

$$\int f d\eta < \int g d\xi$$

Suppose we knew that for each $h \in P$ the map

$$\eta \rightarrow \int h d\eta \quad \eta \in M^1_1(X)$$

was upper semi-continuous when $M^1_1(X)$ is equipped with the $w^*$-topology. Then

$$\int f d\eta \geq \liminf_n \int f dT\mu^+_n = \liminf_n \int f dT\mu^-_n \geq \liminf_n \int g dT\mu^-_n \geq \int g d\xi$$

contradicting (4). Thus it only remains to show that the map defined by (5) is upper semi-continuous. Since for each $h \in P$ the map

$$\varphi_h: x \rightarrow \int h dv_x, \quad x \in X$$

is upper semi-continuous there is a family $\{f_i\}_{i \in I}$ of continuous functions, downwards directed in the pointwise ordering, such that $\varphi_h(x) = \inf_{i \in I} f_i(x)$. Now by definition

$$\int h d\eta = \int \varphi_h d\eta$$

By Lusin's theorem we can find compact sets to which the restriction of $\varphi_h$ is continuous, carrying as large a portion of the mass of $\eta$ as we wish. It then follows that

$$\int h d\eta = \inf_i \int f_i d\eta$$

which shows that the map defined by (5) is upper semi-continuous and this concludes the proof of Theorem 2.
3.

We conclude with a couple of examples and some remarks. First an example of a non-simplicial subspace whose Choquet boundary is not closed and where conditions (*) and (**) are satisfied.

**Example 3.** Let \( \{P_1, P_2, P_3, P_4\} \) be the 4 corners of a square in \( \mathbb{R}^2 \), let \( P_5 \) denote the midpoint of the segment \( P_1P_2 \) and let \( \{Q_n\} \) be a sequence of points converging to \( P_5 \) from outside the square. Let

\[
X = \{P_i : i = 1, \ldots, 5\} \cup \{Q_n : n \in \mathbb{N}\}
\]

and let \( B \) be the space of all continuous functions on \( X \) which are affine on the set \( \{P_i : i = 1, \ldots, 5\} \). Then the Choquet boundary for \( B \) is \( X \setminus \{P_5\} \), the space \( B \) is not simplicial and it is not hard to see that (*) and (**) are satisfied.

In [2, Counterexample 3.11] an example is given of a subspace for which condition (*) is not satisfied. In that example the Choquet boundary is closed so that condition (**) is satisfied. The second example given here is an example of a subspace \( B \) for which points in \( \partial B X \setminus \partial B X \) have more than one boundary representing measure so that condition (**) is not satisfied, but where there exists a simplicial space containing \( B \) and having the same Choquet boundary as \( B \).

**Example 4.** Let \( X \subseteq \mathbb{R}^2 \) be the set

\[
X = \left\{ x_n = \left( \frac{1}{n}, 0 \right) : n \in \mathbb{N}\right\} \cup \{y_1 = (0, -1), y_2 = (0, 0), y_3 = (0, 1)\}
\]

and let

\[
B = \left\{ f \in C(X) : \frac{1}{2}(f(y_1) + f(y_3)) = f(y_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} f(x_n) \right\}.
\]

Then \( \partial B X = X \setminus \{y_2\} \) and \( y_2 \) has two boundary representing measures so that condition (**) is not satisfied. As a simplicial space \( A \) containing \( B \) and having the same Choquet boundary as \( B \) we can take

\[
A = \left\{ f \in C(X) : f(y_2) = \frac{1}{2}(f(y_1) + f(y_3)) \right\}.
\]

In the general case when points of \( \partial B X \setminus \partial B X \) may have more than one boundary representing measure we do not have a necessary and sufficient condition for the existence of a simplicial space \( A \) containing \( B \) and having the same Choquet boundary as \( B \). Further, we do not know whether condition (**) is redundant in Theorem 2 i.e. contained in condition (*), or more
generally even whether condition (*) is sufficient in the general case when points of \( \partial_B^+X \setminus \partial_BX \) may have more than one boundary representing measure.

REFERENCES

