ON THE BURNSIDE RING AND STABLE COHOMOTOPY OF A FINITE GROUP

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0. Introduction.

In this paper we study the connection between permutation representations of finite groups $G$ and stable cohomotopy of the classifying space $BG$, analogous to the connection between the character ring of $G$ and the $K$-theory of $BG$.

Suppose $S$ is a finite $G$-set. Each ordering of $S$ gives a homomorphism $\varphi: G \to \Sigma_{|S|}$, where $\Sigma_n$ denotes the permutation group in $n$ letters and $|S|$ is the cardinality of $S$. Different orderings give conjugate maps, as do isomorphic $G$-sets. Hence the homotopy class of $B\varphi: BG \to B\Sigma_{|S|}$ only depends on the isomorphism class of $S$.

The disjoint union $\bigsqcup_{n \geq 0} B\Sigma_n$ is a monoid and its group completion $\Omega B$ ($\bigsqcup_{n \geq 0} B\Sigma_n$) is homotopy equivalent (as an $H$-space) to the space $QS^0 = \lim_{n \to \infty} Q^nS^n$ of stable self maps of spheres. Let $i: \bigsqcup_{n \geq 0} B\Sigma_n \to QS^0$ be the resulting $H$-map and form the composition

$$\alpha_G(S): BG \to B\Sigma_{|S|} \to \bigsqcup_{n \geq 0} B\Sigma_n \xrightarrow{i} QS^0.$$ Disjoint union and Cartesian product turn the equivalence classes of $G$-sets into a semiring, whose associated ring is the Burnside ring $A(G)$ ([17], [7]), and the correspondence $S \to \alpha_G(S)$ defines an additive map

$$\alpha_G: A(G) \to [BG, QS^0].$$

$[BG, QS^0]$ is by definition the stable cohomotopy $\pi_S^0(BG)$.

The space $QS^0$ admits besides the loop addition the smash product, homotopic to the product given by composition of maps. If $\bigsqcup_{n \geq 0} B\Sigma_n$ is equipped with the monoid structure induced from the homomorphisms $\Sigma_n \times \Sigma_m \to \Sigma_{nm}$, then $i$ respects both structures and $\alpha_G$ is a ring homomorphism.

The space $QS^0$ splits into a disjoint union of homotopy equivalent spaces $Q_nS^0$, $n \in \mathbb{Z}$, where $Q_nS^0$ denotes the subspace of degree $n$ maps. Thus we have

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an augmentation \([BG, QS^0] \rightarrow \mathbb{Z}\). Taking cardinality of \(G\)-sets defines an augmentation of \(A(G)\), and \(\alpha_G\) is clearly augmentation preserving. The \((n+1)\)-fold products become trivial on the \(n\)-skeleton \(B_n G\), so \(\alpha_G\) factors as

\[
\alpha_G: A(G)/I^{n+1}(G) \rightarrow [B_n G, QS^0].
\]

Passing to the limit we get a map from the \(I(G)\)-adic completion

\[
\hat{\alpha}_G: \hat{A}(G) \rightarrow \lim_{\leftarrow} [B_n G, QS^0] = [BG, QS^0]
\]

(the last isomorphism follows from the finiteness of \([B_n G, Q_0 S^0]\)).

The map \(\hat{\alpha}_G\) is analogous to the isomorphism from the completed representation ring \(\hat{R}(G)\) to \(K(BG)\) [1], and some time ago \(G\). Segal made the

**Conjecture.** \(\hat{\alpha}_G: \hat{A}(G) \rightarrow [BG, QS^0]\) is an isomorphism.

The full conjecture seems very hard and is probably out of reach at the moment. In this paper we study the injectivity of \(\hat{\alpha}_G\).

First we reduce the problem to \(p\)-groups by showing

**Theorem A.** If \(\hat{\alpha}_{G_p}\) is injective for the Sylow subgroups \(G_p\) of \(G\), then \(\hat{\alpha}_G\) is injective.

This is proved by showing that \(\hat{A}(G)\) embeds into \(\bigoplus_p \hat{A}(G_p)\) via the restriction maps, and compatibly with \(BG_p \rightarrow BG\).

For cyclic groups the natural map \(\hat{A}(G) \rightarrow \hat{R}(G)\) is injective, and using Atiyah’s result, \(\hat{R}(G) \cong K^*(BG)\), we deduce

**Theorem B.** \(\hat{\alpha}_G\) is injective for cyclic groups \(G\).

One cannot hope to detect the maps \(\alpha_G(x): BG \rightarrow Q_0 S^0\) for \(x \in \text{Ker} (\hat{A}(G) \rightarrow \hat{R}(G))\) by \(K\)-theory, since the space \(Q_0 S^0\) splits as \(J \times \text{cok} J\) with \(\text{cok} J\) a \(K\)-theory point, and at least for groups \(G\) of odd order \([BG, J]\) embeds into \([BG, BU \times \mathbb{Z}] = \hat{R}(G)\). By studying induced maps in homology we prove

**Theorem C.** \(\hat{\alpha}_G\) is injective for elementary abelian groups \(G = (\mathbb{Z}/p)^n\).

For Theorem C we need an induction machine, which tells that if \(\alpha_H\) is injective for all genuine subgroups \(H\) of a \(p\)-group \(G\) and furthermore \(\alpha_G\) is injective on a specific summand \(\mathbb{Z}x \subset A(G)\), then \(\hat{\alpha}_G\) is injective. It is the maps \(\alpha_G(nx)\) that induce nontrivial maps in \(\mathbb{Z}/p\)-homology. We show even more: one gets a host of homologically distinct elements \(B((\mathbb{Z}/p)^n) \rightarrow \text{cok} J_p\) for \(n \geq 2\).
Theorem D. If $G = (\mathbb{Z}/p)^n$, and $\hat{\Delta}_0(G) = \text{Ker} (\hat{\Delta}(G) \to \hat{R}(G))$, then $\hat{\alpha}_G$ maps $\hat{\Delta}_0(G)$ ($2\hat{\Delta}_0(G)$ if $p = 2$) injectively into $[BG, \text{cok} J_p]$.

We note that Theorems A and B combine to show that $\hat{\alpha}_G$ is injective for the groups with cyclic Sylow subgroups. They are all metacyclic. Theorem C enlarges the class of groups for which $\hat{\alpha}_G$ is injective to include e.g. $A_5$, the alternating group on 5 letters.

The smallest groups we cannot settle with our method are $\mathbb{Z}/4 \times \mathbb{Z}/2$, the dihedral D8 and the quaternionic Q8 of order 8. For these groups the maps $\alpha_G(nx)$ induce zero both in homology and $K$-theory. One wonders if connective $K$-theory or unitary bordism theory could settle these cases.

The representation ring $R(G)$ and $K$-theory $K(X)$ admit the structure of a $\lambda$-ring. Atiyah, Tall and Segal [3], [4] have explored the algebraic nature of such rings showing that one gets exponential isomorphisms $\varphi_k: \tilde{I}(G) \overset{\lambda}{\longrightarrow} 1 + \tilde{I}(G)$ for any $p$-group $G$, and $KSO(X) \overset{\lambda}{\longrightarrow} (1 + KSO(X))$ for any finite complex $X$, where $\hat{\lambda}$ denotes the $p$-adic completion.

Now $A(G)$ has also $\lambda$-operations, yielding $\lambda$-operations on stable cohomotopy $\pi^S_0(X) = [X, QS^0]$, see [19]. Unfortunately the $\lambda$-ring $A(G)$ is not "special", and this breaks down the algebraic program above. However, it is interesting to identify the maps $\varphi_k: Q_0S^0 \to SG_p$. We give a character argument to show

Theorem E. $\varphi_k: Q_0S^0 \to SG_p$ is the composition $Q_0S^0 \overset{\iota}{\longrightarrow} J_p \overset{\lambda}{\longrightarrow} SG_p$.

The paper is divided into 4 sections. The first contains generalities on the Burnside ring: its characters, functorial properties and topology. The main theorem is 1.15 which shows that $\hat{\Delta}(G)$ is detected by $p$-groups.

In section 2 we study the $\lambda$-ring structure of $A(G)$ and note that the natural map $A(G) \to R(G)$ is a $\lambda$-homomorphism. We describe the characters of the associated operations $\lambda^n$, $\psi^n$ and $\varphi_k$, we show they induce operations in the (zero degree) stable cohomotopy $\pi^S_0$ (2.12) and prove Theorem E (2.20). The characters of $\lambda^n$ and $\psi^n$ were obtained independently by C. Siebeneicher [20].

The third section is devoted to the study of $\hat{\Delta}_G$ and contains the proofs of theorems A and B (3.2 and 3.3). We set up the induction machinery needed to prove theorems C and D in section 4 (4.15, 4.22 and 4.23).

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1. The Burnside ring.

Let $G$ be a finite group. A $G$-action on a finite set $S$ is a homomorphism from
\( G \) to \( \Sigma_S \), the permutation group of \( S \). A \( G \)-set is a finite set with a \( G \)-action. Two \( G \)-sets \( S \) and \( T \) are isomorphic if there exists a \( G \)-equivariant bijection \( f: S \rightarrow T \), or in other words if the diagram

\[
\begin{array}{ccc}
G & \xrightarrow{\Sigma_S} & \Sigma_T \\
\downarrow & & \downarrow \\
& \Sigma_f & \\
\end{array}
\]

commutes. The equivalence classes of \( G \)-sets form a commutative semiring \( A^+(G) \) under disjoint union and cartesian product. The associated ring \( A(G) \) is called the Burnside ring of \( G \).

The additive structure of \( A(G) \) is easily described. Breaking \( G \)-sets into \( G \)-orbits one sees that \( A(G) \) is a free abelian group with basis consisting of cosets \( G/H \), one for each conjugacy class of subgroups \( H \) of \( G \). We fix a set \( C(G) \) of representatives.

For the multiplicative structure we introduce characters, following W. Burnside (who called them marks [5 p. 236]). Let \( H \) be a subgroup of \( G \) and \( S \) be a \( G \)-set. Then we set

\[
\chi_H(S) = |S^H|,
\]

the number of elements in \( S \) fixed by \( H \). The character \( \chi_H \) extends to give a ring homomorphism

\[
\chi_H: A(G) \rightarrow \mathbb{Z}.
\]

On the additive generators of \( A(G) \) we have

\[
\chi_{H_1}(G/H_2) = |\{gH_2 \mid H_1gH_2 = gH_2\}| = |\{gH_2 \mid g^{-1}H_1g \subseteq H_2\}|
\]

or

\[
(1.2) \quad \chi_{H_1}(G/H_2) = |\{gH_2 \mid H_1^g \subseteq H_2\}|.
\]

This shows that \( \chi_H \) depends only on the conjugacy class of \( H \).

The homomorphisms \( \chi_H \) define together a homomorphism

\[
\chi: A(G) \rightarrow \bigoplus_{C(G)} \mathbb{Z}.
\]

The first statement of the following theorem is due to W. Burnside [5] and the second one to A. Dress (unpublished, cf. [7]).

**Theorem 1.3.** The homomorphism \( \chi: A(G) \rightarrow \bigoplus_{C(G)} \mathbb{Z} \) is an embedding with
finite cokernel. Its image consists of families \((d_H)_{H \in C(G)}\) satisfying the congruences

\[
d_H \equiv - \sum_{H \leq K \leq G, K/H \text{ cyclic}} \varphi([K/H])d_K \pmod{|N_G(H)/H|}
\]

where we set \(d_{K'} = d_K\) for \(K \sim K' \in C(G)\) and \(\varphi\) is the Euler function.

**Proof.** To prove the injectivity, it is enough to show that non-isomorphic \(G\)-sets \(S\) and \(T\) cannot have the same characters. Write \(S = \sum m_H G/H\), \(T = \sum n_H G/H\) with \(H\) running over \(C(G)\) and let \(H_0 \in C(G)\) be maximal with respect to \(m_{H_0} \neq n_{H_0}\). Note from (1.2) that \(\chi_{H_1}(G/H_2)\) is non-zero if and only if \(H_1\) is conjugate to a subgroup of \(H_2\), denoted by \(H_1 \leq H_2\). Then

\[
\chi_{H_0}(S) = m_{H_0} \chi_{H_0}(G/H_0) + \sum_{H_0 \leq H, H + H_0} m_H \chi_{H_0}(G/H)
\]

and

\[
\chi_{H_0}(T) = n_{H_0} \chi_{H_0}(G/H_0) + \sum_{H_0 \leq H, H + H_0} n_H \chi_{H_0}(G/H)
\]

are different as the sum terms coincide but \(m_{H_0} = n_{H_0}\).

Let \(S\) be a \(G\)-set. We want to show that the numbers \(\chi_H(S)\) satisfy the congruences for each subgroup \(H\). If \(H < K\) then the \(K\)-fixed points of \(S\) are contained in the \(N_G(H)/H\)-set \(S^H\) so that we are reduced to the case \(H = e\). By a theorem of W. Burnside [5 p. 191] \(|G|^{-1} \sum_{g \in G} |S^g|\) is an integer, namely the number of \(G\)-orbits in \(S\). This implies the congruence

\[
|S| = \chi_e(S) \equiv - \sum_{g \neq 1} \chi_{g}(S) = - \sum_{K \leq G, K \text{ cyclic}} \varphi([K]) \chi_K(S) \pmod{|G|}
\]

where \(\varphi([K])\) is the number of generators in \(K\).

Conversely, we construct \(x \in A(G)\) with given characters \((d_H)\) by adding increasing orbits. Start with \(d_G\) points with trivial \(G\)-action. Choose a total order on \(C(G)\) extending \(\leq\). Assume we have \(y \in A(G)\) such that \(\chi_H(y) = d_H\) for \(H\) greater than \(H_0\). Since the characters of \(y\) and the numbers \((d_H)\) both fulfils the congruences, we have

\[
\chi_{H_0}(y) \equiv d_{H_0} \pmod{|N_G(H_0)/H_0|},
\]

say \(d_{H_0} = \chi_{H_0}(y) + n|N_G(H_0)/H_0|\). We add \(n\) copies of \(G/H_0\) to \(y\). This does not change the earlier adjusted characters, but

\[
\chi_{H_0}(y + nG/H_0) = \chi_{H_0}(y) + n|N_G(H_0)/H_0| = d_{H_0}
\]

by (1.2). This completes the induction.
Finally $\chi$ has finite cokernel since $\oplus |G|Z \subseteq \text{Im} \chi$ by the congruences. The theorem is proved.

**Remark.** In [17] G. Segal defined a ring $\omega^0_G$ in terms of equivariant stable homotopy. It coincides with $A(G)$ as both are characterized by theorem 1.3. For a proof and generalization to compact Lie groups, see [16, Theorem 3].

If $f: H \to G$ is a homomorphism of finite groups, then the pull-back $f^*S$ of a $G$-set $S$ has the same underlying set with $H$-action

$$H \xrightarrow{f} G \to S.$$

The induced maps $f^*: A(G) \to A(H)$ make $A$ into a contravariant functor. The characters of $f^*x$ are

\begin{equation}
\chi_U(f^*x) = \chi_{f(U)}(x), \quad U \subseteq H, \ x \in A(G).
\end{equation}

In the special case of a subgroup $i: H \to G$ we call $i^*$ the restriction homomorphism, and denote it by $\text{Res}^G_H$.

There is also a covariant induction homomorphism $\text{Ind}^G_H$ or $f_*$ for inclusions of subgroups $f: H \to G$. On the coset basis it is given by

$$\text{Ind}^G_H(H/H_1) = G/H_1, \quad H_1 \subseteq H \subseteq G$$

It is easily checked from (1.2) that the characters of $f_*y$ are

\begin{equation}
\chi_U(f_*y) = \sum_{U^* \subseteq H} \chi_{U^*}(y), \quad U \subseteq G, \ y \in A(H)
\end{equation}

where $g$ runs through representatives of $G/H$.

The homomorphisms $\text{Res}$ and $\text{Ind}$ are related in the same fashion as the restriction and induction maps in representation theory or cohomology of finite groups. If $H$ is a subgroup of $G$, then the Frobenius reciprocity

\begin{equation}
\text{Ind}^G_H(y \cdot \text{Res}^G_H(x)) = \text{Ind}^G_H(y) \cdot x, \quad y \in A(H), \ x \in A(G)
\end{equation}

holds. Further, if $K$ is another subgroup of $G$, let $Kg_1H, \ldots, Kg_rH \subseteq G$ be the double cosets of $K$ mod $(K, H)$. Then we have

\begin{equation}
\text{Res}^G_K \text{Ind}^G_H(x) = \sum_{i=1}^r \text{Ind}^K_{K \cap g_iHg_i^{-1}}(c_{g_i} \cdot \text{Res}^H_{g_i^{-1}Hg_i}(x)), \quad x \in A(H)
\end{equation}

where $c_{g_i}$ is conjugation by $g_i$. The proofs of (1.6) and (1.7) are analogous to the corresponding formulas of representation theory.

Each $G$-set $S$ can be considered as a linear representation of $G$ over a field $k$ by extending the $G$-action on the canonical basis of $k^S$ by linearity. As the trace of a permutation matrix is equal to the number of 1's along the diagonal, the
linear character of \(k^5\) can be read off from the characters \(\chi_H\) of (1.1) for \(H\) cyclic:

\[
\chi_{k^5}(g) = \chi_{\langle g \rangle}(S).
\]

If \(k\) has characteristic 0, then the elements in \(R_k(G)\) are detected by their linear characters. We conclude from theorem 1.3

**Lemma 1.8.** Let \(k\) be a field of characteristic 0. Then the kernel of the natural map \(A(G) \to R_k(G)\) coincides with the kernel of the restriction map

\[
\text{Res}: A(G) \to \bigoplus_{C \subseteq G} A(C)
\]

to the cyclic subgroups of \(G\).

In Segal's conjecture the Burnside ring \(A(G)\) is compared with the stable cohomotopy ring \(\pi_0^S(BG)\). As the latter is complete (see section 3), we study here the algebraic process of completing \(A(G)\).

The character \(\chi_G\) just counts the number of points in a \(G\)-set and defines an augmentation \(\varepsilon: A(G) \to \mathbb{Z}\). This is a split surjection so \(A(G) = \mathbb{Z} \oplus I(G)\) where \(I(G)\) is the augmentation ideal \(\varepsilon^{-1}(0)\).

We give the ring \(A(G)\) the usual \(I(G)\)-adic topology, letting the powers \(I(G)^n\) be a neighbourhood basis of 0. The completion of \(A(G)\) is defined as the inverse limit

\[
\hat{A}(G) = \lim_{\leftarrow n} A(G)/I(G)^n.
\]

We shall study the kernel of \(A(G) \to \hat{A}(G)\). First we recall a result from commutative algebra (see e.g. [22 p. 262, Corollary to Theorem 8]). Let \(A\) be a Noetherian ring with no nilpotents and \(m \subseteq A\) a prime ideal. Then the kernel of the natural map from \(A\) to the \(m\)-adic completion \(\hat{A} = \lim A/m^n\) is

\[
\bigcap_{n=0}^{\infty} m^n = \bigcap_{p_j + m \neq A} p_j
\]

where \(p_j\) runs over such minimal prime ideals of \(A\) that \(p_j + m \neq A\).

The ring \(A(G)\) is Noetherian as a finitely generated abelian group. A. Dress determined in [7] the prime ideal structure of \(A(G)\). There are two types of prime ideals in \(A(G)\): the minimal ones

\[
p_{U,0} = \{x \in A(G) \mid \chi_U(x) = 0\}
\]

for \(U \subseteq G\), and the maximal ones

\[
p_{U,p} = \{x \in A(G) \mid \chi_U(x) \equiv 0 \pmod{p}\}
\]

for \(U \subseteq G\) and \(p\) a prime. Furthermore,
\( p_{U,0} = p_{V,0} \) if and only if \( U \sim V \),
\( p_{U,p} = p_{V,q} \) if and only if \( p = q \) and \( U^p \sim V^p \)

where \( U^p \) is the smallest normal subgroup of \( U \) with \( U/U^p \) a \( p \)-group, and \( p_{U,0} \subseteq p_{U,p} \) together with (1.9) accounts for all inclusions between prime ideals in \( A(G) \).

**Proposition 1.10.** The kernel of \( A(G) \to \hat{A}(G) \) coincides with the kernel of the restriction map

\[
\text{Res}: A(G) \to \bigoplus_{G_p \supseteq G} A(G_p)
\]

to the Sylow subgroups \( G_p \) of \( G \).

**Proof.** It follows from the above that the kernel is

\[
\bigcap_{n=0}^{\infty} I(G)^n = \bigcap_{p_{U,0}+I(G)\subseteq A(G)} p_{U,0}.
\]

Now \( I(G) = p_{e,0} \) and if the ideal \( p_{U,0} + p_{e,0} \) is proper then it is contained in a maximal ideal \( p_{V,p} \). By (1.9) this implies that \( p_{U,p} = p_{e,p} = p_{V,p} \) hence \( U^p = e = V^p \) and \( U \) is a \( p \)-group. Conversely, if \( U \) is a \( p \)-group then \( p_{U,0} + p_{e,0} \subseteq p_{U,p} \). Thus

\[
\text{Ker}(A(G) \to \hat{A}(G)) = \bigcap_{U \subseteq G \text{ p-group}} \text{Ker} \chi_U,
\]

and the claim follows from theorem 1.3.

**Corollary 1.11.** If \( G \) is a \( p \)-group then \( A(G) \to \hat{A}(G) \) is a monomorphism.

In the case of a \( p \)-group \( G \) the \( I(G) \)-adic completion is the familiar \( p \)-adic one:

\[
\hat{A}(G) = \mathbb{Z} \oplus (\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} I(G))
\]

where \( \hat{\mathbb{Z}}_p = \lim_{\leftarrow n} \mathbb{Z}/p^n \) denotes the \( p \)-adic integers:

**Proposition 1.12.** If \( G \) is a \( p \)-group then the \( I(G) \)-adic topology of \( A(G) \) is the same as its \( p \)-adic topology.

**Proof.** We have to prove that for each \( m \) there are integers \( n_1, n_2 \) such that

\[
\begin{align*}
(1) & \quad p^n I(G) \subseteq I(G)^m \\
(2) & \quad I(G)^{n_2} \subseteq p^m I(G).
\end{align*}
\]
The first relation follows from Atiyah’s

**Lemma 1.13.** For any group \( G \), \(|G|I(G)^n \subseteq I(G)^{n+1}\).

(This is a consequence of the reciprocity formula 1.6, see [1p. 269, Proposition 6.13]).

To get the inclusion (2), we note that for any \( H \leq G \) and \( U \leq G \)

\[ \chi_U(G/H \setminus |G/H|) \equiv 0 \pmod{p} \]

since the complement of \((G/H)^U\) consists of non-trivial \( U \)-orbits, hence \( \chi_U(I(G)) \subseteq pZ \). As \( \chi = \bigoplus_{U \leq G} \chi_U \) is a ring homomorphism we have \( \chi(I(G)^n) \subseteq \bigoplus_{e+U \leq G} p^nZ \) and it is enough to prove

\[ \bigoplus_{e+U \leq G} |G|Z \subseteq \chi(I(G)) \].

This follows immediately from the congruences 1.3. The proof of 1.12 is complete.

The completion \( \hat{A}(G_p) \) is now described for \( p \)-groups \( G_p \). Next we shall show that if \( G \) is an arbitrary finite group, then \( \hat{A}(G) \) embeds into the sum \( \bigoplus \hat{A}(G_p) \), taken over the Sylow subgroups \( G_p \) of \( G \). This is done by completing the map of proposition 1.10.

Let \( A \) be a Noetherian ring and \( m \subset A \) an ideal. The \( m \)-adic completion of a finitely generated \( A \)-module \( M \) is defined to be \( \hat{M} = \lim_{\leftarrow n} M/m^nM \). It is a basic fact that Noetherian completion is an exact functor [1 p. 258, Proposition 3.16].

If \( H \leq G \), then \( A(H) \) is an \( A(G) \)-module via the restriction homomorphism \( q = \text{Res}^H_G : A(G) \to A(H) \). In the following proof we distinguish the prime ideals \( p_{U,p} \) of \( A(H) \) and \( A(G) \) by upper indices, so that \( p_{U,p}^H \subseteq A(H) \) and \( p_{U,p}^G \subseteq A(G) \).

**Proposition 1.14.** Let \( H \) be a subgroup of \( G \). Then the \( I(H) \)-adic topology of \( A(H) \) is the same as its \( I(G) \)-adic topology.

**Proof.** It is enough to show that the radicals of the ideals \( q(I(G)) \) and \( I(H) \) coincide [22 p. 256]. This means that each prime ideal \( p \subseteq A(H) \) either contains the both ideals or none. Since \( q(I(G)) \subseteq I(H) \), one way is trivial. Let \( p \) be a prime ideal of \( A(H) \) with \( q(I(G)) \subseteq p \). We claim that \( I(H) \subseteq p \). We know that \( p \) is of the form \( p_{U,0}^H \) or \( p_{U,p}^H \) with some subgroup \( U \leq H \) and some prime \( p \). If \( p = p_{U,0}^H \), then

\[ p_{e,0}^G = I(G) \subseteq q^{-1}(p) = p_{U,0}^G \]
which implies \( U = e \) and \( p = p^H_{e,0} = I(H) \) by (1.9). Similarly, if \( p = p^H_{U,p} \), then

\[
p^G_{e,0} = I(G) \subset g^{-1}(p) = p^G_{U,p}
\]

and \( U \) must be a \( p \)-group by (1.9), whence \( p = p^H_{U,p} = p^H_{e,p} \). In both cases \( I(H) \subset p \), are claimed.

**Theorem 1.15.** Let \( G \) be a finite group and \( \{G_p\} \) its Sylow subgroups. Then the completion of the restriction maps \( \text{Res}_{G_p}^G \) gives an injective homomorphism

\[
0 \to \hat{A}(G) \to \bigoplus_p \hat{A}(G_p).
\]

**Proof.** By 1.10 \( A(G) / \bigcap_{n=0}^\infty I^n(G) \) maps injectively into \( \bigoplus_p A(G_p) \). Both modules have \( I(G) \)-adic topology by 1.14. The claim follows since Noetherian completion is an exact functor.

We close the chapter with some examples. The first two are abelian \( p \)-groups. The last one illustrates the restrictions to Sylow subgroups and completion.

**Example 1.16.** The cyclic group \( \mathbb{Z}/p^n \). It has a unique subgroup of order \( p^{n-m} \), \( 0 \leq m \leq n \). Let \( \eta_m \) be the quotient \( (\eta_0 = 1) \). We have additively

\[
A(\mathbb{Z}/p^n) = \mathbb{Z} \oplus \mathbb{Z} \eta_1 \oplus \cdots \oplus \mathbb{Z} \eta_n.
\]

From the characters

\[
\chi_{\mathbb{Z}/p^n}(\eta_m) = \begin{cases} p^m, & i \leq n-m \\ 0, & i > n-m \end{cases}
\]

one gets the multiplication \( \eta_1 \cdot \eta_m = p^l \eta_m \) for \( l \leq m \).

**Example 1.17.** The elementary abelian group \( (\mathbb{Z}/p)^n \). It can be interpreted geometrically as a vector space over the finite field \( F_p \), with subgroups corresponding to linear subspaces. The number of \( m \)-dimensional planes is

\[
G(m,n) = \frac{(p^n-1)(p^{n-1}-p) \ldots (p^n-p^{m-1})}{(p^m-1)(p^{m-1}-p) \ldots (p^m-p^{m-1})}
\]

\[
= \frac{(p^n-1)(p^{n-1}-1) \ldots (p^n-m+1-1)}{(p^m-1)(p^{m-1}-1) \ldots (p-1)}
\]

\((G \text{ stands for Grassmann).} \ A((\mathbb{Z}/p^n)^m) \) is additively generated by the \( m\)-
dimensional quotient planes $\eta_{m}^{i}, 0 \leq m \leq n, 1 \leq i \leq G(n-m, n)=G(m, n),$ and

$$F_{p}^{n}/V_{1} \times F_{p}^{n}/V_{2} = |F_{p}^{n}/V_{1} + V_{2}|F_{p}^{n}/V_{1} \cap V_{2}.$$  

**Example 1.18.** The alternating group $A_{4}$. The diagram of subgroups is

![Diagram](image)

and the character table is given in table 1.19 where the small letters $1, a, b, c,$ and $d$ denote the cosets $A_{4}/H$ in the given order.

<table>
<thead>
<tr>
<th>$A_{4}/H$</th>
<th>$\chi_{H}$</th>
<th>$A_{4}$</th>
<th>$Z/2 \oplus Z/2$</th>
<th>$Z/3$</th>
<th>$Z/2$</th>
<th>$e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$a$</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>3</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>$b$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>$c$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>$d$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>12</td>
<td></td>
</tr>
</tbody>
</table>

Table 1.19.

The Burnside rings of the Sylow subgroups of $A_{4}$ are described in the preceding examples:

$$A(Z/2 \oplus Z/2) = Z \oplus Z\eta_{1}^{i} \oplus Z\eta_{1}^{2} \oplus Z\eta_{2}^{3} \oplus Z\eta_{2}$$

$$A(Z/3) = Z \oplus Z\xi.$$  

The restriction map $A(G) \rightarrow A(G_{2}) \oplus A(G_{3})$ is read from the character table using (1.4) $\chi_{H}(i^{*}x) = \chi_{H}(x)$. The result is

1 $\rightarrow$ (1, 1)

$a$ $\rightarrow$ (3, $\xi$) $\quad$ $a_{1}$ $\rightarrow$ (0, $\xi$)

$b$ $\rightarrow$ (2, $1 + \xi$) $\quad$ $b_{1}$ $\rightarrow$ ($\eta_{2}$, 0)

$c$ $\rightarrow$ ($\eta_{1}^{i} + \eta_{2}^{i} + \eta_{3}^{i}, 2\xi$) $\quad$ $c_{1}$ $\rightarrow$ ($\eta_{1}^{i} + \eta_{2}^{i} + \eta_{3}^{i}, 0$)

$d$ $\rightarrow$ (3$\eta_{2}, 4\xi$) $\quad$ $d_{1}$ $\rightarrow$ (0, 0)
in the basis \( a_1 = a - 3, b_1 = b - a - 1, c_1 = c - 2a, d_1 = d - 3b - a + 3 \) for \( I(A_4) \). Here \( x \) denotes \( x - \varepsilon(x) \in I(G_p) \). This shows that the image of \( A(G) \) in \( A(G_2) \oplus A(G_3) \) consists precisely of the stable elements. These are the pairs \((x_2, x_3)\) with
\[
(1.20) \quad \begin{align*}
(1) & \quad \varepsilon(x_2) = \varepsilon(x_2) \\
(2) & \quad \chi_{H_1}(x_p) = \chi_{H_2}(x_p), \text{ if } H_1 < G_p \text{ and } H_2 < G_p \text{ are conjugate in } G.
\end{align*}
\]
The condition (2) rules out \( \eta_1^1, \eta_1^2 \) and \( \eta_1^3 \) since they have different characters on the three \( \mathbb{Z}/2 < \mathbb{Z}/2 \oplus \mathbb{Z}/2 \) which are conjugate in \( A_4 \).

It might be interesting to know whether (1.20) characterizes the image of \( A(G) \) in \( \oplus_p A(G_p) \) in general. If the Segal conjecture \( \tilde{A}(G) = \eta_3^0(BG) \) is true, then this holds at least on the completion level by general properties of cohomology theories on \( BG \) (see [11, 1.7]).

Finally, the multiplication table for \( I(A_4) \)

<table>
<thead>
<tr>
<th></th>
<th>( a_1 )</th>
<th>( b_1 )</th>
<th>( c_1 )</th>
<th>( d_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 )</td>
<td>-3a_1</td>
<td>( d_1 )</td>
<td>0</td>
<td>-3d_1</td>
</tr>
<tr>
<td>( b_1 )</td>
<td>-4b_1 - d_1</td>
<td>-4c_1</td>
<td>-d_1</td>
<td></td>
</tr>
<tr>
<td>( c_1 )</td>
<td></td>
<td>6b_1 - 10c_1 + 2d_1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( d_1 )</td>
<td></td>
<td></td>
<td>3d_1</td>
<td></td>
</tr>
</tbody>
</table>

shows that \( \tilde{A}(G) = \mathbb{Z} \oplus \hat{\mathbb{Z}}_3 a_1 \oplus \hat{\mathbb{Z}}_2 b_1 \oplus \hat{\mathbb{Z}}_2 c_1 \) and \( I(G)^\infty = \mathbb{Z}d_1 \).

2. \( \lambda \)-Operations on the Burnside ring.

Let \( G \) be a finite group. If \( k \) is a field and \( V \) is a representation of \( G \) over \( k \) then the exterior powers \( \lambda^n V \) are also \( G \)-representations. We want to construct operations in \( A(G) \) which under the natural map \( A(G) \to R_k(G) \) correspond to the exterior powers. As it is not clear how to make sense of the relation \( x \wedge y = -y \wedge x \) in a \( G \)-set, we consider first the symmetric powers \( s^n V \) where no signs are needed.

Let \( S = \{s_1, \ldots, s_l\} \) be a \( G \)-set. The vector space \( s^n(k^S) \) has a basis consisting of monomials of degree \( n \) in \( s_1 \in s^1(k^S) \), considered as elements of the symmetric algebra \( s(k^S) \). We define the \( n \)th symmetric power of \( S \) as
\[
s^n(S) = S^n/\Sigma_n
\]
with the diagonal \( G \)-action. It is clear that
\[
(2.1) \quad s^1(S) = S
\]
\[
(2.2) \quad s^n(S \cup T) = \sum_{i=0}^{n} s^i(S)s^{n-i}(T).
\]
We assign to $S$ the formal power series

$$(2.3) \quad s_t(S) = 1 + \sum_{n \geq 1} s^n(S)t^n \in A(G)[[t]].$$

It is invertible as the leading coefficient is 1, and $(2.2)$ shows that $s_t(S \cup T) = s_t(S) \cdot s_t(T)$. The homomorphism $s_t$ is uniquely extended to $A(G)$ by $s_t(S - T) = s_t(S) \cdot s_t(T)^{-1}$.

In the representation ring $R_k(G)$ the symmetric powers are connected to the exterior powers by means of the identity

$$\lambda_t(V)s_{-t}(V) = 1.$$

Thus we are lead to

**Definition 2.4.** The $n$th exterior power of $x \in A(G)$, denoted by $\lambda^n(x)$, is the coefficient of $t^n$ in the series $\lambda_t(x) = s_{-t}(x)^{-1}$.

The formulae $(2.1)-(2.3)$ translate to give

$$(2.5)\quad (i) \quad \lambda^0(x) = 1$$

$$(ii) \quad \lambda^1(x) = x$$

$$(iii) \quad \lambda^n(x + y) = \sum_{i=0}^{n} \lambda^i(x)\lambda^{n-i}(y).$$

This is summarised in saying that the operations $\lambda^n$, $n \geq 1$, give $A(G)$ the structure of a $\lambda$-ring [4]. By construction they are natural with respect to induced maps, and $A(G) \to R_k(G)$ is a $\lambda$-homomorphism.

We shall calculate the character of $\lambda^n(x)$, $\chi_U(\lambda^n(x))$. By $(2.5)$ and naturality it is enough to consider a single $G$-orbit $x = G/H$. A point $(s_1, \ldots, s_n) \in (G/H)^n/\Sigma_n$ is fixed under $G$ only if it can be split up to $G$-orbits $G/H$. This implies that $n$ is a multiple of $|G/H|$, and

$$(2.6) \quad \chi_G(s^n(G/H)) = \begin{cases} 0, & n \not\equiv 0 \pmod{|G/H|} \\ 1, & n \equiv 0 \pmod{|G/H|}. \end{cases}$$

Thus $\chi_G(s_t(G/H)) = (1 - t^{|G/H|})^{-1}$. For a subgroup $U$ of $G$ we can break up $S$ into $U$-orbits $S_i$ so

$$(2.7) \quad \chi_U(\lambda_t(S)) = \prod_{S_i \subset S \text{ U-orbits}} (1 - (-t)^{|S_i|}).$$

In particular the degree of $\chi_U(\lambda_t(S))$ is equal to $|S|$, hence $\lambda^n(S) = 0$ if $n > |S|$. Also $\epsilon(\lambda_t(s)) = \chi_e(\lambda_t(s)) = (1 + t)^{|S|}$. Thus $A(G)$ is a finite-dimensional augmented $\lambda$-ring.
We define the Adams operations $\psi^n$: $A(G) \to A(G)$, $n \geq 1$, by

$$\psi_{-i}(x) = -t^{\lambda_i(x)\over \lambda_i(x)} x,$$

where $\psi_i(x) = \sum_{n \geq 1} \psi^n(x)t^n$.

Then (2.5) implies that $\psi^n$ is additive, $\psi^n(x + y) = \psi^n(x) + \psi^n(y)$. As to the characters, the logarithmic differentiation of (2.7) yields

$$\chi_U(\psi_i(X)) = \sum_{S_i \subset S} |S_i| t^{i|S_i|} (1 - t^{i|S_i|})^{-1}.$$

This proves

**Proposition 2.8.** $\chi_U(\psi^n(S)) = \sum_{S_i \subset S} |S_i| n|S_i|$, where $S = \bigcup S_i$ the decomposition into $U$-orbits.

**Corollary 2.9.** The Adams operations are periodic of period dividing the order of $G$.

(Indeed, the length of each $U$-orbit $U/H$ is a divisor of $|G|$).

If $G$ is a $p$-group and $(n, p) = 1$, then the only orbits occurring in 2.8 are the $U$ fixed points. This proves

**Corollary 2.10.** If $G$ is a $p$-group, then $\psi^n = \text{id}$ for $n$ relatively prime to $p$.

The operations $\lambda^n$ have geometrical significance: they induce natural transformations of $\pi_0^G$, the zeroth stable cohomotopy functor.

First recall the Barratt–Quillen theorem. The group completion map $i$: $\coprod_{n \geq 1} B\Sigma_n \to QS^0$ gives a natural transformation of monoid-valued functors

$$\left[X, \coprod_{n \geq 1} B\Sigma_n \right] \to [X, QS^0].$$

Here $A(X) = [X, \coprod_{n \geq 1} B\Sigma_n]$ is the set of isomorphism classes of finite coverings of $X$ organized to a semiring under disjoint union and fibrewise cartesian product of the total spaces, and $\pi_0^G(X) = [X, QS^0]$ is the stable cohomotopy of $X$ (in degree 0), an abelian group with respect to loop sum. The group completion theorem states [18, Proposition 4.1]

**Theorem 2.11.** The transformation $A \to \pi_0^G$ is universal among transformations $\theta: A \to F$, where $F$ is a representable abelian-group-valued homotopy functor on compact spaces, and $\theta$ is a transformation of monoid-valued functors.

We deduce from this the existence of $\lambda$-operations on $\pi_0^G$ along the lines of Segal [19].
Theorem 2.12. There are natural transformation $\lambda^n: \pi_0^0 \to \pi_0^0$ for $n \geq 0$, such that

(i) $\lambda^0(x) = 1$
(ii) $\lambda^1(x) = x$
(iii) $\lambda^n(x + y) = \sum_{i=0}^n \lambda^i(x)\lambda^{n-i}(y)$.

Proof. We define a transformation $\lambda^n: A(X) \to \pi_0^0(X)$. Assume $X$ is connected. An $m$-fold covering $Y \downarrow X$ can be written as $P \times \Sigma_m[m] \downarrow X$, where $P$ is the principal $\Sigma_m$-bundle associated to $Y$ consisting of mappings of $[m] = (1, 2, \ldots, m)$ onto the fibres of $Y$, and $[m]$ has the usual $\Sigma_m$-action. Let $\lambda^n([m]) = S - T \in A(\Sigma_n)$. We associate to $Y \downarrow X$ the difference

$$\lambda^n(Y) = P \times S - P \times T \in \pi_0^0(X)$$

where we have used $A \to \pi_0^0$ from 2.11.

Let us form the mapping

$$\lambda_t = \sum_{n \geq 0} \lambda^n t^n: A(X) \to 1 + \pi_0^0(X)[[t]]^+ = \prod_{n \geq 1} \pi_0^0(X).$$

It is a monoid homomorphism, when we use multiplication of power series on the right. As $1 + \pi_0^0(X)[[t]]^+$ is a representable abelian-group-valued functor, $\lambda_t$ extends by theorem 2.11 to a group homomorphism

$$\lambda_t: \pi_0^0(X) \to 1 + \pi_0^0(X)[[t]]^+. $$

This completes the proof of theorem 2.12.

In the articles [3] and [4] Atiyah, Tall and Segal showed that special $p$-adic $\lambda$-rings possess certain canonical exponential isomorphisms between the additive group $\hat{I}(G)$ and the multiplicative group $1 + \hat{I}(G)$. Unfortunately the Burnside ring $A(G)$ is special only if $G$ is cyclic: The Adams operations $\psi^n$ are ring homomorphisms in special $\lambda$-rings, but Siebeneicher showed that this is not true in $A(G)$ for any non-cyclic $G$ [20, p. 232]. On the other hand, $A(G)$ embeds as a sub-$\lambda$-ring of $R(G)$ if $G$ is cyclic.

However, it is interesting to study the exponential map $e_k$. We first do the algebra and then identify the resulting geometric map $Q_0S^0 \to SG[1/k]$ with the composition

$$Q_0S^0 \to \text{Im} J_p \to SG \left[ \frac{1}{k} \right].$$

Philosophically this is a negative result: the $\lambda$-operations on $A(G)$ do not give any information on the fibre of $e$, the space usually denoted $\text{cok } J_p$. 
Let $G$ be a finite group. We shall encounter series of the form $\lambda_a(x) = 1 + \sum a^n \lambda^n(x)$. To show their convergence in $\hat{A}(G)$ we introduce a new topology on $A(G)$. Define the Grothendieck operations by

\[(2.13)\quad \gamma^n(x) = \lambda^n(x + n - 1).\]

If $\gamma_i(x) = 1 + \sum_{n \geq 1} \gamma^n(x) e^n$, then

\[(2.14)\quad \gamma_i(x) = \lambda_{e(1-i)}(x), \quad \gamma_i(x + y) = \gamma_i(x) \gamma_i(y).\]

The $\gamma$-operations are convenient on the augmentation ideal $I(G)$ as the generators $G/H - e(G/H)$ have finite $\gamma$-dimension but infinite $\lambda$-dimension. In fact, (2.7) and (2.14) imply

\[\chi_U(\gamma_i(S - e(S))) = \prod_{S_i \subset S, \text{U-orbit}} [(1 - t)^{|S_i|} - (-t)^{|S_i|}]\]

for any $G$-set $S$.

Define the $\gamma$-filtration by

\[(2.15)\quad I_n \text{ is the group generated additively by } \gamma^{n_1}(x_1) \cdots \gamma^{n_r}(x_r) \text{ with } x_i \in I(G), \sum n_i \geq n.\]

Then $I_m \cdot I_n \subset I_{m+n}$, $I_0 = A(G)$ and $I_1 = I(G)$. Thus the filtration $(I_n)_{n \geq 0}$ defines a topology on $A(G)$, the $\gamma$-topology.

**Proposition 2.16.** If $G$ is a $p$-group, then the $p$-adic, $I(G)$-adic and $\gamma$-topologies on $A(G)$ are equivalent.

**Proof.** We proved in 1.12 that the first two topologies coincide. Atiyah [1, Corollary 12.3] shows that the $\gamma$-topology is equivalent to the $I(G)$-adic if $I(G)$ has a finite number of generators, each of finite $\gamma$-dimension.

Let $G$ be a $p$-group $x \in I(G)$ and $x \in \hat{Z}_p$. Then the series $\gamma_a(x) = 1 + \sum_{n \geq 1} a^n \gamma^n(x)$ converges in the $\gamma$-topology, hence also in $\hat{A}(G)$. More generally, if $x \in A$ where $A$ is a finitely generated $\hat{Z}_p$-algebra, then $\gamma_a(x)$ exists in $1 + \hat{I}(G) \otimes \hat{Z}_p A$. We fix a prime $k$ different from $p$ and apply this to $A = \hat{Z}_p[\xi]$, where $\xi$ is a primitive $k$th root of 1.

**Definition 2.17.** $\varrho_k(x) = \prod_{\nu \neq 1} \lambda_{-\nu}(x), \quad x \in I(G).$

A priori $\varrho_k(x)$ belongs to $1 + \hat{I}(G) \otimes \hat{Z}_p[\xi]$. But it is invariant under the action of the Galois group of $\hat{Q}_p(\xi)/\hat{Q}_p$, so actually $\varrho_k(x) \in 1 + \hat{I}(G)$.

We compute the character of $\varrho_k(x)$. A substitution in (2.7) yields
\[
\chi_U(\varrho_k(S-\varepsilon(S))) = \prod_{u^+ = 1}^{u^+} \left( \prod_{S \subseteq S} (1 - u^{[S]}) \right) (1 - u)^{-|S|}.
\]

As the sizes of the \(U\)-orbits \(S_i\) are 1 or multiples of \(p\) and \((k, p) = 1\), \(1 - u^{[S_1]}\) runs through the same values as \(1 - u\), when \(S_i\) is fixed. Noting that \(\prod_{u^+ = 1}^{u^+} (1 - u) = k\) we get

\[
(2.18) \quad \chi_U(\varrho_k(S-\varepsilon(S))) = k^{o_U(S)-\varepsilon(S)}
\]

where \(o_U(S)\) is the number of \(U\)-orbits in \(S\).

Next we show that \(\varrho_k\) can be obtained by a direct operation on \(G\)-sets.

**Proposition 2.19.** For a \(G\)-set \(S\) let \(\theta_k(S)\) be the underlying set of the vector space \(F^2_k\) with the linear \(G\)-action extending the permutation of the basis. Then \(\theta_k\) satisfies

(i) \[\theta_k(S + T) = \theta_k(S) \theta_k(T)\]

(ii) \[\varepsilon \theta_k(S) = k^{\varepsilon(S)}\]

(iii) \[\theta_k\] is natural

(iv) \[\theta_k(S) = \prod_{u^+ = 1}^{u^+} \lambda_{-u}(S)\]

on \(A^+(G)\).

**Proof.** Properties (i)–(iii) are obvious. To prove (iv) it is enough to check \(\chi_U(\theta_k(S))\) for \(U = G\) by naturality and for a transitive \(G\)-set \(S\) by (i). A point \(\sum_{x \in S} a_x x\), \(a_x \in F_k\), is fixed under \(G\) only if it is of the form \(a \sum_{x \in S} x\), thus

\[
\chi_G(\theta_k(S)) = |F_k| = k
\]

but \(\chi_G(\prod \lambda_{-u}(S)) = k^{o_S} = k\).

**Remark.** There is no problem about the convergence of \(\lambda_{-u}(S)\) in (iv), since \(\lambda_\xi(S)\) is a polynomial.

We return now to the stable cohomotopy interpretation. Let \(p\) and \(k\) be different primes. The operation \(\theta_k : A^+(\Sigma_n) \to A^+(\Sigma_n)\) induces a natural transformation \(\theta_k : A \to A\) as in theorem 2.12: if \(Y \downarrow X\) is an \(n\)-fold covering, write it as \(Y = P \times \Sigma_n[n]\) with some principal \(\Sigma_n\)-bundle \(P\) and set \(\theta_k(Y) = P \times \Sigma_n \theta_k[n]\). By (2.19) (i) and (ii) the composite

\[
A \overset{\theta_k}{\to} A \to \pi_S^0
\]
is exponential and maps $n$-fold coverings of $X$ to the component $[X, Q_kS^0]$. In order to apply theorem 2.11 the elements $\theta_k(Y) \in \pi^2_3(X)$ have to be invertible in the composition product, in particular the maps in $Q_kS^0$ must have an inverse of degree $k^{-n}$. This can be accomplished by forming the localization $\lambda: QS^0 \to QS_p^0$ of the space $QS^0$ at $p$ [21, sections 2 and 4]. Denote the 1-component of $QS_p^0$ by $SG_p$. Then the transformation

$$q_k: A(X) \to [X, SG_p]$$

which takes an $n$-fold covering $Y \downarrow X$ to

$$X \xrightarrow{\theta_k(Y)} Q_kS^0 \xrightarrow{\lambda} Q_kS_p^0 \xrightarrow{k^{-n}} SG$$

extends to a homomorphism

$$(*) \quad q_k: \pi^2_3(X) \to [X, SG_p]$$

by theorem 2.11.

The restriction of $(*)$ to $\pi^2_3$ corresponds to an $H$-map $q_k: Q_0S^0 \to SG_p$, defined up to homotopy. The space $SG_p$ splits as a product $J_p \times \text{cok} J_p$ (this will be discussed in section 4), and we point out here

**Theorem 2.20.** The map $q_k: Q_0S^0 \to SG_p$ factors $Q_0S^0 \xrightarrow{e} J_p \xrightarrow{\varphi} SG_p$.

**Proof.** Compare proposition 2.19 to [14, p. 236].

The natural homomorphism $A(G) \to R(G)$ is a $\lambda$-ring homomorphism. For the elementary abelian groups its kernel is large. We evaluate the Adams operations on $A((\mathbb{Z}/p)^n)$ in the concluding example.

**Example 2.21.** Elementary abelian groups $(\mathbb{Z}/p)^n$.

Let $G = (\mathbb{Z}/p)^n$. Each generator $G/H$ of $A(G)$ is the image of the regular representation under $\pi^*: A(G/H) \to A(G)$. By naturality it is thus enough to find $\psi^k(\eta^n)$, where we denote $\eta_n = G/e$ (see 1.17). From 2.8 we get

$$\chi_H(\psi^k(\eta_n)) = \begin{cases} 0, & k \not\equiv 0 \pmod{|H|} \\ p^n, & k \equiv 0 \pmod{|H|} \end{cases}, \quad H \leq (\mathbb{Z}/p)^n$$

which depends only on the size of $H$. This suggests that we begin with the sum of all cosets of cardinality $p^m$,

$$\eta_{m^*} = \sum_{i} \eta_i^* = \sum_{|H| = p^{n-m}} G/H$$

which has the characters $\chi_H(\eta_{m^*}) = 0$ if $|H| > p^{n-m}$, $\chi_{\mathbb{Z}/p^n-m}(\eta_{m^*}) = p^m$, and then correct $\chi_H$ for smaller $H$ by adding linear combinations of $\eta_{m+k}^*$, $k > 0$. An inductive calculation shows that the element.
(2.22) \[ a_m = \sum_{k=0}^{n-m} (-1)^k p^{k(k-1)/2} G(k, m+k) \frac{\eta_{m+k}^\text{tot}}{p^{m+k}} \in A(G) \left[\frac{1}{p}\right], \]

where \( G(k, m+k) \) is defined in example 1.17, has characters

\[ \chi_{\{\mathbb{Z}/p\}}(a_m) = \begin{cases} 1, & k = n-m \\ 0, & k \neq n-m. \end{cases} \]

By the first formula we then get

\[ \psi^p\sigma(h)(\eta_n) = \sum_{i=n-m}^n p^i a_i = p^m \sum_{k=0}^m (-1)^k p^{(k)} G(k, n-m+k-1) \eta_{n-m+k}^\text{tot} \]

if \((p, h) = 1, 0 \leq m < n\) and

\[ \psi^p\sigma(h)(\eta_n) = p^n. \]

3. The map \( \alpha_G \).

In this section we study the injectivity of the map \( \hat{\alpha}_G : \hat{A}(G) \to \pi_0^G(BG) \) from the completion of the Burnside ring of a finite group \( G \) to the stable cohomotopy of its classifying space \( BG \).

Recall the definition of \( \alpha_G \). Each \( G \)-set \( S \) with \( G \)-action \( \varrho : G \to \Sigma_{|S|} \) gives rise to a map \( \alpha_G(S) : BG \to QS^0 \) by

\[ \xymatrix{ BG \ar[r]^-{B\varrho} & B\Sigma_{|S|} \ar[r]^-{\bigvee_{n \geq 1}} & B\Sigma_n \ar[r]^-{\iota} & QS^0, } \]

where \( \iota \) is the group completion map. The homotopy class of \( \alpha_G(S) \) depends only on the class of \( S \) in \( A(G) \), and the correspondence \( S \mapsto \alpha_G(S) \) extends to a ring homomorphism

\[ \alpha_G : A(G) \to [BG, QS^0] \]

(by definition, \( \pi_0^G(BG) = [BG, QS^0] \)).

Alternatively, we can define \( \alpha_G(S) \) as the image of the covering \( EG \times S \downarrow BG \) in \( \pi_0^G(BG) \) (cf. 2.11). This is quite analogous to the homomorphism

\[ \alpha : R(G) \to K^*(BG) \]

studied by Atiyah in [1]: if \( \varrho : G \to \text{Gl}(n, \mathbb{C}) \) is a complex representation of \( G \), then \( \alpha(\varrho) \) is the class of the vector bundle \( EG \times_G (\mathbb{C}^n, \varrho) \) in \( K^0(BG) \). It is no surprise that \( \alpha_G \) and \( \alpha \) are connected via the natural map \( A(G) \to R(G) \):

**Proposition 3.1.** Let \( G \) be a finite group. Then the diagram

\[ \begin{array}{ccc} A(G) & \xrightarrow{\alpha_G} & \pi_0^G(BG) \\ \downarrow & & \downarrow e_* \\ R(G) & \xrightarrow{\alpha} & K^*(BG) \end{array} \]
commutes, where \( e : QS^0 \to BU \times \mathbb{Z} \) is induced from a unit of the unitary spectrum.

\[ \begin{diagram}
BG & \xrightarrow{B\varrho} & B\Sigma_n & \xrightarrow{i} & Q_nS^0 & \xrightarrow{e} & QS^0 \\
& & \downarrow{BP} & & \downarrow{e_n} & & \downarrow{e} \\
BG & \xrightarrow{BP \cdot B\varrho} & BU_n & \xrightarrow{(n)} & BU \times \mathbb{Z} & &
\end{diagram} \]

But the right hand squares commute [12, Corollary 5.31].

**Remark.** We shall give in section 4 a closer description of the map \( e \) (see 4.18).

In section 1 we considered the \( I(G) \)-adic topology on \( A(G) \). If \( X \) is a CW-complex with \( n \)-skeleton \( X^n \), we filter \( \pi_S^0(X) \) by

\[
(F) \quad F^n\pi_S^0(X) = \text{Ker} (\pi_S^0(X) \to \pi_S^0(X^{n-1})).
\]

Then \( F^n \cdot F^m \subset F^{n+m} \) by diagonal approximation. J. W. Milnor's original construction of \( BG \) gave a CW-complex with finite skeletons \( B_nG \). As the stable homotopy groups \( \tilde{\pi}_S^0(S^n) \) are finite, so are the groups \( \tilde{\pi}_S^0(B_nG) \). Hence \( \pi_S^0(BG) = \lim_{\rightarrow} \pi_S^0(B_nG) \), and \( \pi_S^0(BG) \) is complete in the filtration topology.

It follows from the definition that \( \alpha_G(I(G)) \subset [BG, QS^0] = F^1\pi_S^0(BG) \), and since \( \alpha_G \) is a ring homomorphism, \( \alpha_G(I(G)^\ast) \subset F^n\pi_S^0(BG) \). Thus \( \alpha_G \) is continuous and induces a homomorphism

\[ \hat{\alpha}_G : \hat{A}(G) \to \pi_S^0(BG) \]

between the completions.

All the maps of 3.1 are continuous homomorphisms, when \( R(G) \) is equipped with augmentation ideal topology and \( K^\ast(BG) \) with a filtration topology similar to (F). Passing to completions we have

\[ \begin{diagram}
\hat{A}(G) & \xrightarrow{\hat{\alpha}_G} & \pi_S^0(BG) \\
& \downarrow{e_*} & \\
\hat{R}(G) & \xrightarrow{\hat{\alpha}} & K^\ast(BG)
\end{diagram} \]

The main result of Atiyah [1] states that \( \hat{\alpha} \) is an isomorphism. Therefore we can conclude the injectivity of \( \hat{\alpha}_G \) if \( \hat{A}(G) \) embeds into \( \hat{R}(G) \). If \( G \) is cyclic then \( A(G) \to R(G) \) is injective by Lemma 1.8. If moreover the order of \( G \) is a prime
power $p^n$, then the augmentation ideals of both rings have the $p$-adic topology by Proposition 1.12 and [4, p. 277]. But then $\hat{A}(G) \to \hat{R}(G)$ is injective, since the $p$-adic completion is an exact functor. We have proved

**Theorem 3.2.** Let $G$ be a cyclic group of prime power order. Then $\hat{\alpha}_G$ is injective.

We can express theorem 3.2 by saying that the maps $\alpha_G(x)$ for cyclic $G$ are detected by $K$-theory. Indeed, the proof of 3.1 shows that $\alpha(x): BG \to BU \times \mathbb{Z}$ factors as

$$BG \xrightarrow{\alpha_G(x)} QS^0 \xrightarrow{\epsilon} BU \times \mathbb{Z}.$$

Since the map induced by $\alpha(x)$ in $K$-theory is non-trivial, if $x \neq 0$, so must be the one induced by $\alpha_G(x)$, too.

Next we invoke theorem 1.15 to show that the injectivity of $\hat{\alpha}_G$ can be deduced from that of $\hat{\alpha}_{G_p}$ for all Sylow subgroups $G_p$ of $G$. Consider the commutative diagram

$$\begin{array}{ccc}
\hat{A}(G) & \xrightarrow{\hat{\alpha}_G} & \pi^0_S(BG) \\
\downarrow \text{Res} & & \downarrow \\
\bigoplus_p \hat{A}(G_p) & \xrightarrow{\oplus \hat{\alpha}_{G_p}} & \bigoplus_p \pi^0_S(BG_p)
\end{array}$$

By Theorem 1.15, Res is injective. If the maps $\hat{\alpha}_{G_p}$ are injective for all $G_p \leq G$, then $\hat{\alpha}_G$ must be injective. Hence

**Theorem 3.3.** Let $G$ be a finite group and $\{G_p\}$ its Sylow subgroups. If $\hat{\alpha}_{G_p}$ is injective for all $G_p \leq G$, then $\hat{\alpha}_G$ is injective.

Theorem 3.3 reduces the study of $\hat{\alpha}_G$ to $p$-groups $G$. First, we note

**Lemma 3.4.** Let $G$ be a $p$-group. Then $\hat{\alpha}_G$ is injective if and only if $\alpha_G$ is injective.

(Indeed, as $A(G)$ embeds into $\hat{A}(G)$ by Corollary 1.11, one way is trivial and the converse follows since the $p$-adic completion is an exact functor and $I(G)$ and $\hat{\pi}^0_S(BG)$ both have $p$-adic topology, $I(G)$ by Proposition 1.12 and $\hat{\pi}^0_S(BG)$ being profinite with $BG$ $p$-local [21].)

The smallest non-trivial $p$-group is the cyclic $\mathbb{Z}/p$, where we can apply Theorem 3.2. Suppose inductively that $\alpha_H$ is injective for all genuine subgroups
H of G. By naturality of α an element in A(G) which has a non-zero restriction to some \( H < G \), cannot lie in the kernel of \( \alpha_G \). Applying Theorem 1.3 we get

**Lemma 3.5.** Let \( G \) be a p-group. Suppose \( \alpha_H \) is injective for all genuine subgroups \( H < G \). Then \( \alpha_G \) is injective on \( \ker \chi_G \).

To handle the rest, we have

**Proposition 3.6.** Let \( G \) be a p-group. There exists an element \( x \in A(G) \) with \( \chi_G(x) = p \) and \( \chi_H(x) = 0 \) for \( H < G \). It is induced from an epimorphism \( G \to (\mathbb{Z}/p)^d \).

**Proof.** The existence of \( x \) follows from the congruences of 1.3. However, we prefer to construct it directly.

Let \( \Phi(G) \) be the Frattini subgroup of \( G \), that is, the intersection of all maximal subgroups of \( G \). We recall some elementary facts about \( \Phi(G) \) [8, III § 3]:

1) \( \Phi(G) = G^p[G, G] \),
2) \( \Phi(G) \triangleleft G \) and \( G/\Phi(G) \) is a maximal elementary abelian quotient of \( G \), say \( (\mathbb{Z}/p)^d \), and
3) the elements of \( \Phi(G) \) are redundant in any set of generators for \( G \).

One can also characterize the quotient \( G/\Phi(G) \) as \( H_1(G; \mathbb{Z}/p) \). In \( A(\mathbb{Z}/p)^d \) we write down the element

\[
y = p - \eta_1^{\text{tot}} + \eta_2^{\text{tot}} - \ldots + (-1)^{d} p^{(d-1)} \eta_d
\]

(See example 2.21, \( y = pa_0 \) in (2.22)). If \( \pi: G \to G/\Phi(G) \) denotes the projection, then \( \pi^*(y) \) has the required properties. Clearly \( \chi_G(\pi^*(y)) = (\chi_{\mathbb{Z}/p})^d(y) = p \). If \( H < G \), then \( \pi(H) < (\mathbb{Z}/p)^d \) and \( \chi_H(\pi^*(y)) = \chi_{\mathbb{N}_H}(y) = 0 \). Indeed, if \( \pi(H) = (\mathbb{Z}/p)^d \), then \( H\Phi(G) = G \), which implies \( H = G \) by 3) above. This completes the proof of proposition 3.6.

**Lemma 3.7.** \( \mathbb{Z}x = \bigcap_{H < G} \ker \text{Res}^G_H \) and it is a \( \lambda \)-ideal of \( A(G) \).

**Proof.** The second claim follows from the first, since the maps \( \text{Res}^G_H \) are \( \lambda \)-homomorphisms. By definition \( \text{Res}^G_H(x) = 0 \) for each \( H < G \). For the other containment suppose \( \text{Res}^G_H(y) = 0 \) for each \( H < G \); we must show \( \chi_G(y) \equiv 0 \) (mod \( p \)). Let \( H < G \) be a subgroup of index \( p \). As \( H < N(H), H \) must be normal in \( G \). The congruences of 1.3 become

\[
0 = \chi_H(y) \equiv -\varphi(p)\chi_G(y) = -(p-1)\chi_G(y) \pmod{p}
\]
thus $\chi_G(y) \equiv 0 \pmod{p}$.

We are ready to state the final result.

**Theorem 3.8.** Let $G$ be a $p$-group. Suppose that

1) $\alpha_H$ is injective for all $H < G$

2) $\alpha_G$ is injective for the $\lambda$-ideal $\mathbb{Z}x$ described in Proposition 3.6.

Then $\hat{\alpha}_G$ is injective.

**Proof.** We first show that $\alpha_G$ is injective on $\mathbb{Z}x \oplus \text{Ker } \chi_G$. If $m \in \mathbb{Z}$, $\chi_G(y) = 0$ and $\alpha_G(mx + y) = 0$, then also

$$0 = \alpha_G(mx + y)\alpha_G(y) = \alpha_G(y^2)$$

since $xy = 0$ (all characters are 0). By Lemma 3.5 $y^2 = 0$, so $y = 0$. Thus $\alpha_G(mx) = 0$ and $m = 0$ by assumption 2).

Now any element in $A(G)$ can be written as $n + z$ with $0 \leq n < p$ and $z \in \mathbb{Z}x \oplus \text{Ker } \chi_G$ since the latter ideal consists of $z$ such that $\chi_G(z) \equiv 0 \pmod{p}$. Suppose $\alpha_G(n + z) = 0$, then

$$0 = \alpha_G(z)\alpha_G(n + z) = \alpha_G(nz + z^2)$$

where $nz + z^2 \in \mathbb{Z}x \oplus \text{Ker } \chi_G$. From the above $nz = -z^2$, and taking characters we get

$$\chi_H(z) = 0 \text{ or } -n \text{ for all } H \leq G.$$ 

As $\chi_G(z) \equiv 0 \pmod{p}$ we must have $\chi_G(z) = 0$. But then $\chi_H(z) = 0$ for all $H < G$: if $H < G$ is a maximal subgroup with $\chi_H(z) = -n$ then by 1.3

$$-n = \chi_H(z) \equiv -\sum \phi(K/H)\chi_K(z) = 0 \pmod{|N(H)/H|}$$

which is impossible, since $|N(H)/H|$ is a positive power of $p$. Thus $z = 0$ and $\alpha_G(n) = 0$ implies $n = \deg \alpha_G(n) = 0$.

This completes the proof of Theorem 3.8.

4. Homological study of $\alpha_G$.

In this section we shall study the maps induced by $\alpha_G(x) : B \mathbb{Z} \to \mathbb{Z}^n$ in homology for elementary abelian groups $G$. As a corollary we get that $\alpha_{(\mathbb{Z}/p)^n}$ is injective. We obtain also information relative to the splitting $\mathbb{Q}_0S_p^0 \cong J_p \times \text{cok } J_p$. We suppress the index $G$ and write $\alpha$ for $\alpha_G$.

Consider $\alpha(S)$ for a $(\mathbb{Z}/p)^n$-set $S$. The map $\alpha$ is additive, so we can restrict to transitive sets: $S = (\mathbb{Z}/p)^n/H$ where $H \leq (\mathbb{Z}/p)^n$. Both $H$ and the quotient $(\mathbb{Z}/p)^n\cong (\mathbb{Z}/p)^n/H$ are elementary abelian, and $S$ is induced from the regular representation $\eta_m = (\mathbb{Z}/p)^n/1$ of $(\mathbb{Z}/p)^n$. Thus $\alpha(S)$ factors
(4.1) \[ B(Z/p)^n \to B(Z/p)^m \xrightarrow{\alpha(\eta_m)} QS^0 \]

Recall that the composition product in \( QS^0 \) corresponds to the product in \( \coprod B\Sigma_n \) coming from the homomorphisms

\[ \psi_{n,m} : \Sigma_n \times \Sigma_m \to \Sigma_{nm} \]

defined as

\[ \psi_{n,m}(g, h)(i, j) = (g(i), h(j)). \]

Here \( \Sigma_{nm} \) is regarded as the permutation group of pairs \((i, j), 1 \leq i \leq n, 1 \leq j \leq m.\) This requires a linear ordering of the pairs; we use the lexicographic one.

We can express \( \eta_m \) inductively in terms of \( \psi_{n,m}. \) The first \( \eta_1 \) is just the inclusion \( Z/p \subset \Sigma_p \) as cyclic permutations. Then \( \eta_2 = \psi_{p,p} \circ (\eta_1 \times \eta_1), \) and generally

\[ \eta_n : (Z/p)^n = Z/p \times (Z/p)^{n-1} \xrightarrow{\eta_1 \times \eta_{n-1}} \Sigma_p \times \Sigma_p^{n-1} \xrightarrow{\psi_{p,p}^{n-1}} \Sigma_{p^n}. \]

Hence \( \alpha(\eta_n) \) can be written as the composition

(4.2) \[ \alpha(\eta_n) : B(Z/p)^n = (B\Sigma_p)^n \xrightarrow{(B\eta)^n} (B\Sigma_p^n)^n \xrightarrow{(\psi_{p,p})^{n-1}} QS^n. \]

We shall need certain facts about the homology of \( QS^0 \) with \( Z/p \)-coefficients. General references for this are [10] and [12] for \( p = 2 \) and [6] for \( p > 2. \) Here is a summary.

The space \( QS^0 \) has two products: the loop sum \( * \) and the composition product \( \cdot. \) They induce products on \( H_*(QS^0; Z/p), \) denoted similarly. They are homomorphisms \( Q^b : H_*(QS^0; Z/p) \to H_*(QS^0; Z/p) \) with the following properties [modifications for the case \( p = 2 \) are stated inside square brackets]:

(4.3) Degree: \( Q^b \) raises degree by \( 2b(p-1) [b] \)

(4.4) Evaluation: \( Q^b x = 0 \) if \( 2b < \deg x \) \( [b < \deg x] \)
\( Q^b x = x \ast_p \) if \( 2b = \deg x \) \( [b = \deg x] \)

(4.5) Cartan formula: \( Q^b(x \ast y) = \sum_{i+j=b} Q^i x \ast Q^j y. \)

(4.6) Adem relations: If \( a > pb \) then

\[ Q^a Q^b x = \sum (-1)^{a+t} \binom{(p-1)(t-b)-1}{pt-a} Q^{a+b-t} Q^t x; \]

if \( p > 2, a \geq pb \) and \( \beta \) denotes the mod \( p \) Bockstein, then

\[ Q^a \beta Q^b x = \sum (-1)^{a+t} \binom{(p-1)(t-b)}{pt-a} \beta Q^{a+b-t} Q^t x. \]
\[ + \sum (-1)^{a+t} \left( \frac{(p-1)(t-b) - 1}{pt-a-1} \right) Q^{a+b-t} \beta Q^t x. \]

In all cases the summation is over \( t \) such that \((p+1)t \geq a+b\).

(4.7) Nishida relations: If \( P^{r}_* \) is dual to the reduced \( p \)-th power \( P^r [\text{the square } \text{Sq}^r] \) then

\[ P^a_\ast Q^b x = \sum_{t \geq 0} (-1)^{a+t} \left( \frac{(p-1)(b-a)}{a-pt} \right) Q^{b-a+t} P^t_\ast x; \]

if \( p > 2 \) then

\[ P^a_\ast \beta Q^b x = \sum_{t \geq 0} (-1)^{a+t} \left( \frac{(p-1)(b-a) - 1}{a-pt} \right) \beta Q^{b-a+t} P^t_\ast x \]

\[ + \sum_{t \geq 0} (-1)^{a+t} \left( \frac{(p-1)(b-a) - 1}{a-pt-1} \right) Q^{b-a+t} P^t_\ast \beta x. \]

Let \([k] \in H_0(QS^0; \mathbb{Z}/p)\) denote the component of maps of degree \( k \). E. Dyer and R. Lashof showed that the homology ring \( H_\ast(QS^0; \mathbb{Z}/p) \) was generated by successive operations of \( Q^s, \beta Q^s \) on \([1]\) as an algebra under \( \ast \). To make a precise statement, we introduce the Dyer–Lashof algebra \( R(p) \).

Let \( \mathcal{F} \) be the free graded associative algebra generated by the symbols \( Q^s, \beta Q^s, s \geq 0 \) with degrees \( 2s(p-1) \) and \( 2s(p-1) - 1 \) respectively [if \( p = 2 \), \( \mathcal{F} \) is generated by \( Q^s, s \geq 0 \), with degree \( s \)]. The monomials in \( \mathcal{F} \) can be written as

\[ \beta^{s_1} Q^{s_2} \ldots \beta^{s_k} Q^{s_k} \]

with \( \varepsilon_i = 0 \) or 1 and \( s_i \geq \varepsilon_i \). Denote them by \( Q^I \), where \( I = (\varepsilon_1, s_1, \ldots, s_k, \varepsilon_k) \). We say that \( I \) is admissible if \( s_1 \leq ps_2 - \varepsilon_2, \ldots, s_{k-1} \leq ps_k - \varepsilon_k \), and we define the length and excess of \( I \) by \( l(I) = k \) and

\[ e(I) = (2s_1 - \varepsilon_1) - \sum_{j=2}^{k} (2s_j(p-1) - \varepsilon_j) \quad (p > 2) \]

\[ e(I) = s_1 - \sum_{j=2}^{k} s_j \quad (p = 2) \]

The quotient of \( \mathcal{F} \) by the ideal generated by the Adem relations and by monomials with \( e(I) < 0 \) is the Dyer–Lashof algebra \( R(p) \).

The formulas (4.3)–(4.6) tell that \( R(p) \) acts on \( H_\ast(QS^0; \mathbb{Z}/p) \). In fact the set

(4.8) \[ X = \{ Q^I[1], \ I \text{ admissible}, e(I) + \varepsilon_1 > 0 \} \]

forms a basis for the \( \ast \)-algebra \( H_\ast(QS^0; \mathbb{Z}/p) \) up to component shift. Indeed, let \( \mathbb{Z}/p[Z] \) be the group ring of \( Z = \pi_0(QS^0) \). Then
\[ H_\ast(QS^0; \mathbb{Z}/2) = PX \otimes \mathbb{Z}/2[\mathbb{Z}] \]
\[ H_\ast(QS^0; \mathbb{Z}/p) = PX^+ \otimes EX^- \otimes \mathbb{Z}/p[\mathbb{Z}], \quad \text{if } p > 2, \]
where \( P \) and \( E \) denote the polynomial and exterior algebras, respectively, and \( X^+ (X^-) \) is the even (odd) degree part of \( X \).

The composition product is related to the operations \( Q^b \) by May's formula:
\[
Q^b(x) \cdot f = \sum_{t \geq 0} Q^{b+t}(x \cdot P^tf) \quad \text{and, if } p > 2
\]
(4.9)
\[
\beta Q^b(x) \cdot f = \sum \beta Q^{b+t}(x \cdot P^tf) - (-1)^{\deg x} \sum Q^{b+t}(x \cdot P^t \beta f)
\]

After these preparations we turn to the evaluation of (4.2) in homology. The map \( i : B\Sigma_p \rightarrow QS^0 \) is obtained from the Dyer–Lashof map \( \theta_p \) as the composite
\[
i : B\Sigma_p = E\Sigma_p \times (\ast)^p \rightarrow E\Sigma_p \times (QS^0)^p \xrightarrow{\theta_p} QS^0,
\]
where \( \ast \) goes to the identity map in the 1-component \( Q_1S^0 \). By (4.2), \( \alpha(\eta_1) \) is of the form \( BZ/p \xrightarrow{B\eta_1} B\Sigma_p \rightarrow QS^0 \). This is precisely the map used in the definition of \( Q^e \) [6, pp. 7–8]; if \( e_m \in H_m(BZ/p; \mathbb{Z}/p) \) denotes the standard generator then
\[
(4.10) \quad \alpha(\eta_1)_\ast(e_m) = Q_m[1] = \begin{cases} 
(-1)^s Q_x[1] & \text{if } m = 2s(p-1) \\
(-1)^s \beta Q_x[1], & \text{if } m = 2s(p-1) - 1 \text{ for } p \text{ odd} \\
0 & \text{otherwise} 
\end{cases}
\]
for \( p = 2. \)

It follows from (4.2) and (4.10) that \( \alpha(\eta_1)_\ast \) takes the generators \( e_{i_1} \otimes \ldots \otimes e_{i_n} \) of \( H_\ast(B(\mathbb{Z}/p)^n; \mathbb{Z}/p) \) to products of the form
\[
\pm \beta^{e_i} Q_x[1] \cdot \ldots \cdot \beta^{e_n} Q_x[1].
\]

We would like to express these elements in the \( \ast \)-product basis (4.8). A two-fold product, for example \( Q^a[1] \cdot Q^b[1] \), becomes
\[
(4.11) \quad Q^a[1] \cdot Q^b[1] = \sum_{t \geq 0} Q^{a+t}(P^t Q^b[1])
\]
\[
= \sum_{t \geq 0} (-1)^t \binom{(p-1)(b-t)}{t} Q^{a+t} Q^{b-t}[1]
\]
by (4.9) and (4.7). Applying the Adem relations (4.6) it can be written as a linear combination of admissible terms \( Q^aQ'[1] \). If the excess is negative then \( Q'[1] = 0 \) by (4.4). Similarly it is shown by induction on the length of the product that
Lemma 4.12.

$$\beta^e Q^n[1] \cdot \ldots \cdot \beta^e Q^n[1] = \sum \lambda_i Q^l[1]$$

where $I$ ranges over admissible sequence of length $n$ and excess $\geq 0$.

The terms $Q^l[1]$ with $e(l) + e_1 = 0$ decompose as $\ast$-products of shorter $Q^l[1]$'s (4.4).

We shall now find the special case of Lemma 4.12 in the lowest degree where we can get an admissible $Q^l[1]$ of length $n$ and excess $> 0$ involving no Bocksteins $\beta$. It is clearly $Q^p Q^{p^n-1} \ldots Q^l[1]$ in dimension $2(p^n+1-1)$ with excess 2 [if $p = 2$ then $d = 2^{n+1}-1$ and $e = 1$]. We give the proofs of the next two lemmas only for $p = 2$. The (easier) case $p = 2$ follows by trivial modifications.

Lemma 4.13. $Q^p[1] \cdot Q^{p^n-1}[1] \cdot \ldots \cdot Q^l[1] = Q^p Q^{p^n-1} \ldots Q^l[1]$.

Proof. To begin with, $Q^p[1] \cdot Q^l[1] = Q^p Q^l[1]$ by (4.11). Suppose by induction that the claim holds for $n$. Since $x_n = Q^p Q^{p^n-1} \ldots Q^l[1]$ is primitive, so is also $P^t_\ast x_n$. If $t > 0$, then according to (4.7) $P^t_\ast x_n$ is a linear combination of $Q^l[1]$'s of length $n$ and degree $< 2(p^n+1-1)$, without Bocksteins. By the minimality of $x_n$, $P^t_\ast x_n$ is $\ast$-decomposable. By a general theorem of Hopf algebras [15, Proposition 4.23] $P^t_\ast x_n$ must then be a $\ast - p$th power, especially

$$\deg P^t_\ast x_n = 2(p^n+1-1) - 2t(p-1) \equiv -2 + 2t \equiv 0 \pmod{p}$$

so that $t \equiv 1 \pmod{p}$. Now we can apply May's formula (4.9) to get

$$Q^{p^n+1}[1] \cdot Q^p[1] \cdot \ldots \cdot Q^l[1] = Q^{p^n+1}[1] \cdot x_n$$

$$= \sum_{i \geq 0} Q^{p^{n+1}+i} P^i_\ast x_n = Q^{p^{n+1}} x_n = Q^{p^{n+1}} Q^p \ldots Q^l[1]$$

since $Q^s(x^*) = 0$ only if $s \equiv 0 \pmod{p}$ in virtue of the Cartan formula (4.5).

Now we can prove that the map $\alpha: A((\mathbb{Z}/p)^n) \to \pi_0^s(B(\mathbb{Z}/p)^n)$ is injective on $\mathbb{Z}n$ and thereby on the whole of $A((\mathbb{Z}/p)^n)$.

Proposition 4.14. $\alpha(m_\eta_n)$ is homologically non-trivial for all non-zero integers $m$.

Proof. Let first $m > 0$. Then using the diagonal formula

$$\psi(e_{2i}) = \sum e_{2i_1} \otimes \ldots \otimes e_{2i_m}, \quad i_1 + \ldots + i_m = i$$

for $B\mathbb{Z}/p \to (B\mathbb{Z}/p)^m$ and Lemmas 4.12 and 4.13 we obtain

$$\alpha(m_\eta_n)(e_{2mp^{n+1}}(p-1) \otimes \ldots \otimes e_{2m(p-1)}) = (Q^p Q^{p^{n-1}} \ldots Q^l[1])^m + \ldots$$
where the other terms are of the form \( Q^{l_1}[1] \ldots Q^{l_\nu}[1] \) with \( l(l_j) = n \) and \( \deg(l_j) < 2(p^{n+1} - 1) \) for at least one \( j \). Since \( Q^\nu Q^{n-1} \ldots Q^1[1] \) is a polynomial generator, they cannot cancel the first term.

If \( m < 0 \), then apply the loop inverse \( \chi_* \), and note that \( \chi_*(x) = x \star [−2 \deg x] \) on primitive elements \( x \).

**Theorem 4.15.** \( \hat{\alpha}: \hat{A}((\mathbb{Z}/p)^n) \rightarrow \pi^1_0 \mathbb{P}(B(\mathbb{Z}/p)^n) \) is injective for all primes \( p \).

**Proof.** By Theorems 3.2 and 3.8 we are reduced to showing that \( \alpha \) is injective on \( \mathbb{Z}x \), where

\[
x = p - \eta_1^{\text{tot}} + \ldots + (-1)^n p^{n-1} \eta_n.
\]

The argument of Proposition 4.14 applies also here, since the terms \( \eta_i^{\text{tot}} \) contribute in homology only by \( \ast \)-products of \( Q^j[1] \) with \( l(l) < n \) (cf. (4.1)).

This completes the proof of theorem 4.15.

Let \( Q_0 S^0 \) be the 0-component of \( QS^0 \). Let \( X_p \) denote the localization of the space \( X \) at a prime \( p \) [21]. D. Sullivan has showed that the space \( Q_0 S^0 \) splits locally

\[
Q_0 S^0_p \cong J_p \times \text{cok} J_p.
\]

The space \( J_2 \) is defined as the fibre of \( \psi^2 - 1: BO_2 \rightarrow BSpin_2 \). At odd primes \( J_p \) is the fibre of \( \psi^k - 1: BU_p \rightarrow BU_p \), where \( k \) is a prime power generating the group of units in \( \mathbb{Z}/p^2 \). The homotopy groups of \( J_p \) are essentially the \( p \)-primary part of the image of the \( J \)-homomorphism \( O \rightarrow G \) in the stable homotopy of spheres. To describe the second factor \( \text{cok} J_p \) we recall the discrete models for \( J_p \) due to D. Quillen [14, chapter VIII].

First, let \( p = 2 \). Let \( F_3 \) denote the finite field with 3 elements. Let \( N_n(F_3) \) be the group of orthogonal transformations of the quadratic space \( (F_3^n, x_1^2 + \ldots + x_n^2) \) for which the determinant and the spinor norm [14, p. 164] agree. We encounter now a similar situation to the construction of \( QS^0 \) from the symmetric groups: there are sum and product maps on the disjoint union

\[
\coprod_{n \geq 0} BN_n(F_3)
\]

coming from direct sum and tensor product of quadratic spaces.

Let \( \overline{F}_3 \) be an algebraic closure of \( F_3 \) and choose an embedding \( \mu: \overline{F}_3^* \rightarrow \mathbb{C}^* \).

If \( G \) is a finite group and \( \varrho: G \rightarrow \text{Gl}_n(\overline{F}_3) \) a representation of \( G \), then the complex-valued function on \( G \)

\[
\chi(g) = \sum_{i=1}^n \mu(\lambda_i)
\]
where \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of \( \varphi(g) \), is the character of a unique element in the complex representation ring \( R(G) \). Moreover, Quillen has proved that if \( \varphi \) takes values in \( O_n(F_3) \), then \( \chi \) is the character of an element in the real representation ring \( RO(G) \).

We lift the standard representations of \( N_n(F_3) \) in \( F_3^3 \) in the above way to virtual representations in \( RO(N_n(F_3)) \) and apply \( \alpha: RO(G) \to KO(BG) \) to get maps

\[
v_n: BN_n(F_3) \to BO \times (n).
\]

They are compatible with the sum and product on (4.17), giving rise to an \( H \)-map

\[
v: \Omega B \left( \bigoplus_{n \geq 0} BN_n(F_3) \right) \to BO \times Z
\]

from its group completion. Now the Adams operation \( \psi^3 \) is characterized by its action on the characters \( (\psi^3 \chi)(g) = \chi(g^3) \), so \( \psi^3 \circ v_n = v_n \) as the Frobenius map \( \lambda \to \lambda^3 \) just permutes the eigenvalues of any representation realizable over \( F_3 \).

Let \( J_2^S \) denote the zero component of \( \Omega B \left( \bigoplus_{n \geq 0} BN_n(F_3) \right) \) localized at 2. Then \( v: J_2^S \to BO_2 \) lifts to an \( H \)-map \( J_2^S \to J_2 \), which can be proved to be a homotopy equivalence e.g. by cohomological methods. From now on we identify \( J_2^S \) and \( J_2 \).

Let

\[
e: QS^0 \cong \Omega B \left( \bigoplus_{n \geq 0} BS_n \right) \to \Omega B \left( \bigoplus_{n \geq 0} BN_n(F_3) \right)
\]

be induced from the functor which takes a finite set \( S \) to the vector space \( F_3^S \). We restrict \( e \) to the zero component and localize to get \( e: Q_0S^0_2 \to J_2 \). We shall also use the analogous map \( e: Q_0S^0_2 \to BO_2 \), induced from the functor \( S \mapsto R^S \). Then the triangle

\[
(4.18)
\]

\[
\begin{array}{c}
Q_0S^0_2 \\
\downarrow e \\
BO_2
\end{array}
\]

\[
\begin{array}{c}
\quad \quad J_2 \\
\end{array}
\]

clearly commutes. The space \( \text{cok} J_2 \) is defined as the fibre in

\[
(4.19)
\]

\[
\text{cok} J_2 \to Q_0S^0_2 \xrightarrow{e} J_2
\]

There exists a splitting \( \alpha_2: J_2 \to Q_1S^0_2 \), and \( \alpha_2 \circ [-1] \) gives (4.16).

At odd primes \( p \) the model for \( J_2 \) is constructed from general linear groups.
over the finite field $F_k$: $J_p$ is equivalent to the zero component of the group completion of

\[(4.17') \quad \prod_{n \geq 0} B\text{Gl}_n(F_k), \text{ } k \text{ a prime power generating } (\mathbb{Z}/p^2)^*.\]

As before defines $e: Q_0S^0_p \to J_p$ and gets a commutative diagram

\[(4.18') \quad \begin{array}{ccc}
Q_0S^0_p & \xrightarrow{e} & J_p \\
\downarrow e & & \downarrow \\
BU_p & & 
\end{array}\]

and $\text{cok } J_p$ is defined as the fibre in

\[(4.19') \quad \text{cok } J_p \to Q_0S^0_p \xrightarrow{e} J_p.\]

Let now $G$ be a $p$-group. We shall in the following consider the abelian groups $[BG, X_p]$, where $X = Q_0S^0$, $BU$, $U$ for all $p$ and in addition to these, $X = BO$ and $SO$ for $p = 2$. We claim that in all cases

\[ [BG, X_p] = [BG, X]. \]

If $X = Q_0S^0$ this holds because $Q_0S^0$ has finite homotopy groups, so $Q_0S^0 = \prod_p Q_0S^0_p$, and $BG$ is $p$-local (even $p$-complete) [21, section 3].

For the other spaces we recall the results of Atiyah [1] and Atiyah–Segal [2]. Consider the representable $K$-theory and the theory $K^*(; Z_{(p)})$ defined by the unitary spectrum and its localization at $p$. For any finite CW-complex $Y$ we have

\[ K^*(Y; Z_{(p)}) \cong K^*(Y) \otimes Z_{(p)}. \]

The formula is valid also for $Y = BG$ since it follows from [1] and [2] that $\lim^1$ of the inverse systems $K^*(B_nG)$ and $K^*(B_nG) \otimes Z_{(p)}$ vanishes, so that $K^* (BG) = \lim K^*(B_nG)$ and $K^*(BG; Z_{(p)}) = \lim K^*(B_nG) \otimes Z_{(p)}$. For any group $G$

\[ K^0(BG) = \tilde{R}(G) \text{ and } K^1(BG) = 0 \]

[1, p. 270] and in the case of $p$-groups the completion is the $p$-adic one: $\tilde{R}^0(BG) = I(G) \otimes \hat{Z}_p$ [4 p. 277]. Since these groups are clearly unaffected by $\otimes Z_{(p)}$, we get

\[(4.20) \quad [BG, BU] = [BG, BU_p] = I(G) \otimes \hat{Z}_p, \quad [BG, U_p] = 0 \]

where $I(G)$ is the augmentation ideal of $R(G)$.

If $p = 2$, then using the Real $K$-theory $KR^*$ instead of $K^*$ and [2, p. 17] we obtain in the same fashion

\[(4.21) \quad [BG, BO] = [BG, BO_2] = I(G) \otimes \hat{Z}_2, \]
for 2-groups $G$, where $I(G)$ is the augmentation ideal of $R_{R}(G)$.

Let $G$ be a $p$-group. After these preliminaries we turn to the question: when does a map $\hat{\alpha}(x): BG \to Q_{o}S^{0}$ lift to $\text{cok} J_{p}$ in the fibration (4.19), (4.19'). In order for $e \circ \hat{\alpha}(x): BG \to Q_{o}S^{0} \to J_{p}$ to be nullhomotopic, it is necessary in the light of (4.18) and (4.18') that the image of $\hat{\alpha}(x)$ under $e_{*}: \tilde{\pi}_{*}^{0}(BG) \to \tilde{R}_{*}^{0}(BG)$ is zero. From Proposition 3.1, this is equivalent to

$$x \in \hat{A}_{0}(G) = \text{Ker} (\hat{A}(G) \to \hat{R}(G)).$$

If $p$ is odd, this condition is also sufficient, since in the mapping sequence of the fibration $J_{p} \to BU_{p} \xrightarrow{\psi^{-1}} BU_{p}$

$$[BG, U_{p}] \to [BG, J_{p}] \to [BG, BU_{p}]$$

the first group is trivial (4.20), so $x \in \hat{A}_{0}(G)$ maps to zero already in $[BG, J_{p}]$.

In particular, if $G$ is an elementary abelian group $(Z/p)^{n}$ with odd $p$, we know that all the maps $\hat{\alpha}(x): BG \to Q_{o}S^{0}$, $x \in \hat{A}_{0}(G)$ are homotopically distinct (Theorem 4.15). Thus $\hat{\alpha}$ lifts to a monomorphism $\hat{\alpha}'$

$$\hat{A}_{0}(G)$$

$$\hat{\alpha}'$$

$$\hat{\alpha}$$

$$[BG, \text{cok} J_{p}] \to [BG, Q_{o}S^{0}] \to [BG, J_{p}].$$

**Theorem 4.22.** Let $p$ be an odd prime and $G$ the elementary abelian group $(Z/p)^{n}$. Then the ideal

$$\hat{A}_{0}(G) = \text{Ker} (\hat{A}(G) \to \hat{R}(G))$$

maps monomorphically into $[BG, \text{cok} J_{p}]$.

Let then $G$ be a 2-group and $x \in \hat{A}_{0}(G)$. Then the image of $\hat{\alpha}(x)$ is 0 in $[BG, BU_{2}]$. To see when $e \circ \hat{\alpha}(x): BG \to J_{2}$ is non-trivial, we consider the maps $J_{2} \to BO_{2} \xrightarrow{c} BU_{2}$, where $c$ is complexification. The map $[BG, BO_{2}] \to [BG, BU_{2}]$ corresponds to the completion of $R_{R}(G) \subset R(G)$ by (4.20), (4.21) and [2, p. 17], which is injective. In the mapping sequence of $J_{2} \to BO_{2} \xrightarrow{\psi^{-1}} BSpin_{2}$

$$[BG, Spin_{2}] \to [BG, J_{2}] \to [BG, BO_{2}]$$

the first group is a subgroup of $[BG, SO_{2}]$ as $SO_{2} \cong RP^{\infty} \times Spin_{2}$, hence it is a vector space over $Z/2$. Thus (at least) 2$x$ maps to zero in $[BG, J_{2}]$. We have proved the first half of
Theorem 4.23. Let $G$ be the elementary abelian 2-group $(\mathbb{Z}/2)^n$. Then the ideal $2\hat{A}_0(G)$ maps monomorphically into $[BG, \text{cok } J_2]$. There are elements in $\hat{A}_0(G)$ which do not lift to cok $J_2$.

Proof. Consider the critical element $x \in A_0(G)$ with $\chi_G(x) = 2$ and $\chi_H(x) = 0$ for all genuine subgroups $H < G$. We claim that $1 - x$ can be written as a product in terms of the $2^n - 1$ quotients $\eta_i^1 = G / (\mathbb{Z}/2)^{n-1}$:

$$1 - x = \prod_{i=1}^{2^n-1} (\eta_i^1 - 1).$$

Indeed, check the characters. First $\chi_H(\eta_i^1) = 2$ or 0 according to whether $H \leq (\mathbb{Z}/2)^{n-1}$ or $H \not\leq (\mathbb{Z}/2)^{n-1}$, the hyperplane defining $\eta_i^1$. Therefore we get

$$\chi_G\left(\prod_{i=1}^{2^n-1} (\eta_i^1 - 1)\right) = (-1)^{2^n-1} = -1.$$

On the other hand each hyperplane containing $H$ corresponds to a line inside $H^\perp$. If $H < G$ the number of these, $|H|^\perp - 1$, is odd, so

$$\chi_H\left(\prod_{i=1}^{2^n-1} (\eta_i^1 - 1)\right) = 1^{\text{odd}}(-1)^{\text{even}} = 1, \quad H < G.$$

Thus $\alpha(1-x)$ is a composition product of maps of the form

$$BG \xrightarrow{B\eta_1} B\mathbb{Z}/2 \xrightarrow{i_2} Q_2S^0 \xrightarrow{[1]} SG.$$

But the map $i_2[1] : B\mathbb{Z}/2 \to SG$ is homotopy equivalent to the composite

$$\mathbb{R}P^\infty \to SO \xrightarrow{J} SG$$

[6, p. 120] so that $\alpha(1-x)$ factors through $J : SO \to SG$. Let $e_1 : SG \to J^\circ$ be the 1-component of the map $e$ defined just before (4.18). We showed in 4.14 that $\alpha(-x)$, hence $\alpha(1-x)$ induces a non-trivial map in homology. It is well-known that the composite

$$H_*(SO) \xrightarrow{J_*} H_*(SG) \xrightarrow{e_1_*} H_*(J)$$

is injective [6, p. 120 and Theorem 12.5 p. 185]. Then $e_1 \circ \alpha(1-x)$, hence $e \circ \alpha (-x)$ must be homologically non-trivial.

This completes the proof of Theorem 4.23.

Remark 4.24. Theorems 4.22 and 4.23 enable us to get hold of elements in $H_*(\text{cok } J_p; \mathbb{Z}/p)$. Let us consider the first case $A_0(\mathbb{Z}/2 \oplus \mathbb{Z}/2) = \mathbb{Z}x$ $(A_0(\mathbb{Z}/2) = 0!)$. It is most convenient to evaluate $f = \alpha(1-x)$, since from the preceding proof

$$1 - x = (\eta_1^1 - 1)(\eta_2^1 - 1)(\eta_3^1 - 1)$$
where \( \eta_i^1 : \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \mathbb{Z}/2 \) are projections to the first and second factor for \( i = 1, 2, \) and \( \eta_i^1 \) takes the quotient modulo the diagonal subgroup \( \Delta \mathbb{Z}/2 \subset \mathbb{Z}/2 \oplus \mathbb{Z}/2. \)

The maps \( f_i = x(\eta_i^1 - 1): \mathbb{P}^\infty \times \mathbb{P}^\infty \to \mathbb{P}^\infty, \; i = 1, 2, 3, \) have the effect
\[
f_{1\ast}(e_m \otimes e_n) = \delta_{n0}x_m, \; f_{2\ast}(e_m \otimes e_n) = \delta_{m0}x_n
\]
and
\[
f_{3\ast}(e_m \otimes e_n) = \binom{m+n}{m}x_{m+n}
\]
on homology (cf. 4.10). Here \( x_k = Q^k[1] * [-1] \in H_* (SG; \mathbb{Z}/2), \) and adding up we get
\[
(4.25) \quad f_* (e_m \otimes e_n) = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m+n-i-j}{m-i}x_i \cdot x_j \cdot x_{m+n-i-j}.
\]
As a special case of this formula \( f_* (e_{2n} \otimes e_1) = p_{2n+1}, \) where the polynomial
\[
p_{2n+1} = x_{2n+1} + \sum_{i=1}^{n} x_i x_{2n+1-i}
\]
is the standard primitive element of degree \( 2n+1 \) in the subalgebra \( E(x_1, x_2, \ldots) \subset H_*(SG). \)

Let \( \tilde{f} \) denote \( \alpha(-x) = f \ast [-1]. \) We know from theorem 4.23 that \( 2\tilde{f} : (\mathbb{P}^\infty)^2 \to Q_0 S^0 \) lifts to \( \text{cok} \; J_2. \) Therefore the elements
\[
C_{4n+2} = (2\tilde{f})_* (e_{4n} \otimes e_2) = \tilde{f}_* (e_{2n} \otimes e_1) \ast \tilde{f}_* (e_{2n} \otimes e_1)
\]
\[
= p_{2n+1} \ast p_{2n+1} \ast [-2]
\]
\( n \geq 1, \) lie in \( \text{Ker} \; e_* . \) Since they are primitive, the lie in \( H_* (\text{cok} \; J_2, \mathbb{Z}/2). \) (The elements \( C_{2n-2} \) have a connection with the Arf invariant conjecture: they are spherical if and only if there are stable homotopy classes in \( \pi_{2n-2}^S(S^0) \) of Arf invariant one [9].)

\textbf{Remark 4.26.} We succeeded in proving that \( \delta_G \) is injective for elementary abelian \( G \) by evaluating the maps \( \alpha_G(nx) \) in homology. Let us indicate briefly where this program fails for more complicated groups. The smallest ones we have not covered are the following three groups of order 8:

\[
\mathbb{Z}/4 \oplus \mathbb{Z}/2 = \langle x, y \mid x^4 = y^2 = 1, xy = yx \rangle
\]
\[
D8 = \langle x, y \mid x^4 = y^2 = 1, y^{-1}xy = x^3 \rangle
\]
\[
Q8 = \langle x, y \mid x^4 = 1, y^2 = x^2, y^{-1}xy = x^3 \rangle
\]
(D8 and Q8 are the dihedral and the quaternion groups). In all cases the Frattini subgroup \( \Phi(G) = G^2 \) is \( \mathbb{Z}/2 \) generated by \( x^2 \). The cohomology of \( G \) (with \( \mathbb{Z}/2 \) coefficients) can be computed from the spectral sequence of the central extension
\[
1 \to \Phi(G) \to G \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to 1.
\]
The \( E^2 \)-term is
\[
H^*(\mathbb{Z}/2) \otimes H^*(\mathbb{Z}/2 \oplus \mathbb{Z}/2) = P(t) \otimes P(t_1, t_2).
\]
We choose \( t_1 \) and \( t_2 \) as the generators of the cohomology of \( \langle \pi(x) \rangle \) and \( \langle \pi(y) \rangle \). The differentials are determined by the characteristic class \( d_2(t) \in H^2(\mathbb{Z}/2 \oplus \mathbb{Z}/2) \), which is
\[
t_1^2, t_1^2 + t_1 t_2 \quad \text{and} \quad t_1^2 + t_1 t_2 + t_2^2,
\]
respectively.

The critical elements \( BG \to QS^0 \) are compositions of
\[
BG \xrightarrow{B\pi} B(\mathbb{Z}/2 \oplus \mathbb{Z}/2) = (\mathbb{R}P^\infty)^2
\]
with the maps \( \alpha(nx) : (\mathbb{R}P^\infty)^2 \to QS^0 \). We evaluated \( \tilde{f} = \alpha(-x) \) in the preceding remark. From (4.25) we get
\[
(4.27) \quad \tilde{f}_*(e_n \otimes e_m) = \tilde{f}_*(e_m \otimes e_n), \quad \tilde{f}_*(e_n \otimes e_0) = 0 \quad (n > 0).
\]
Consider now e.g. the cohomology of \( G = D8 \). In its spectral sequence
\[
d_3(t^2) = d_3(Sq^1 t) = Sq^1 d_2 t = Sq^1 (t_1^2 + t_1 t_2) = t_1^2 t_2 + t_1 t_2^2 = t_2 d_2 t = 0
\]
so that \( E^3 = E^\infty \) and
\[
H^*(D8) = P(s) \otimes P(t_1, t_2)/(t_1^2 + t_1 t_2)
\]
where \( s \in H^2(D8) \) is any element whose image is \( t^2 \in H^2(\mathbb{Z}/2) \), and \( t_1 \) and \( t_2 \) come from \( H^*(\mathbb{Z}/2 \oplus \mathbb{Z}/2) \). Thus the image of \( B\pi^* \) in \( H^n(D8) \) is generated by the elements \( t_1^n = t_1^{n-1} t_2 = \ldots = t_1 t_2^{n-1} \) and \( t_2^n \). Dually the image of \( B\pi_\ast \) in \( H^n(\mathbb{Z}/2 \oplus \mathbb{Z}/2) \) is generated by
\[
e_n \otimes e_0 + e_{n-1} \otimes e_1 + \ldots + e_1 \otimes e_{n-1} \quad \text{and} \quad e_0 \otimes e_n.
\]
From (4.27) \( \tilde{f}_*(\text{Im } B\pi_\ast) = 0 \). Hence all maps \( \alpha_{D8}(nx) = (-n\tilde{f}) \circ B\pi \) vanish in \( \mathbb{Z}/2 \)-homology.

A similar computation shows that \( \tilde{f} \circ B\pi \) induces the zero map \( H_\ast(Q8) \to H_\ast(Q_0 S^0) \). In fact here \( \text{Im } B\pi^* = 0 \) from dimension 4 on. Finally for \( G = \mathbb{Z}/4 \oplus \mathbb{Z}/2 \) we get that \( (\tilde{f} \circ B\pi)_\ast \) is non-trivial precisely in dimension 3. But then \( (2\tilde{f} \circ B\pi)_\ast \) vanishes.

By Proposition 3.1 these maps induce 0 also in \( K \)-theory. We pose the
Question. Are the maps

$$f_\ast: BG \to \mathbb{RP}_\infty \times \mathbb{RP}_\infty \to Q_0 S^0,$$

where $G=\mathbb{Z}/4 \oplus \mathbb{Z}/2, D_8, Q_8$ and $x=2-\eta_1^1-\eta_1^2-\eta_1^3+\eta_2 \in A_0(\mathbb{Z}/2 \oplus \mathbb{Z}/2)$ homotopic to zero?

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