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FLAT AND PROJECTIVE MODULES

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In this paper we study rings in which finitely generated flat modules are projective.

First we show that if the associated Lie ring of the ring R, is nilpotent, then cyclic flat left modules are projective if and only if cyclic flat right modules are projective. Next we give an example of a ring R with solvable Lie ring and a non projective cyclic flat left module, but having all cyclic flat right modules projective.

In the second part of the paper we study the trace ideal of the finitely generated flat module and we prove that for any ideal I of a commutative ring R, there exist a commutative ring S and a cyclic flat S-module, M, such that R is a subring of S and the trace ideal of M is IS.

In the last part of the paper we study, when projectivity for finitely generated flat modules can be lifted modulo the Jacobson radical.

1.

In [2] it is shown that if $M=R/I$ is a cyclic flat left R-module, then there exists a family of elements $a_i \in I$ ($i \in \mathbf{N}$) satisfying $a_i a_{i+1}=a_i$ for all i such that M is R-projective if and only if there exists an $i \in \mathbf{N}$ such that

$$Ra_i = Ra_{i+1} = \ldots = Ra_{i+k} = \ldots .$$

Furthermore it follows from $Ra_i=Ra_{i+1}$, that a_{i+1} is idempotent and hence $a_{i+2}a_{i+1}=a_{i+2}$.

Let us also recall that every cyclic flat left R-module is projectove if and only if each ascending chain of principal left ideals

$$Ra_1 \subseteq \ldots \subseteq Ra_n \subseteq \ldots, \quad \text{where } a_n=a_n a_{n+1} ,$$

terminates.

2.

If R is a ring, the associated Lie ring is denoted $G(R)$ and the product in $G(R)$ is $[x,y]=xy-yx$.

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C_n is defined by induction as follows, $C_0 = G(R)$ and

$$C_{n+1} = \{[x,y] \mid x \in G(R) \text{ and } y \in C_n\} .$$

D_n is also defined by induction. $D_0 = G(R)$ and

$$D_{n+1} = \{[x,y] \mid x,y \in D_n\} .$$

Recall that $G(R)$ is said to be nilpotent if $C_n = 0$ for some n and solvable if $D_n = 0$ for some n.

PROPOSITION 1. *Let R be a ring and suppose $G(R)$ is nilpotent. Every cyclic flat right R-module is projective if and only if every cyclic flat left R-module is projective.*

PROOF. Suppose that every cyclic flat right R-module is projective. To show that every cyclic flat left R-module is projective it suffices to prove that each ascending chain

$$(*) \qquad Ra_1 \subseteq \ldots \subseteq Ra_n \subseteq \ldots, \qquad a_n = a_n a_{n+1} ,$$

terminates.

By our assumption there exists an n such that $C_n = 0$. Note that $[a_i, a_{i+1}] = a_{i+1}a_i - a_i$ and $[a_{i+1}a_i - a_i, a_i] = a_{i+1}a_i^2 - a_i^2$. Now it is easily seen that for all n

$$a_{i+1}a_i^n - a_i^n \in C_n .$$

Thus $a_{i+1}a_i^n = a_i^n$ for all i.

Let I denote the right ideal generated by a_i^n, $i \in \mathbf{N}$.

R/I is a flat right R-module, since $a_{i+1}^n a_i^n = a_i^n$, and consequently R/I is projective. From the remarks in section 1 it follows that there exists an m_0 such that a_m^n is idempotent for $m \geqq m_0$ and moreover $a_m^n x_m = a_{m+1}^n$ for a suitable x_m. If we multiply this last equation by a_m^n on the left, we get $a_{m+1}^n = a_m^n$. Since $a_{m+1}a_{m+2}^n = a_{m+1}$, we get the following

$$Ra_{m+1} \subseteq Ra_{m+2}^n = Ra_{m+1}^n = Ra_m^n \subseteq Ra_m$$

and the proof of proposition 1 is completed.

We finish this section by an example.

EXAMPLE 1. The ring R defined below has the following properties:

(i) Cyclic flat right modules are projective.
(ii) $G(R)$ is solvable.

(iii) There exists a non-projective cyclic flat left R-module.

(iv) $R/\text{rad}\,(R)$ is commutative and there exists a cyclic flat, non-projective $R/\text{rad}\,(R)$-module. (Here rad (R) denotes the prime-radical of R).

Construction of the ring R. Let R be the $\mathbf{Z}_2$-algebra on the generators x_i, $i \in \mathbf{N}$, and defining relations

(1) $$x_i x_{i+1} = x_i \quad \text{for all } i \in \mathbf{N}\,.$$

(2) $$x_{i+j} x_{i+k} x_i = x_{i+j} x_i \quad \text{for all } i, j \text{ and } k \in \mathbf{N}\,.$$

We have a homomorphism from R to the commutative ring $S = \mathbf{Z}_2[t_i]_{i \in \mathbf{N}}$, where $t_i t_{i+1} = t_i$ for all $i \in \mathbf{N}$, sending x_i to t_i, thus $x_i \neq 0$.

Let I denote the keft ideal in R generated by the x_i's. R/I is a cyclic flat left R-module and non-projective, since I is not generated by a single one of the x_i's.

We prove that $G(R)$ is solvable by proving that $D_2 = 0$. It is obvious that it suffices to prove that $[[y_1, y_2], [y_3, y_4]] = 0$, when y_j is a monomial in the x_i's. From (1) and (2) it follows that any monomial in the x_i's either is of the form $x_{i+k} x_i^p$ or of the form x_j^q. It follows now that D_1 is generated by elements $(1 + x_j) x_i^p$, $j > i$, and $(x_{i+k} + x_{i+t}) x_i^p$. It is now straightforward to check that $D_2 = 0$.

By considering the two cases $i \geqq j$ and $i < j$ one can easily prove the following two equations

$$(x_{i+1} x_i - x_i) x_{j+k} x_j^p (x_{i+1} x_i - x_i) = 0$$

$$(x_{i+1} x_i - x_i) x_j^p (x_{i+1} x_i - x_i) = 0\,.$$

Consequently $(x_{i+1} x_i - x_i) R (x_{i+1} x_i - x_i) = 0$, so $x_{i+1} x_i - x_i$ belongs to rad (R). Hence $R/\text{rad}\,(R)$ is a commutative ring and since S has no nilpotent elements it is easily seen that $R/\text{rad}\,(R)$ is isomorphic to S. The module $S/\sum_i S t_i$ is a cyclic flat non-projective S-module, thus (iv) follows.

The argument that shows that (i) holds is similar to that in [2, p. 209], but slightly more complicated. We have to prove that any ascending chain of principal right ideals of the form

$$(a_1) \subseteqq \ldots \subseteqq (a_n) \subseteqq \ldots, \qquad \text{where } a_{n+1} a_n = a_n\,,$$

terminates. As noted, any a_i is a sum of monomial of one of the two forms $x_{i+k} x_i^p$, x_j^q, $p, q \in \mathbf{N} \cup \{0\}$.

Let us first assume that infinitely many of the a_i's have a non-zero constant term, i.e. constant term 1. Then we can suppose that we have an ascending chain of principal right ideals of the form

(**) $$(1 + b_1) \subseteqq \ldots \subseteqq (1 + b_n) \subseteqq \ldots\,,$$

where $(1+b_{n+1})(1+b_n)=(1+b_n)$ for all n and all b_n's have zero constant term. We get that $b_{n+1}b_n=b_{n+1}$ for all n. In b_{n+1} we pick a monomial $x_m x_k^p$ such that k is chosen maximal and among the monomials of the form $x_i x_k^q$ we pickone with q minimal, then it follows that b_n must be a sum of monomials, where at least one of these is of the form $x_n x_{k+r}^q$, now it is easily seen that (**) must terminate.

We can now assume without loss of generality that all a_n's have zero constant term. Among all monomials appearing in the a_i's pick $a_m^r a_t^q$, $t<m$ and $r=0$ or 1, such that m is minimal and among all these monomials pick one where t is maximal and next pick one with q minimal. Suppose this monomial appears in the representation of a_n as a sum of monomials. It is now straightforward to check that the relation $a_{n+1}a_n=a_n$ can not hold.

One might recall that for a commutative ring cyclic flat modules are projective if and only if cyclic flat modules over the ring moculo the prime-radical are projective.

3.

It is well-known and quite easy to prove that the trace ideal of a projective module is an idempotent ideal. If moreover the ring is commutative the trace ideals of the projective modules are precisely the pure ideals of the ring, [3]. Here we prove the following result:

PROPOSITION 2. *Let R be a commutative ring and $\mathfrak{a}$ any ideal in R. Then there exists a commutative ring S, containing R as a subring, and a cyclic flat S-module M such that the trace ideal of M equals $\mathfrak{a}S$.*

PROOF. Let S_1 be the polynomial ring in countable many indeterminates X_i, $i \in \mathbf{N}$, and let I be the ideal generated by

$$X_n X_{n+1} - X_n \ (n \in \mathbf{N}) \quad \text{and} \quad aX_n \ (a \in \mathfrak{a},\ n \in \mathbf{N})\,.$$

Let S be the factor ring S_1/I, and M the factor module $S/\sum_i Sx_i$, where x_i denotes the image of X_i under the canonical homomorphism $S_1 \to S_1/I$.

M is a flat S-module by the results in section 1.

We have a R-homomorphism from S_1 to R given by $X_1 \to 0$. Each element in I is mapped to 0, so it follows that R is a subring of S.

It remains to show that the trace ideal of M is $\mathfrak{a}S$. Let us first note that if $S/\mathfrak{b}$ is a cyclic module, the trace ideal must be contained in $\bigcap_{b\in\mathfrak{b}} S(1-b)$. If M_0 is a cyclic flat module and $r \in \bigcap_{b\in\mathfrak{b}} S(1-b)$, then r must belong to the trace ideal of M_0. To prove the last claim we just have to prove that $rb=0$ for all $b \in \mathfrak{b}$. Let

$b \in \mathfrak{b}$, and choose $b_1 \in \mathfrak{b}$, such that $bb_1 = b$ (this is possible since M is flat). We can write $r = t(1-b_1)$ for a suitable $t \in R$, and if we multiply this equation by b, we get $rb = 0$.

Now let us return to the proof of proposition 2. We have to show that

$$\bigcap_{i\in \mathsf{N}} S(1-x_i) = \mathfrak{a}S .$$

If $a \in \mathfrak{a}$, then $a = ax_i$ for all i, consequently $\mathfrak{a}S$ is contained in $\bigcap_{i\in \mathsf{N}} S(1-x_i)$.

So assume on the contrary that $b \in \bigcap_{i\in \mathsf{N}} S(1-x_i)$. We must have $bx_i = 0$, since b belongs to the trace ideal of M. Let us write $b = b_0 + b_1$ with $b_0 \in R$ and $b_1 \in \sum_i Sx_i$. We can choose an element x_j such that $b_1 x_j = b_1$, thus we have

$$b_1 x_k = b_1 \quad k \geqq j .$$

Now we get the following equation

$$bx_k = b_0 x_k + b_1 = 0;$$

consequently $b_1 = -b_0 x_k$, and hence $b = b_0(1-x_j) = b_0(1-x_{j+1})$ and $b_0(x_j - x_{j+1}) = 0$.

In the ring S_1 we must have $b_0(x_j - x_{j+1}) \in I$. Therefore

$$b_0(x_j - x_{j+1}) = af_1(x_1,\ldots,x_m) + \sum_n g_n(x_1,\ldots,x_{p_n})(x_n x_{n+1} - x_n) .$$

If we reduce the coefficients in each polynomial modulo $\mathfrak{a}$, we get

$$b_0(x_j - x_{j+1}) = \sum g_n(x_1,\ldots,x_{p_n})(x_n x_{n+1} - x_n) .$$

From this last equation it follows that $b_0 = 0$, i.e. $b_0 \in \mathfrak{a}$, hence $b = b_0(1-x_j) \in \mathfrak{a}S$.

REMARK. If in proposition 2 $\mathfrak{a}$ is a nilpotent ideal, then the trace ideal of the flat module M is nilpotent. Thus one can not conclude that the trace ideal of a finitely generated flat module is idempotent.

4.

In this last section we study the following open problem:

Is it true for all rings R and all finitely generated flat left R-modules M with M/JM R/J-projective, that M is R-projective? (J denotes the Jacobson radical of R).

The problem is known to have a positive solution in each of the following cases

(a) R is commutative [5].
(b) Each prime factor ring of R is a Goldie ring [4].
(c) J is equal to the prime-radical of R [4].

By Morita technique it suffices to solve the problem in case M is a cyclic module. Moreover it follows from the results in [4] that one might assume without loss of generality that the cyclic module $M = R/I$ has $\mathrm{hd}_R\, M \leqq 1$, i.e. I is projective. In case idempotents can be lifted modulo J we might also note that the problem has a positive solution, because in this case

$$I = Re \oplus (I \cap R(1-e)),$$

where e is an idempotent in I such that $e+J$ generates $I+J$. Since I is projective, $I \cap R(1-e)$ is also projective, but this module is equal to $J(I \cap R(1-e))$, thus zero. So I is finitely generated, and consequently M is projective.

We are going to prove that to solve the problem one may assume R/J isomorphic to $K \oplus K$, where K either is a finite field or the ring of integers.

Suppose that there exists a ring R having a cyclic flat non-projective module M, such that M/JM is R/J-projective. By the results in [4] it follows that we might assume that the ring is a prime ring. From the remarks in the introduction of this paper we get that there exists a family of element $(a_i)_{i \in N}$, such that $a_i a_{i+1} = a_i$ and $a_m \equiv a_m a_{n_0}$ modulo J for $m \geqq n_0$. If we leave out a finite number of the a_i's we get elements a_i, $i \in \mathbf{N}$, such that $a_i = a_i a_{i+1}$ and $a_m - a_m a_1 \in J$ for all m.

Let the prime ring of R be denoted by K; K is either a finite field or the ring of integers.

We let C_0 denote the least subring of R containing K and the a_i's. C_1 denotes the least subring of R containing C_0 and the inverses of the elements

$$u_{c_1,t} = 1 - c_1(a_t - a_t a_1) \quad (c_1 \in C_1,\ t \in \mathbf{N}).$$

C_n is defined by induction: C_{n+1} is the least subring of R containing C_n and the inverses of the elements

$$u_{c_n,t} = 1 - c_n(a_t - a_t a_1) \quad (c_n \in C_n,\ t \in \mathbf{N}).$$

For all n, we have $C_n \subseteqq C_{n+1}$; S denotes the union of all the C_n's.

The module $S/\sum_i Sa_i$ is a cyclic flat S-module, since $a_i = a_i a_{i+1}$. It is not projective, because S is a subring of R ([1, Theorem 3.1]), and by the construction of the ring S we have $a_n - a_n a_1 \in J(S)$; hence the module $S/\sum_i Sa_i$ is projective modulo $J(S)$.

If we can prove that $S/J(S)$ is isomorphic to $K\oplus K$, then we have reduced the general problem to the case where R/J is isomorphic to two copies of a finite field or to two copies of the ring of integers.

Let us first note that

$$u_{c_n,t}^{-1}(1-u_{c_n,t}) = 1-u_{c_n,t}^{-1} \in J(S) \quad \text{for all } n \text{ and } t .$$

Thus any elements'in S is congruent to an element in C_0 modulo $J(S)$. Having this in mind it is readily checked that $a_n S(1-a_n) \in J(S)$ for all $n \in \mathbf{N}$.

Let us assume that K is a finite field. In case K is the ring of integers a similar argument works.

For any primitive ideal P in S, we must have $a_i \in P$ or $(1-a_i) \in P$. If $a_i \in P$ for some i, then $a_1 = a_1 a_1 \in P$. Now, for any $m \in \mathbf{N}$, we get the following

$$a_m \in Sa_1 + J(S) \subsetneqq Sa_1 + P \subsetneqq P .$$

S modulo the twosided ideal generated by the a_i's is clearly isomorphic to K, so P must equal that ideal.

On the other hand, if $(1-a_i) \in P$ for all i, then

$$u_{c_n,t} = 1-c_n a_t(1-a_1)$$

is congruent to 1 modulo P. Thus we get that S/P is isomorphic to K.

We have now proved that S has two primitive ideals P_1 and P_2; both ideals are maximal left and right ideals. Hence $S/J(S)$ is isomorphic to $S/P_1 \oplus S/P_2$, which is isomorphic to $K\oplus K$.

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