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A PROOF OF THE MACKEY–BLATTNER–NIELSEN THEOREM

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1. Introduction.

Let G be a locally compact group, H and K closed subgroups, u a (strongly continuous unitary) representation of H on the Hilbert space $h(u)$, and v a representation of K on $h(v)$. Let U denote the left regular and V the right regular representation of G on $L^2(G)$. Let $\operatorname{ind} u$ denote the representation of G induced by u, as defined e.g. in [13], and p_u the corresponding representation of $L^\infty(G/H)$ on $h(\operatorname{ind} u)$. We prove that the von Neumann algebra

$$[L^\infty(G)\otimes 1\otimes 1]' \cap (V\otimes u\otimes 1)(H)' \cap (U\otimes 1\otimes v)(K)'$$

on $L^2(G)\otimes h(u)\otimes h(v)$ is isomorphic to the von Neumann algebra

$$[p_u(L^\infty(G/H))\otimes 1]' \cap ((\operatorname{ind} u)\otimes v)(K)'$$

on $h(\operatorname{ind} u)\otimes h(v)$.

With an appropriate definition of the representation $\operatorname{ind}' v$ (say) of G induced "from the right" by v and the corresponding representation p_v' of $L(K\setminus G)$ on $h(\operatorname{ind}' v)$, we get a symmetric result, and so as a corollary we find, that the von Neumann algebra

$$[p_u(L^\infty(G/H))\otimes 1]' \cap ((\operatorname{ind} u)\otimes v)(K)'$$

on $h(\operatorname{ind} u)\otimes h(v)$ is isomorphic to the von Neumann algebra

$$[1\otimes p_v'(L^\infty(K\setminus G))]' \cap (u\otimes \operatorname{ind}' v)(H)'$$

on $h(u)\otimes h(\operatorname{ind}' v)$. Nielsen's theorem [13], cf. [14], is the case $h(v)=\mathbf{C}$, $v\equiv 1$, and the Mackey–Blattner theorem [10], [1], is the subcase $K=G$.

Our proof uses [7] and the Maréchal [11], [12] and Vesterstrøm–Wils [15] theory of disintegration. In [8] we give new proofs of the relevant results of that theory, based on the notion of essential values of a measurable map instead of on the existence of a lifting. So Nielsen's theorem is proved without the use of liftings.

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We use freely [2], [3], [4], [5] and [6].

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2. Preliminaries.

Let T be a locally compact space, μ a positive Radon measure on T, and S a topological space. We call a map f of T into S Lusin measurable if for any $\varepsilon > 0$ any compact subset of T is the union of a compact set on which f is continuous and a set with measure less than ε, cf. [2]. If h is a Hilbert space and $S = \mathscr{L}(h)$, we call f a measurable field, if $t \mapsto f(t)\xi$ is Lusin measurable for each $\xi \in h$. By a disintegration of an operator $B \in \mathscr{L}(L^2(\mu, h))$ (which we identify with $\mathscr{L}(L^2(\mu)) \otimes \mathscr{L}(h)$) we mean a bounded map $b \colon T \to \mathscr{L}(h)$, such that b and b^* are measurable fields, and such that for each $f \in \mathscr{L}^2(\mu, h)$, $t \mapsto b(t)(f(t))$ is a function in $\mathscr{L}^2(\mu, h)$ whose class (with respect to equality l.a.e.) is the image under B of the class of f. In [12] and [15], cf. [8], it is proved that any $B \in \mathscr{L}(L^2(\mu, h))$ commuting with all multiplication operators by functions in $\mathscr{L}^\infty(\mu)$ has a disintegration.

Now assume given a locally compact group G with left Haar measure dg and module Δ_G, and a closed subgroup H with left Haar measure $d\gamma$ and module Δ_H. Let π denote the quotient map $G \to G/H$. Let λ be a quasiinvariant measure on G/H and $\varkappa$ a corresponding continuous function on $G \times (G/H)$ with values in $]0, \infty[$, such that

$$\int_{G/H} \varphi(g^{-1}x)\, d\lambda(x) = \int_{G/H} \varphi(x)\varkappa(g, x)\, d\lambda(x), \qquad \varphi \in \mathscr{K}(G/H) .$$

Recall that $\varrho(g) = \varkappa(g, H)$ defines a continuous function on G with

$$\varrho(g\gamma) = \varrho(g)\Delta_H(\gamma)\Delta_G(\gamma)^{-1}, \qquad g \in G,\ \gamma \in H ,$$

and $\varrho(e) = 1$, and $\lambda^\# = \varrho dg$.

Given a representation u of H on $h(u)$ define the induced representation $\operatorname{ind} u$ of G and the representation p_u of $L^\infty(G/H) = L^\infty(\lambda)$ on $\mathscr{F}(u)$ and $h(\operatorname{ind} u)$ as usual, see e.g. [13] or [7].

Let K be a closed subgroup of G with left Haar measure dk, and let λ_K be a quasiinvariant measure on $K \setminus G$ and $\varkappa_K$ a corresponding continuous function on $(K \setminus G) \times G$ with values in $]0, \infty[$, such that

$$\int_{K \setminus G} \varphi(xg)\, d\lambda_K(x) = \int_{K \setminus G} \varphi(x)\varkappa_K(x, g^{-1})\, d\lambda_K(x) .$$

Given a representation v of K define the representation $\operatorname{ind}' v$ induced "from the right", cf. [9], on the Hilbert space $h(\operatorname{ind}' v)$ of classes corresponding to the space $\mathscr{F}'(v)$ of Lusin measurable functions f on G with values in $h(v)$ satisfying

$$\forall g \in G \quad \forall k \in K: \quad f(kg) = v(k)(f(g)),$$

and with norms transferable to λ_K square integrable functions on $K \setminus G$, by

$$((\operatorname{ind}' v)(g)f)(k) = \varkappa_K(Kk, g)^{\frac{1}{2}} f(kg), \qquad k, g \in G,\ f \in \mathscr{F}'(v)$$

and define

$$(p'_v(\varphi)f)(g) = \varphi(Kg)f(g), \qquad g \in G,\ f \in \mathscr{F}'(v),\ \varphi \in \mathscr{L}^\infty(K \setminus G).$$

Let U denote the left regular representation of G on $L^2(G)$, given by

$$(U(g)f)(k) = f(g^{-1}k), \qquad g, k \in G,\ f \in \mathscr{L}^2(G),$$

and let V denote the right regular representation given by

$$(V(g)f)(k) = \Delta_G(g)^{\frac{1}{2}} f(kg), \qquad g, k \in G,\ f \in \mathscr{L}^2(G).$$

Let p denote the representation of $L^\infty(G)$ as multiplication operators on $L^2(G)$. Let q denote the representation of $L^\infty(G/H)$ on $L^2(G)$ defined by

$$(q(\varphi)f)(g) = \varphi(gH)f(g), \qquad g \in G,\ f \in \mathscr{L}^2(G),\ \varphi \in \mathscr{L}^\infty(G/H),$$

and let r denote the representation of $L^\infty(G/H)$ on $L^2(G/H)$ defined by

$$(r(\varphi)f)(x) = \varphi(x)f(x), \qquad x \in G/H,\ f \in \mathscr{L}^2(G/H),\ \varphi \in \mathscr{L}^\infty(G/H).$$

LEMMA 1. *The von Neumann algebra* $p(L^\infty(G)) \cap V(H)'$ *on* $L^2(G)$ *is equal to* $q(L^\infty(G/H))$.

PROOF. It is obvious that $q(L^\infty(G/H)) \subseteq p(L^\infty(G)) \cap V(H)'$. Let T be an operator in $p(L^\infty(G)) \cap V(H)'$. Choose $f \in \mathscr{L}^\infty(G)$ such that $T = p(f)$. For any $\gamma \in H$ and $\varphi \in \mathscr{K}(G)$, $\varphi \geqq 0$, we have

$$\Delta_G(\gamma) \int_G \varphi(g)\varrho(g\gamma)f(g\gamma)\,dg$$

$$= \int_G \varphi(g\gamma^{-1})\varrho(g)f(g)\,dg = \int_G f(g)\Delta_H(\gamma)\Delta_G(\gamma)^{-1}\varrho(g\gamma^{-1})\varphi(g\gamma^{-1})\,dg$$

$$= \Delta_H(\gamma)(TV(\gamma^{-1})((\varrho\varphi)^{\frac{1}{2}}) \mid V(\gamma^{-1})((\varrho\varphi)^{\frac{1}{2}}))$$

$$= \Delta_H(\gamma)(T((\varrho\varphi)^{\frac{1}{2}}) \mid (\varrho\varphi)^{\frac{1}{2}}) = \Delta_H(\gamma) \int_G \varphi(g)\varrho(g)f(g)\,dg.$$

Therefore there exists a measure ν on G/H with $\nu^{\#} = \varrho f dg$, cf. [4 p. 44, Prop. 4b, or p. 56, Lemme 5]. Then ν is absolutely continuous with respect to λ, so $\nu = F\lambda$ for some locally integrable function F on G/H, and $f = F \circ \pi$ locally almost everywhere, since $\varrho f dg = (F\lambda)^{\#} = (F \circ \pi)\varrho dg$.

LEMMA 2. *Let h be a Hilbert space. Any operator A in the von Neumann algebra $q(L^{\infty}(G/H)) \otimes \mathscr{L}(h)$ on $L^2(G) \otimes h$ has a disintegration a satisfying*

$$\forall g \in G \quad \forall \gamma \in H: \quad a(g\gamma) = a(g) .$$

PROOF. Let O denote the C*-algebra of bounded maps $a: G/H \to \mathscr{L}(h)$ such that a and a^* are measurable fields. Given $a \in O$, let $a(\lambda)$ denote the operator on $L^2(G/H, h)$ defined by $(a(\lambda)f)(x) = a(x)(f(x))$, and let $a(G)$ denote the operator on $L^2(G, h)$ defined by $(a(G)f)(g) = a(gH)(f(g))$. Then $a \mapsto a(\lambda)$ is a homomorphism of O onto $r(L^{\infty}(G/H)) \otimes \mathscr{L}(h)$, with kernel

$$\{a \in O \mid \ \forall \xi \in h: a(x)\xi = 0 \text{ l.a.e.}\}$$

(see [12], [15], [8]) and $a \mapsto a(G)$ is a homomorphism of O onto a sub *algebra of $q(L^{\infty}(G/H)) \otimes \mathscr{L}(h)$ containing $q(L^{\infty}(G/H)) \otimes 1$ and $1 \otimes \mathscr{L}(h)$ and so weakly dense in $q(L^{\infty}(G/H)) \otimes \mathscr{L}(h)$, with the same kernel. To show that the injective homomorphism $a(\lambda) \mapsto a(G)$ is onto $q(L^{\infty}(G/H)) \otimes \mathscr{L}(h)$, it is enough to show that it is weak-weak continuous on normbounded sets. It is enough to show that $a(\lambda) \mapsto (a(G)\varphi \mid \varphi)$ is weakly continuous for a dense set of vectors $\varphi \in L^2(G, h)$. So let $f \in \mathscr{L}^2(G)$ and $\xi \in h$ be given; then

$$F(gH) = \left(\int_H |f(g\gamma)|^2 \varrho(g\gamma)^{-1} \, d\gamma \right)^{\frac{1}{2}}$$

defines a function in $\mathscr{L}^2(G/H)$, [4 p. 57], and

$$\begin{aligned} (a(G)f\xi \mid f\xi) &= \int_G |f(g)|^2 (a(gH)\xi \mid \xi) \, dg \\ &= \int_{G/H} |F(x)|^2 (a(x)\xi \mid \xi) \, d\lambda(x) = (a(\lambda)F\xi \mid F\xi) . \end{aligned}$$

3. Nielsen's Theorem.

LEMMA 3. *Any operator A in the von Neumann algebra*

$$[p(L^{\infty}(G)) \otimes 1]' \cap (V \otimes u)(H)'$$

on $L^2(G) \otimes h(u)$ has a disintegration a satisfying

$$\forall g \in G \quad \forall \gamma \in H: \qquad a(g\gamma) = u(\gamma)^{-1}a(g)u(\gamma) .$$

PROOF. Choose a unitary ϱ-extension $P: G \to u(H)''$ of u, with P and P^* measurable fields, [7, Theorem 1(a)]. Define an operator $Q \in \mathscr{L}(L^2(G, h(u)))$ by $(Qf)(g) = P(g)(f(g))$, $f \in \mathscr{L}^2(G, h(u))$. Then Q is unitary, and commutes with $p(L^\infty(G)) \otimes 1$, and

$$Q(V(\gamma) \otimes u(\gamma)) = (V(\gamma) \otimes 1)Q, \qquad \gamma \in H;$$

therefore

$$\begin{aligned} &Q([p(L^\infty(G)) \otimes 1]' \cap (V \otimes u)(H)')Q^{-1} \\ &= [p(L^\infty(G)) \otimes 1]' \cap [V(H) \otimes 1]' \\ &= [p(L^\infty(G))' \otimes \mathscr{L}(h(u))] \cap [V(H)' \otimes \mathscr{L}(h(u))] \\ &= (p(L^\infty(G))' \cap V(H)') \otimes \mathscr{L}(h(u)) \\ &= q(L^\infty(G/H)) \otimes \mathscr{L}(h(u)) . \end{aligned}$$

Here we have used that $p(L^\infty(G))$ is maximal abelian, and Lemma 1. When

$$A \in [p(L^\infty(G)) \otimes 1]' \cap (V \otimes u)(H)' ,$$

then by Lemma 2 QAQ^{-1} has a disintegration b satisfying

$$\forall g \in G \quad \forall \gamma \in H: \qquad b(g\gamma) = b(g) ,$$

and A has the disintegration $g \mapsto a(g) = P(g)^{-1}b(g)P(g)$ satisfying

$$\forall g \in G \quad \forall \gamma \in H: \; a(g\gamma) = u(\gamma)^{-1}a(g)u(\gamma) .$$

THEOREM. *Let u and v be representations of the closed subgroups H and K of the locally compact group G on the Hilbert spaces $h(u)$ and $h(v)$ respectively. The von Neumann algebra*

$$[p(L^\infty(G)) \otimes 1 \otimes 1]' \cap (V \otimes u \otimes 1)(H)' \cap (U \otimes 1 \otimes v)(K)'$$

on $L^2(G) \otimes h(u) \otimes h(v)$ is isomorphic to the von Neumann algebra

$$[p_u(L^\infty(G/H)) \otimes 1]' \cap ((\mathrm{ind}\, u) \otimes v)(K)'$$

on $h(\mathrm{ind}\, u) \otimes h(v)$.

PROOF. Let $u \otimes 1$ denote the representation $\gamma \mapsto u(\gamma) \otimes 1$ of H on $h(u) \otimes h(v)$; note that $h(\mathrm{ind}\,(u \otimes 1))$ is naturally identified with $h(\mathrm{ind}\, u) \otimes h(v)$, and that with this identification $p_{u \otimes 1} = p_u \otimes 1$.

Let B denote the C*-algebra of bounded maps $b: G \to \mathscr{L}(h(u) \otimes h(v))$, such that b and b^* are measurable fields, and

$$\forall g \in G \quad \forall \gamma \in H: \quad b(g\gamma) = (u\otimes 1)(\gamma)^{-1} b(g)(u\otimes 1)(\gamma) .$$

When $b \in B$ let $b(G)$ denote the operator on $L^2(G, h(u)\otimes h(v))$ defined by

$$(b(G)f)(g) = b(g)(f(g)), \quad g \in G,\ f \in \mathscr{L}^2(G, h(u)\otimes h(v)) ,$$

and let $\tilde{b}$ denote the operator on $h(\text{ind}\,(u\otimes 1))$ defined by

$$(\tilde{b}f)(g) = b(g)(f(g)), \quad g \in G,\ f \in \mathscr{F}(u\otimes 1) .$$

Then $b \mapsto b(G)$ is a homomorphism of B onto

$$[p(L^\infty(G))\otimes 1\otimes 1]' \cap (V\otimes u\otimes 1)(H)' ,$$

by Lemma 3, with kernel

$$\{b \in B \mid \quad \forall \xi \in h(u)\otimes h(v):\ b(g)\xi = 0 \text{ l.a.e.}\} ,$$

and $b \mapsto \tilde{b}$ is a homomorphism of B onto $p_{u\otimes 1}(L^\infty(G/H))$, by [7] Proposition 2, with the same kernel, by [7] Lemma 1. The counter image of $(U\otimes 1\otimes v)(K)'$ under $b \mapsto b(G)$ is

$$\{b \in B \mid \quad \forall k \in K \quad \forall \xi \in h(u)\otimes h(v):\ b(kg)\xi = [1\otimes v(k)]b(g)[1\otimes v(k)^{-1}]\xi \text{ l.a.e.}\} ,$$

and so is the counter image of $((\text{ind}\, u)\otimes v)(K)'$ under $b \mapsto \tilde{b}$.

COROLLARY. *The von Neumann algebra*

$$[p_u(L^\infty(G/H))\otimes 1]' \cap ((\text{ind}\, u)\otimes v)(K)' \quad \textit{on} \quad h(\text{ind}\, u)\otimes h(v)$$

is isomorphic to the von Neumann algebra

$$[1\otimes p'_v(L^\infty(K\setminus G))]' \cap (u\otimes \text{ind}'\, v)(H)' \quad \textit{on} \quad h(u)\otimes h(\text{ind}'\, v) .$$

PROOF. By symmetry

$$[1\otimes p'_v(L^\infty(K\setminus G))]' \cap (u\otimes \text{ind}'\, v)(H)'$$

is also isomorphic to

$$[p(L^\infty(G))\otimes 1\otimes 1]' \cap (V\otimes u\otimes 1)(H)' \cap (U\otimes 1\otimes v)(K)' .$$

Nielsen's theorem is the case $h(v) = \mathbf{C}$, $v \equiv 1$ of the corollary.

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