ESSENTIAL FUNCTION ALGEBRAS WITH LARGE SILOV BOUNDARY

P. J. DE PAEPE

1. Introduction.

In this note we construct, starting with a function algebra $A$ on its homomorphism space $\Delta A$, a function algebra $B$ on $\Delta B$ which shares a number of properties with $A$ but such that the Silov boundary $\partial B$ of $B$ equals $\Delta B$. The easiest way to do this is to consider $X = \Delta A \times [0, 1]$ and to take $B$ the algebra of all continuous functions $f$ on $X$ such that $f \upharpoonright \Delta A \times \{0\}$ belongs to $A$. Then $\Delta B = \partial B = X$ but $B$ is not essential on $\Delta B$. Our construction provides an algebra $B$ which is essential on $\Delta B$ and such that $\Delta B$ can be considered as a subset of $C^{n+2}$ if $\Delta A \subset C^n$.

2. Notation and definitions.

Let $X$ be a compact Hausdorff space. $C(X)$ will be the algebra of continuous complex-valued functions on $X$ with the supremum norm. A function algebra $A$ on $X$ is a closed subalgebra of $C(X)$ containing the constants and separating the points of $X$. The maximal ideal space of $A$ is denoted by $\Delta A$ and its Silov boundary by $\partial A$. If $K$ is a compact subset of $X$ and $f \in A \ f \upharpoonright K$ denotes the restriction of $f$ to $K$ and $[A \upharpoonright K]$ will be the closure in $C(K)$ of the restrictions to $K$ of elements of $A$. $A$ is called essential on $X$ if the minimal closed subset $K$ of $X$ with the property $f \in C(X)$ and $f \upharpoonright K = 0$ implies $f \in A$ equals $X$.

For a compact subset $X$ in $C^n$ $P(X)$ will be the closure in $C(X)$ of the algebra of polynomials on $X$ and $R(X)$ is the closure in $C(X)$ of the algebra of rational functions having poles outside of $X$.

For information on function algebras we refer to [6].

3. The construction.

Let $Y$ be a Swiss cheese (e.g. [5]) such that $R(Y) \neq C(Y)$ and $\Delta R(Y) = \partial R(Y) = Y$. $R(Y)$ is essential on $Y$. Let $A$ be a function algebra on a metrizable compact space $X$ with $\Delta A = X$. Choose a dense sequence $\{x_n\}$ in $X$.

Received June 8, 1978.
Consider in $X \times \mathbb{C}^2$ the following subsets:
\[
\tilde{X} = \{(x,0,0) : x \in X\}
\]
\[
Y_n = \{(x_n,t,y) : y \in tY, 0 \leq t \leq a_n\}
\]
where $tY = \{ty : y \in Y\}$, $n \in \mathbb{N}$ and $\{a_n\}$ is a decreasing sequence of positive numbers having 0 as limit. Let $Z$ be the union of the sets $\tilde{X}$ and $\{Y_n\}$ in $X \times \mathbb{C}^2$. Let $B$ be the algebra of all continuous functions $f$ on $Z$ such that

(i) the function on $X$ defined by $x \mapsto f(x,0,0)$ belongs to $A$ (abbreviation $f|\tilde{X} \in A$)

(ii) the function on $Y_{n,\alpha} = Y_n \cap \{t = \alpha\}$ defined by $y \mapsto f(x_n,\alpha,y)$ belongs to $R(\alpha Y)$ for all fixed $n$ and $0 < \alpha \leq a_n$ (abbreviation $f|Y_{n,\alpha} \in R(Y_{n,\alpha})$).

Then it is clear that

1. $Z$ is compact in $X \times \mathbb{C}^2$ and connected if $X$ is,
2. $B$ is a function algebra on $Z$,

and we will show

3. $\Delta B = Z$,
4. $\partial B = Z$,
5. $B$ is essential on $Z$.

4. Proofs.

Proof of Assertion 3. Consider the function $t$ on $Z$ and let $\varphi \in \Delta B$. Then $\varphi(t) \in [0,a_1]$, let $\varphi(t) = \alpha$. Let $f_\alpha$ be a continuous function on $[0,a_1]$ of the variable $t$ peaking at $\alpha$ and consider $f_\alpha$ as a function on $Z$. Then $f_\alpha \in B$. Using this function it is easily seen that

\[
\varphi \in \Delta[B|Z \cap \{t = \alpha\}].
\]

If $\alpha = 0$, then $\varphi \in \Delta[B|\tilde{X}] = \tilde{X}$, since $B|\tilde{X} \simeq A$.

If $\alpha \neq 0$ then $Z \cap \{t = \alpha\}$ consists of a finite union of disjoint sets $Y_{n,\alpha}$:

\[
Z \cap \{t = \alpha\} = Y_{1,\alpha} \cup \ldots \cup Y_{n(\alpha),\alpha},
\]

with $n(\alpha)$ the greatest integer such that $a_{n(\alpha)} \geq \alpha$. Let $\mu$ be a Jensen representing measure for $\varphi \in \Delta[B|Z \cap \{t = \alpha\}]$ on $Z \cap \{t = \alpha\}$. If $\mu(Y_{n,\alpha}) > 0$ for some $n$ then consider a function $g_{n,\alpha} \in B$ such that $g_{n,\alpha} = 1$ on $Y_{n,\alpha}$, $g_{n,\alpha} = 0$ on $Z \setminus Y_n \cap \{t \geq \alpha/2\}$ and $0 \leq g_{n,\alpha} < 1$ on $Z \setminus Y_{n,\alpha}$. Now

\[
\log|\varphi(1 - g_{n,\alpha})| \leq \int_{Z \cap \{t = \alpha\}} \log|1 - g_{n,\alpha}| d\mu = -\infty
\]

since $\mu(Y_{n,\alpha}) > 0$. So because $\mu$ represents $\varphi$ we have $\mu(Y_{n,\alpha}) = 1 = \|\mu\|$. Hence
\[ \varphi \in \mathcal{A}(B \mid Y_{n,z}) = Y_{n,z} \]

since \( B \mid Y_{n,z} \cong R(Y_{n,z}) \). So if \( \varphi \in \mathcal{A}B \) then \( \varphi \in Z \). Hence \( \mathcal{A}B = Z \).

**Proof of Assertion 4.** Consider a peak point \( y_0 \in Y \) for the algebra \( R(Y) \) with peak function \( h \) (these points are dense in \( Y \)). Now \( z = (x_n, \alpha, \alpha y_0) \) where \( 0 < \alpha \leq a_n \) is in \( Z \) and the function \( h(y/\alpha)g_{n,z} \) peaks at \( z \) and belongs to \( B \). Moreover points \( z \) of the above type are dense in \( Z \). Hence \( \partial B = Z \).

**Proof of Assertion 5.** Let \( K \) be a proper compact subset of \( Z \) then \( Z \setminus K \) contains an open subset of some \( Y_{n,a} \), \( a > 0 \). Since \( R(Y_{n,a}) \) is essential on \( Y_{n,a} \), not every element of \( C(Z) \) vanishing on \( K \) belongs to \( B \), hence \( B \) is essential on \( Z \).

5. Remarks.

In [4] Glicksberg posed the following question: if \( B_1 \subset B_2 \subset C(\mathcal{A}B_1) \) and \( \partial B_1 = \partial B_2 \), must \( \mathcal{A}B_1 = \mathcal{A}B_2 \) ?

The answer is negative and a well-known example is the following: let \( A_1 \) and \( A_2 \) be function algebras on \( X \) such that \( A_1 \subset A_2 \) and \( X = \mathcal{A}A_1 \neq \mathcal{A}A_2 \). Then the algebras \( B_1 \) and \( B_2 \) on \( X \times [0,1] \) of all continuous extensions to \( X \times [0,1] \) of elements of \( A_1 \), respectively \( A_2 \), on \( X \times \{0\} \) provide a counter example.

Using our construction it is possible to obtain \( B_1 \) and \( B_2 \) essential algebras on \( \mathcal{A}B_1 \): start with \( A_1 \) (on a metrizable space \( X \)) and construct \( B_1 \) as in section 3. Let \( B_2 \) be all continuous functions \( f \) on \( Z \) such that \( f \mid Y_{n,a} \in R(Y_{n,a}) \), \( 0 < \alpha \leq a_n \), \( n \in \mathbb{N} \), and \( f \mid \tilde{X} \in A_2 \) (notation of section 3). Then

\[ \mathcal{A}B_2 = Z \cup \{ (z,0,0) : z \in \mathcal{A}A_2 \setminus X \} \neq \mathcal{A}B_1 = Z. \]

But \( \partial B_1 = \partial B_2 = Z \).

In [1] and [2] Csordas and Reiter asked for a solution of the following problem: is there a non-separating essential function algebra \( B \) on a (connected) space \( X \) for which \( \mathcal{A}B = \partial B = X \)? (a function algebra \( A \) on \( X \) is called separating if for every proper compact subset \( Y \) of \( X \) and every point \( x \) in \( X \setminus Y \) there exists \( f \in A \) such that \( f(x) \notin f(Y) \)). The answer is positive and given by Eifler [3].

Starting with a non-separating function algebra \( A \) on a connected metrizable space \( X \) with \( \mathcal{A}A = X \) and using the construction in section 3 we find a function algebra \( B \) satisfying all statements in the question of Csordas and Reiter. For \( A \) one can take \( P(X) \) where \( X \) is the unit polydisc in \( C^2 \), \( B \) then becomes a function algebra on a compact set in \( C^4 \). Eifler’s example shows some similarity with ours: the crucial ingredient is again the algebra \( R(Y) \), \( Y \) a Swiss cheese. His algebra \( B \) is a subalgebra of \( R(Y) \) which is a function algebra on a quotient space of \( Y \).
REFERENCES


MATHEMATISCH INSTITUUT
UNIVERSITEIT VAN AMSTERDAM, THE NETHERLANDS

Author's present address:
INSTITUUT VOOR PROPEDEUTISCHE WISKUNDE
UNIVERSITEIT VAN AMSTERDAM, THE NETHERLANDS