SCATTERED C*-ALGEBRAS II

HELGE ELBRØND JENSEN

This note is a continuation of [3], in which scattered C*-algebras were introduced as those C*-algebras, for which each positive functional is a countable sum of pure functionals. This corresponds in the commutative case to an algebra $C_0(X)$, where $X$ is a locally compact, scattered space.

In [3] it was proved, among other things, that if a C*-algebra $A$ has a decomposition series $(I_p)_{0 \leq p \leq n}$ such that each algebra $I_{p+1}/I_p$ is isomorphic with an elementary C*-algebra, then $A$ is scattered. In the following we shall prove the converse of this result.

**Lemma 1.** Let $A$ be a C*-algebra with continuous trace. If $A$ is scattered, then $A$ is a scattered space.

**Proof.** We will prove, that each bounded Radon measure on $A$ has an atom. Then result then follows from [4, 19.7.6].

Let $\mu$ be such a measure. From [1, 4.5] we see, that there exists a compact set $E \subseteq A$ with $\mu(E) > 0$ and an element $e \in A$, such that $\pi(e)$ is a one-dimensional projection for all $\pi \in E$. Moreover, the function $\pi \rightarrow tr\pi(xe)$ is continuous on $E$ for each $x \in A$, where $tr$ denotes the ordinary trace. Define a positive functional $\varphi$ on $A$ by

$$\varphi(x) = \int_E tr(\pi(xe)) d\mu(\pi), \quad x \in A.$$ 

Since $A$ is scattered, we have $\varphi = \sum \lambda_i \psi_i$, where each $\psi_i$ is a pure state. Therefore, since $\varphi(e) = \mu(E) > 0$, there is a pure state $\psi$ and a number $\lambda > 0$, such that $\lambda \psi \leq \varphi$ and $\psi(e) = \alpha > 0$.

Now, let $\pi_0$ denote the irreducible representation corresponding to $\psi$, and suppose, that $\mu(\pi_0) = 0$. For each $\varepsilon > 0$ we can find an open set $D \subset A$ with $\mu(D) < \varepsilon$ and $\pi_0 \in D$, and a continuous function $f: \hat{A} \rightarrow [0, 1]$, such that $f(\pi_0) = 1$ and $f(\pi) = 0$ for $\pi \notin D$. Let $f(e)$ denote the unique element in $A$, for which $\pi(f(e)) = f(\pi)\pi(e)$ for all $\pi \in \hat{A}$. (Dauns–Hofmann’s theorem, see f.ex. [2] about this).

We then have

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\[
\varphi(f \cdot e) = \int_E f(\pi) \text{tr} \pi(e) d\mu(\pi) < \varepsilon
\]

and for some vector \( \xi \) in the Hilbert space for \( \pi_0 \)

\[
\psi(f \cdot e) = (\pi_0(f \cdot e)\xi | \xi) = (\pi_0(e)\xi | \xi) = \psi(e) = \alpha.
\]

Therefore, since \( \lambda \psi \leq \varphi \), we have \( \lambda \alpha < \varepsilon \), which is impossible for subsets \( D \) with sufficiently small measure. Consequently \( \mu(\pi_0) > 0 \), and the lemma is proved.

**Theorem 2.** A C*-algebra \( A \) is scattered if and only if \( A \) has a decomposition series \( (I_p)_{0 \leq p \leq \infty} \), such that each algebra \( I_{p+1}/I_p \) is elementary.

**Proof.** One half of the theorem follows from [3, proposition 2.6]. Next, suppose, that \( A \) is scattered. Since each quotient algebra of a scattered C*-algebra is scattered [3, proposition 2.4] the conclusion follows by transfinite induction, if it is proved, that \( A \) has an elementary ideal. But \( A \) is of type I [3, theorem 2.3] and therefore has an ideal \( J \) with continuous trace, and this ideal is scattered [3, proposition 2.4]. By lemma 1 there is an isolated point in \( \hat{J} \), which gives the desired conclusion (as in [3, lemma 3.3]).

From this, theorem and the proof we get

**Corollary 3.** A C*-algebra \( A \) is scattered, if and only if \( A \) is of type I and \( \hat{A} \) is scattered.

If \( A \) is separable, then the above condition \( A \) is of type I can be omitted. Because, if \( \hat{A} \) is scattered, then Prim \( (A) \) is scattered, so \( A \) has a decomposition series as in theorem 2 by [1, 4.7.3.].

Finally it should be remarked, that C*-algebras as the ones discussed here previously have been considered by Wojtaszyk [5] in the separable case. The C*-algebras investigated in [5] are those with separable dual space, which by theorem 2 and [3, theorem 3.1] are exactly the separable, scattered C*-algebras.
REFERENCES


TECHNICAL UNIVERSITY OF DENMARK
2800 LYNGBY
DENMARK