SIMPLE INJECTIVE MODULES
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Rosenberg and Zelinsky [6] first addressed the question of characterizing for a ring \( R \) those simple modules \( _RT \) having injective hulls of finite length. In that work they also announced a very special case obtained by Kaplansky: if \( R \) is commutative, then every simple \( R \)-module is injective iff \( R \) is von Neumann regular. A few years later Villamayor (see [4]) characterized those (not necessarily commutative) rings \( R \) — now known as \( V \)-rings — over which every simple left module is injective by the property that every left ideal is an intersection of maximal left ideals. (Also see [2] and [3] for further treatment of \( V \)-rings and their bibliography.)

In this note we return to that which is common to both the Rosenberg–Zelinsky and the \( V \)-ring studies. That is, we consider the problem of characterizing, for a ring \( R \), those simple modules \( _RT \) that are themselves injective. As one application, we prove that if \( R \) is a \( V \)-ring, then so is the endomorphism ring of every finitely generated projective module \( PR \).

If \( M \) is a left \( R \)-module, then for each \( X \subseteq M \) and \( A \subseteq R \), we set
\[
(A : X) = \{ r \in R \mid rX \subseteq A \}. 
\]

1. Characterizations of simple injectives.

Let \( R \) be a ring (with identity). Then a left \( R \)-module \( _RT \) is simple iff
\[
T \cong R/M
\]
for some maximal left ideal \( M \) of \( R \). Indeed, if \( _RT \) is simple, then
\[
T \cong R/(0:t)
\]
for each \( 0 \neq t \in T \).

**Definition.** Let \( _RI \preceq _RR \) be a left ideal and let \( a \in R \). A left ideal \( L \preceq _RR \) **supports** \( a \) **on** \( I \) if
\[
L \cap Ra = Ia.
\]

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For each left ideal \( I \) of \( R \) and each \( a \in R \) the left ideal \( Ia \) is a support for \( a \) on \( I \). So since the collection of supports for \( a \) on \( I \) is clearly inductive, each support for \( a \) on \( I \) is contained in a maximal support for \( a \) on \( I \).

1.1. **Lemma.** Let \( M \) be a maximal left ideal of \( R \) and let \( a \in R \). Then

\[ a \in Ma \]

iff \( R \) is a (necessarily unique) maximal support for \( a \) on \( M \).

**Proof.** Since \( Ra \supseteq Ma \), we have

\[ a \in Ma \iff Ra = Ma \iff R \cap Ra = Ma . \]

Now our main result is the following characterization of simple injective modules.

1.2. **Theorem.** Let \( _RM \) be a maximal left ideal of \( R \). Then the simple left \( R \)-module \( R/M \) is injective iff for each \( a \in R \)

\[ L + Ra = R \]

for every maximal support \( L \) of \( a \) on \( M \).

**Proof.** \((\Rightarrow)\) If \( a \in Ma \), then by Lemma 1.1, \( R \) is the unique maximal support for \( a \) on \( M \). So we may assume that \( a \notin Ma \). This means that

\[ R/M \cong Ra/Ma \]

so that \( Ra/Ma \) is injective. Let \( L \) be a maximal support for \( a \) on \( M \). Since

\[ L \cap Ra = Ma , \]

we infer that

\[ (Ra + L)/Ma \cong Ra/Ma \oplus L/Ma . \]

So by the maximality of the support \( L \), we have \( Ra + L = R \).

\((\Leftarrow)\). Let \( I \) be a left ideal of \( R \) and let

\[ \varphi: I \to R/M \]

be a non zero homomorphism. Let \( a \in I \) with

\[ \varphi(a) = 1 + M . \]

Then

\[ \ker \varphi = Ma \cap I = Ma . \]
So $a \notin Ma$. Now let $R_L \leq_R R$ be a left ideal maximal with respect to $Ma \leq L$ and $a \notin L$.

Then since $I(Ma) \cong R/M$ is simple, since $L \cap Ra \leq Ra \leq I$ and since $Ma \leq L$, we have

$$L \cap Ra = Ma.$$  

It is clear, since $a \notin Ma$, that $L$ is a maximal support for $a$ on $M$. So by hypothesis,

$$L + Ra = R.$$  

Therefore,

$$\psi : l + ra \mapsto \phi(ra) \quad (l + ra \in L + Ra)$$

is a well defined homomorphism $\psi : R \rightarrow R/M$ extending $\phi$. So by the Injective Test Lemma ([1, p. 205]) $R/M$ is injective.

If $R$ is a V-ring, then every left ideal $I$ of $R$ is fully idempotent (i.e., $I^2 = I$). A local generalization of this fact is the following.

1.3. **Corollary.** Let $M$ be a maximal left ideal of $R$ with $R/M$ injective. Then for each $a \in R$

$$aR \subseteq M \Rightarrow a \in Ma.$$  

**Proof.** Let $aR \subseteq M$ and let $L$ be a maximal support for $a$ on $M$. Then by Theorem 1.2

$$L + Ra = R.$$  

Thus,

$$a \in RaR \subseteq RaL + RaRa \subseteq L + Ma \subseteq L;$$

and so

$$a \in L \cap Ra = Ma.$$  

As we shall show later (Corollary 1.7) if $R$ is commutative, then the converse of this last corollary holds. In general, however, the converse is false. Indeed, if $R$ is von Neumann regular, then

$$aR \subseteq M \Rightarrow a \in Ma$$

for all left ideals $M$ of $R$. But von Neumann regular rings need not be V-rings. (See, e.g., [2], [3], or [1, Exercise 18.4].)
Let $R$ be a ring and let $P$ be a two sided ideal of $R$. Then each simple left $R/P$ module is a simple $R$-module. The following result characterizes those injective simple $R/P$ modules that are injective as $R$-modules.

1.4. **Theorem.** Let $P$ be an ideal of $R$ and let $M$ be a maximal left ideal of $R$ with $P \subseteq M$. If $R/M$ is $R/P$ injective, then $R/M$ is $R$ injective iff for all $a \in R$

$$a \in P \Rightarrow a \in Ma.$$ 

**Proof.** $(\Rightarrow)$. If $a \in P$, then $aR \subseteq P \subseteq M$, so $a \in Ma$ by Corollary 1.3.

$(\Leftarrow)$. Let $a \in R$ and let $L$ be a maximal support for $a$ on $M$. Then by Theorem 1.2 it will suffice to show that $L + Ra = R$.

First suppose $P \subseteq L$. Then

$$L \cap (Ra + P) = (L \cap Ra) + P = Ma + P,$$

so $L/P$ is a maximal support for $a + P$ on $M/P$ in $R/P$. Since $R/M$ is $R/P$ injective, it follows from Theorem 1.2 that $L + Ra = L + Ra + P = R$.

On the other hand suppose that $P \nsubseteq L$. Then $P + L$ does not support $a$ on $M$. So there is an $x \in R$ with

$$xa \in P + L$$

and $xa \notin Ma$.

But then $Ra/ Ma \cong R/M$ is simple, and for some $r \in R$,

$$a - rx \in Ma < P + L,$$

so $a \in P + L$. Then $a - y \in P$ for some $y \in L$. So by hypothesis

$$a - y \in M(a - y) \leq L,$$

and $a \in L \cap Ra = Ma$. Therefore, by Lemma 1.1, $L = R$.

Recall that if $M$ is a maximal left ideal of $R$, then $(M:R)$ is a primitive ideal of $R$ with $(M:R) \subseteq M$.

1.5. **Corollary.** Let $M$ be a maximal left ideal of $R$. If $R/(M:R)$ is a V-ring, then the simple $R$-module $R/M$ is injective iff $a \in Ma$ for all $a \in (M:R)$.

Let $J = J(R)$ be the Jacobson radical of $R$. If $J = 0$, then $R$ is semi-artinian if every primitive factor ring of $R$ is artinian. Every primitive ring with a polynomial identity is artinian; so for example, if $R$ satisfies a polynomial identity, then $R/J$ is semi-artinian.

1.6. **Corollary.** Let $M$ be a maximal left ideal of $R$. If $R/J$ is semi-artinian, then $R/M$ is injective iff $a \in Ma$ for all $a \in (M:R)$.
Proof. Since \( R/(M : R) \) is primitive, it is artinian and simple, and hence it is a V-ring.

1.7. **Corollary.** Let \( R \) be commutative. Then a simple module \( _RT \) is injective iff \( a \in (0: T)a \) for all \( a \in (0: T) \).

2. **Endomorphism rings of projectives.**

Throughout this section let \( P_R \) be a finitely generated projective module with endomorphism ring

\[
S = \text{End} \left( P_R \right).
\]

If \( P_R \) is a generator, then by Morita equivalence (see [1, Chapter 6]) the categories of left \( R \) and left \( S \) modules are equivalent. In particular, an \( R \)-module \( _RT \) is injective (projective) iff the \( S \)-module

\[
P \otimes_R T
\]

is injective (projective). From this it is immediate, for example, that if \( P \) is a generator and \( R \) is a quasi Frobenius, then so is \( S \). On the other hand, when \( P_R \) is not a generator, the nature of \( P \otimes_R T \) cannot always be readily determined from that of \( T \). However, if \( _RT \) is also simple, we can be fairly definitive.

2.1. **Theorem.** If \( _RT \) is a simple injective (projective) \( R \)-module, then \( P \otimes_R T \) is either zero or a simple injective (projective) \( S \)-module.

Proof. Since the result is true when \( P_R \) is a generator, we may assume that \( P = eR \) for some idempotent \( e \in R \) and that \( S \cong eRe \). In particular,

\[
P \otimes_R T \cong eT.
\]

Suppose then that \( _RT \) is simple and that \( eT \neq 0 \). Then for each \( 0 \neq et \in eT \),

\[
eRe(et) = e(Ret) = eT,
\]

and so \( eT \) is \( eRe \)-simple.

Suppose now that \( T \) is projective. If \( eT \neq 0 \), then as \( R \)-modules, since \( T \) is simple,

\[
Re \cong T \oplus V
\]

for some \( _RV \). Thus, as \( eRe \) modules,

\[
eRe \cong eT \oplus eV,
\]

and \( eT \) is \( eRe \) projective.
Finally, for the interesting case, assume that \( _R T \) is injective. Suppose \( eT \neq 0 \). Say \( 0 \neq et \in T \). Set
\[
M = (0:et) .
\]
Then \( M \) is a maximal left ideal of \( R \) with \( e \notin M \) and
\[
Me \subseteq M \quad \text{and} \quad T \cong R/M .
\]
So as an \( eRe \) module
\[
eT \cong eRe/eMe .
\]
Let \( a \in eRe \) and let \( L \subseteq eRe \) be a maximal support for \( a \) on \( eMea \). We claim that \( RL \cap Ra \subseteq Ma \). For let \( x_1, \ldots, x_n \in L \) and suppose
\[
r_1x_1 + \ldots + r_nx_n = sa \in RL \cap Ra .
\]
If \( se \in M \), then \( sa = sea \in Ma \). Otherwise, if \( se \notin M \), then since \( _RM \) is maximal, \( 1 - yse \in M \) for some \( y \in R \), so since \( a = eae \),
\[
a - eysea \in eMea .
\]
But then since \( L = eLe \),
\[
ey(r_1x_1 + \ldots + r_nx_n) = eyr_1ex_1 + \ldots + eyr_nex_n
\]
\[
= eysea \in L \cap eRea ,
\]
and so since \( L \) supports \( a \) on \( eMe \),
\[
eysea \in eMea .
\]
Thus, \( a \in eMea \) whence \( sa \in Ma \) as claimed.

Now since \( RLe = RL \) and \( Rae = Ra \supseteq Ma \), we have
\[
[RL + Ma + R(1-e)] \cap Ra = Ma .
\]

Thus there is a left ideal \( K \) of \( R \) maximal with respect to
\[
RL + Ma + R(1-e) \leq K \quad \text{and} \quad K \cap Ra = Ma .
\]
So \( K \) is a maximal support for \( a \) on \( M \). But by hypothesis \( R/M \) is injective, so by Theorem 1.2,
\[
K + Ra = R \quad \text{and} \quad eKe + eRea = eRe .
\]

We claim next that \( eKe = L \). Certainly \( L \subseteq eKe \). But since \( R(1 - e) \subseteq K \), we have
\[
K = Ke + R(1 - e) ,
\]
so \( eKe \subseteq K \). Thus
\[ L \cap eRe \subseteq eKe \cap eRe \subseteq e(K \cap Ra) \subseteq eMa = eMea, \]
so since \( L \) is a maximal support, \( L = eKe \). But then
\[ L + eRea = eKe + eRea = eRe \]
and thus, by Theorem 1.2, \( eT \) is injective.

A ring \( R \) is a GV-ring (see [5] for the basic theory of GV-rings) in case each simple left \( R \)-module is either projective or injective.

2.2. Corollary. If \( R \) is a V-ring (GV-ring), then \( S \) is a V-ring (GV-ring).

Proof. It will suffice to prove that every simple \( S \)-module is isomorphic to
\[ P \otimes_R T \]
for some simple \( R \)-module \( T \). Again we may assume \( P = eR \) and \( S = eRe \).

Let \( L \) be a maximal left ideal of \( eRe \). Then there is a maximal left ideal \( M \) of \( R \) with
\[ RL + R(1 - e) \subseteq M \quad \text{and} \quad e \notin M. \]
Since \( eMe = L \), we have
\[ e(R/M) = e(Re/Me) \cong eRe/eMe = eRe/L. \]

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