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NEW CHARACTERIZATIONS OF SPACELIKE HYPERPLANES IN THE STEADY STATE SPACE

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Abstract

In this article, we deal with complete spacelike hypersurfaces immersed in an open region of the de Sitter space $\mathbb{S}_1^{n+1}$ which is known as the steady state space $\mathscr{H}^{n+1}$. Under suitable constraints on the behavior of the higher order mean curvatures of these hypersurfaces, we are able to prove that they must be spacelike hyperplanes of $\mathscr{H}^{n+1}$. Furthermore, through the analysis of the hyperbolic cylinders of $\mathscr{H}^{n+1}$, we discuss the importance of the main hypothesis in our results. Our approach is based on a generalized maximum principle at infinity for complete Riemannian manifolds.

1. Introduction

Let $\mathbb{L}^{n+2}$ denote the $(n+2)$-dimensional Lorentz-Minkowski space $(n \geq 2)$, that is, the real vector space $\mathbb{R}^{n+2}$ endowed with the Lorentz metric defined by

$$\langle v, w \rangle = \sum_{i=1}^{n+1} v_i w_i - v_{n+2} w_{n+2},$$

for all $v, w \in \mathbb{R}^{n+2}$. We define the $(n+1)$-dimensional de Sitter space $\mathbb{S}_1^{n+1}$ as the following hyperquadric of $\mathbb{L}^{n+2}$

$$\mathbb{S}_1^{n+1} = \{p \in \mathbb{L}^{n+2} : \langle p, p \rangle = 1\}.$$

The induced metric from $\langle \cdot, \cdot \rangle$ makes $\mathbb{S}_1^{n+1}$ a Lorentzian manifold with constant sectional curvature one. Moreover, for all $p \in \mathbb{S}_1^{n+1}$, we have

$$T_p(\mathbb{S}_1^{n+1}) = \{v \in \mathbb{L}^{n+2} : \langle v, p \rangle = 0\}.$$

Let $a \in \mathbb{L}^{n+2} \setminus \{0\}$ be a past-pointing null vector, that is, $\langle a, a \rangle = 0$ and $\langle a, e_{n+2} \rangle > 0$, where $e_{n+2} = (0, \dots, 0, 1)$. Then, the open region of the de Sitter space $\mathbb{S}_1^{n+1}$, given by

$$\mathscr{H}^{n+1} = \{p \in \mathbb{S}_1^{n+1} : \langle p, a \rangle > 0\}$$

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is the so-called *steady state space*.

The importance of considering $\mathscr{H}^{n+1}$ comes from the fact that, in cosmology, $\mathscr{H}^4$ is the steady state model of the universe proposed by Bondi & Gold [8] and by Hoyle [16], when looking for a model of the universe which looks the same not only at all points and in all directions (that is, spatially isotropic and homogeneous), but also at all times. For more details, we recommend for the readers to see [15, §5.2] or [24, §14.8].

From a mathematical point of view, the interest in the study of spacelike hypersurfaces immersed in a Lorentzian space is motivated by their nice Bernstein-type properties. In this direction, several authors have approached the problem of characterizing spacelike hyperplanes of $\mathscr{H}^{n+1}$ which are totally umbilical spacelike hypersurfaces isometric to the Euclidean space $\mathbb{R}^n$ and give a complete foliation of $\mathscr{H}^{n+1}$. We refer to readers, for instance, the works [1], [6], [10], [11], [12], [21].

Proceeding in this branch, our purpose in this article is to apply a suitable extension of a maximum principle at infinity of [25] due to Caminha in [9] (cf. Lemma 5) in order to establish new characterization results concerning these spacelike hyperplanes of $\mathscr{H}^{n+1}$. For this, we will assume certain appropriate constraints on the behavior of the higher order mean curvatures.

This paper is organized in the following manner: initially, in §2, we recall some standard facts concerning the higher order mean curvatures, the Newton transformations and the linearized operators naturally attached to a spacelike hypersurface immersed in $\mathscr{H}^{n+1}$. Afterwards, we quote some key lemmas that will be essential to prove our main results. Finally, in §3 we establish our characterization theorems for hypersurfaces of $\mathscr{H}^{n+1}$ (see Theorems 2, 3 and 4). Furthermore, through the analysis of the hyperbolic cylinders of $\mathscr{H}^{n+1}$, we discuss the importance of the main hypothesis in our results (cf. Example 1).

2. Preliminaries and key lemmas

Considering the context of the previous section, we will deal with connected spacelike hypersurfaces $\psi\colon \Sigma \to \mathscr{H}^{n+1}$ with a future-pointing orientation N, that is, a normal unit timelike vector field N such that $\langle N, e_{n+2}\rangle < 0$. Let us denote by $A\colon \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ the Weingarten endomorphism of Σ^n with respect to N. We recall that at each point $p \in \Sigma^n$, the Weingarten operator A restricts to a self-adjoint linear map $A_p\colon T_p\Sigma \to T_p\Sigma$. For $0 \le r \le n$, let $S_r(p)$ denote the r-th elementary symmetric function on the eigenvalues of A_p. Hence, one gets n smooths functions $S_r\colon \Sigma \to \mathbb{R}$, such that

$$\det(tI - A) = \sum_{k=0}^{n} (-1)^k S_k t^{n-k},$$

where $S_0 = 1$ by convention. If $p \in \Sigma^n$ and $\{e_k\}$ is a basis of $T_p\Sigma$ formed by eigenvectors of A_p, with corresponding eigenvalues $\{\lambda_k\}$, one immediately sees that

$$S_r = \sigma_r(\lambda_1, \dots, \lambda_n),$$

where $\sigma_r \in \mathbb{R}[X_1, \dots, X_n]$ is the r-th elementary symmetric polynomial on the indeterminates $X_1, \dots, X_n$.

In the sequel, with this setting, we define the r-th mean curvature H_r of Σ^n, $0 \leq r \leq n$, by

$$\binom{n}{r} H_r = (-1)^r S_r = \sigma_r(-\lambda_1, \dots, -\lambda_n).$$

We observe that $H_0 = 1$, while $H_1 = -\frac{1}{n} S_1$ is the usual mean curvature H of Σ^n.

For $0 \leq r \leq n$, one defines the r-th Newton transformation P_r on Σ^n by setting $P_0 = I$ (the identity operator) and, for $1 \leq r \leq n$, via the recurrence relation

$$P_r = \binom{n}{r} H_r I + A P_{r-1}. \tag{2.1}$$

It is also immediate to check that $P_r e_i = (-1)^r S_r(A_i) e_i$, so that an easy computation (cf. [7, Lemma 2.1]) gives the following lemma.

LEMMA 1. *With the above notations, the following formulae hold:*

(a) $S_r(A_i) = S_r - \lambda_i S_{r-1}(A_i)$;

(b) $\mathrm{tr}(P_r) = (-1)^r \sum_{i=1}^n S_r(A_i) = (-1)^r (n-r) S_r = c_r H_r$;

(c) $\mathrm{tr}(A P_r) = (-1)^r \sum_{i=1}^n \lambda_i S_r(A_i) = (-1)^r (r+1) S_{r+1} = -c_r H_{r+1}$,

where $c_r = (n-r)\binom{n}{r}$.

Associated to each Newton transformation P_r one has the second-order linear differential operator L_r, given by

$$L_r f = \mathrm{tr}(P_r \operatorname{Hess} f).$$

In particular, for $r = 0$, we get the well-known Laplacian operator $L_0 = \Delta$. Furthermore, if the ambient space has constant sectional curvature, Rosenberg proved in [22] (see also [4] for more details) that $L_r f = \mathrm{div}(P_r \nabla f)$, where div stands for the divergence on Σ^n and ∇ denotes the field gradient of $f \in C^\infty(\Sigma)$.

The next lemma gives a geometric condition which guarantees the ellipticity of L_1 (cf. Lemma 3.2 of [3]).

LEMMA 2. *Let $\psi\colon \Sigma^n \to \mathscr{H}^{n+1}$ be a spacelike hypersurface in $\mathscr{H}^{n+1}$. If $H_2 > 0$ on Σ^n, then L_1 is elliptic or, equivalently, P_1 is positive definite (for an appropriate choice of orientation N).*

When $r \geq 2$, the following lemma establishes sufficient conditions to guarantee the ellipticity of L_r (cf. [3, Lemma 3.3]).

LEMMA 3. *Let $\psi\colon \Sigma^n \to \mathscr{H}^{n+1}$ be a spacelike hypersurface in $\mathscr{H}^{n+1}$. If there exists an elliptic point of Σ^n, with respect to an appropriate choice of orientation N, and $H_{r+1} > 0$ on Σ^n, for $2 \leq r \leq n-1$, then for all $1 \leq k \leq r$ the operator L_k is elliptic or, equivalently, P_k is positive definite (for an appropriate choice of orientation N, if k is odd).*

The next lemma was done by Alías, Brasil Jr. and Colares [2] when they studied spacelike hypersurfaces in conformally stationary spacetime, in this case the authors gave a important result about the existence of an elliptic point. Here, for our purpose we rewrite as follows (cf. [2, Lemma 5.4]).

LEMMA 4. *Let $\mathscr{H}^{n+1}$ be a steady state space endowed with a complete closed conformal timelike vector field V, and let Σ^n be a complete spacelike hypersurface in $\mathscr{H}^{n+1}$. Suppose that the divergence* $\operatorname{div} V$ *of V on $\mathscr{H}^{n+1}$ does not vanish at a point of Σ^n where the restriction $|V|_\Sigma = \sqrt{-\langle V, V\rangle}|_\Sigma$ of $|V|$ to Σ^n attains a local minimum. Then, there exists an elliptic point in Σ^n.*

In [25] Yau, generalizing a previous result due to Gaffney [14], established the following version of Stokes' Theorem on an n-dimensional, complete noncompact Riemannian manifold Σ^n:

THEOREM 1. *If $\omega \in \Omega^{n-1}(\Sigma)$ is an integrable $(n-1)$-differential form on Σ^n, then there exists a sequence B_i of domains on Σ^n such that $B_i \subset B_{i+1}$, $\Sigma^n = \bigcup_{i\geq 1} B_i$ and*

$$\lim_{i\to+\infty} \int_{B_i} d\omega = 0.$$

Suppose Σ^n is oriented by the volume element $d\Sigma$, and let $\mathscr{L}^1(\Sigma)$ be the space of Lebesgue integrable functions on Σ^n. If $\omega = \iota_X d\Sigma$ is the contraction of $d\Sigma$ in the direction of a smooth vector field X on Σ^n, then Caminha obtained the following consequence of Yau's result (cf. [9, Proposition 2.1]).

LEMMA 5. *Let X be a smooth vector field on the n-dimensional complete oriented Riemannian manifold Σ^n, such that* $\operatorname{div} X$ *does not change sign on Σ^n. If $|X| \in \mathscr{L}^1(\Sigma)$, then* $\operatorname{div} X = 0$.

3. Characterizing spacelike hyperplanes of $\mathscr{H}^{n+1}$

We will consider in $\mathscr{H}^{n+1}$ the following timelike vector field

$$V = -\langle x, a\rangle x + a, \tag{3.1}$$

where $a \in \mathbb{L}^{n+2}$ is the fixed nonzero null vector which appears in the definition of $\mathscr{H}^{n+1}$. It is not difficult to check that

$$\overline{\nabla}_U V = -\langle x, a\rangle U,$$

for all $U \in \mathfrak{X}(\mathscr{H}^{n+1})$, where $\overline{\nabla}$ stands for the Levi-Civita connection of $\mathscr{H}^{n+1}$. Hence, V is a closed and conformal vector field globally defined on $\mathscr{H}^{n+1}$ (cf. [20, Example 2 in §4]; see also [17, §5]). Furthermore, the n-dimensional distribution $\mathscr{D}$ defined on $\mathscr{H}^{n+1}$ by

$$x \in \mathscr{H}^{n+1} \longmapsto \mathscr{D}(x) = \{v \in T_x\mathscr{H}^{n+1} : \langle V(x), v\rangle = 0\}$$

determines a codimension one spacelike foliation $\mathscr{F}(V)$ which is oriented by V (for more details, see [20, Proposition 1]).

It follows from [18, Example 1] that the leaves of $\mathscr{F}(V)$ are given by

$$\mathscr{L}_\tau = \{x \in \mathscr{H}^{n+1} : \langle x, a\rangle = \tau\}, \quad \tau > 0,$$

which are totally umbilical hypersurfaces of $\mathscr{H}^{n+1}$ isometric to the Euclidean space $\mathbb{R}^n$, and having constant mean curvature one with respect to the future-pointing orientation

$$N_\tau = -x + \frac{1}{\tau}a, \quad x \in \mathscr{L}_\tau.$$

In order to proceed, we recall two particular functions naturally attached to a spacelike hypersurface Σ^n immersed on $\mathscr{H}^{n+1}$, namely, the height and angle function. Such functions are defined, respectively, by $\ell_a = \langle \psi, a\rangle$ and $f_a = \langle N, a\rangle$, where $a \in \mathbb{L}^{n+2}$ is again the fixed nonzero null vector which appear in the definition of $\mathscr{H}^{n+1}$ and N stands for the future-pointing orientation of Σ^n. It is not difficult to check that $\nabla \ell_a = a^\top$ and $\nabla f_a = -A(a^\top)$, where $a^\top$ stands for the orthogonal projection of a onto the tangent bundle $T\Sigma$. Moreover, using Gauss and Weingarten formulas, we obtain

$$\nabla_X \nabla \ell_a = -f_a AX - \ell_a X,$$

for all $X \in \mathfrak{X}(\Sigma)$. Therefore, from Lemma 1 jointly with the above considerations, we can deduce the formula for the operator L_r acting on the height, that is,

$$L_r \ell_a = c_r(-\ell_a H_r + f_a H_{r+1}), \tag{3.2}$$

where $c_r = (n-r)\binom{n}{r}$.

Now, we are in position to state and prove our first result.

THEOREM 2. *Let* $\psi\colon \Sigma^n \to \mathscr{H}^{n+1}$ *be a complete spacelike hypersurface of* $\mathscr{H}^{n+1}$ *with bounded second fundamental form. Suppose that, for some* $1 \le r \le n-1$*, the* r*-th mean curvature* H_r *of* Σ^n *satisfies*

$$H_{r+1} \ge H_r > 0.$$

If $|a^\top| \in \mathscr{L}(\Sigma)$*, then* Σ^n *is a hyperplane* $\mathscr{L}_\tau$ *for some* $\tau > 0$*.*

PROOF. From (3.2) jointly with the fact that $f_a > 0$ (with appropriate choice of orientation N of ψ), we have that

$$L_r(\ell_a) = -c_r H_r \ell_a + c_r H_{r+1} f_a \ge c_r H_r(-\ell_a + f_a), \tag{3.3}$$

where $c_r = (n-r)\binom{n}{r}$.

Next, we observe that ℓ_a is always a positive function and as $a^\top$ stands for the orthogonal projection of a onto the tangent bundle $T\Sigma$, that is, $a^\top = a + f_a N - \ell_a \psi$, we deduce that

$$|a^\top|^2 = f_a^2 - \ell_a^2. \tag{3.4}$$

In particular, we have that $f_a \ge \ell_a$.

Hence, from (3.3) jointly with our assumption that $H_r > 0$, we obtain $L_r(\ell_a) \ge 0$ on Σ^n. So, since we are also supposing that $|a^\top| \in \mathscr{L}(\Sigma)$, from Lemma 5 we get that $L_r(\ell_a) = 0$ on Σ^n. Consequently, using again that $H_r > 0$, we conclude that $f_a = \ell_a$. Hence, returning to equation (3.4), we infer that $|a^\top|^2 = |\nabla \ell_a|^2 = 0$, it follows that $\nabla \ell_a = 0$. Therefore, ℓ_a is constant on Σ^n, which means that Σ^n is a hyperplane of $\mathscr{H}^{n+1}$.

Before presenting our next result, we will need to establish the following definition: a spacelike hypersurface $\psi\colon \Sigma^n \to \mathscr{H}^{n+1}$ is said to be *locally tangent from above* to a hyperplane $\mathscr{L}_\tau$ of $\mathscr{H}^{n+1}$, when there exist a point $p \in \Sigma^n$ and a neighborhood $\mathscr{U} \subset \Sigma^n$ of p such that $\ell_a(p) = \tau$ and $\ell_a(q) \ge \tau$ for all $q \in \mathscr{U}$. In this setting, we get the following:

THEOREM 3. *Let* $\psi\colon \Sigma^n \to \mathscr{H}^{n+1}$ *be a complete spacelike hypersurface of* $\mathscr{H}^{n+1}$ *with bounded second fundamental form. Suppose that, for some* $1 \le r \le n$*, the* r*-th mean curvature* H_r *of* Σ^n *satisfies*

$$H_r \ge 1.$$

If $|a^\top| \in \mathscr{L}^1(\Sigma)$ *and* Σ^n *is locally tangent from above to a hyperplane* $\mathscr{L}_\tau$*, then* Σ^n *is the hyperplane* $\mathscr{L}_\tau$*.*

PROOF. Let us consider the vector field on $\mathscr{H}^{n+1}$ defined in the beginning of this section, namely

$$V(p) = -\langle p, a\rangle p + a.$$

It is not difficult to check that V is a complete closed conformal vector field, with $\operatorname{div} V(p) = (n+1)\langle p, a\rangle$ and $|V|_\Sigma = \ell_a$. Thus, since we are supposing that Σ^n is locally tangent from above to a hyperplane $\mathscr{L}_\tau$, we have that $|V|_\Sigma$ attains a minimum local on Σ^n. Consequently, we can apply Lemma 4 to guarantee the existence of an elliptic point on Σ^n. Hence, since we are also supposing that $H_r \geq 1$, it follows from Lemma 3 that the Newton tensor P_j is positive definite and, consequently, H_j is positive for all $2 \leq j \leq r-1$. But, being $H_2 > 0$, we can also apply Lemma 2 to infer that P_1 is positive definite and, hence, that H is positive.

Now, we define the following vector field tangent to Σ^n

$$X = \sum_{i=0}^{r-1} \frac{1}{c_i} P_i(\nabla \ell_a), \tag{3.5}$$

where $c_i = (i+1)\binom{n}{r}$. From (3.2) we obtain

$$\begin{aligned}\operatorname{div} X &= \sum_{i=0}^{r-1} \frac{1}{c_i} L_i(\ell_a) = (H_r f_a - H_{r-1}\ell_a) + \cdots + (H f_a - \ell_a)\\ &= H_r f_a + H_{r-1}(f_a - \ell_a) + \cdots + H(f_a - \ell_a) - \ell_a.\end{aligned} \tag{3.6}$$

Consequently, since we are assuming that $H_r \geq 1$, from (3.6) we conclude that

$$\operatorname{div} X \geq (H_r + \cdots + H)(f_a - \ell_a). \tag{3.7}$$

On the other hand, taking into account that the angle function satisfies $f_a > 0$ on Σ^n, from (3.4) it follows that the height and angle functions attached to the vector a satisfy

$$f_a \geq \ell_a, \tag{3.8}$$

on Σ^n. Thus, using that H_j is positive, for all $1 \leq j \leq r$, together with equation (3.8), we deduce that $\operatorname{div} X \geq 0$. Moreover, since we are supposing that the second fundamental form of Σ^n is bounded and that $|\nabla \ell_a| = |a^\top| \in \mathscr{L}^1(\Sigma)$, from (2.1) and (3.5) we have that $|X| \in \mathscr{L}^1(\Sigma)$. Hence, we can apply Lemma 5 to get that $\operatorname{div} X = 0$ on Σ^n and, returning to (3.7), $f_a = \ell_a$ on Σ^n. Now, we observe that

$$|\nabla \ell_a|^2 = f_a^2 - \ell_a^2 = 0,$$

on Σ^n. Therefore, the height function ℓ_a is constant and, since we are assuming that Σ^n is locally tangent from above to $\mathscr{L}_\tau$, Σ^n must be just the hyperplane $\mathscr{L}_\tau$.

REMARK 1. Taking into account Lemma 2, we note that when $r = 2$ in Theorem 3 it is not necessary suppose that the complete spacelike Σ^n be locally tangent from above to a hyperplane $\mathscr{L}_\tau$.

By considering a complete spacelike hypersurface $\psi\colon \Sigma^n \to \mathscr{H}^{n+1}$ as being the boundary of a suitable domain of $\mathscr{H}^{n+1}$, from [13, Theorem 1] we get the following tangency principle (see also Theorem 3.6 of [5] and of [21, Theorem 2], for a version corresponding to constant mean curvature spacelike hypersurfaces).

PROPOSITION 1. *Let Σ_1 and Σ_2 be complete spacelike hypersurfaces immersed in $\mathscr{H}^{n+1}$ with mean curvatures H_{Σ_1} and H_{Σ_2}, respectively. Suppose that Σ_1 lies below Σ_2. If, in a neighborhood of a common tangent point $p \in \Sigma_1 \cap \Sigma_2$, we have that $H_{\Sigma_2} \le \alpha \le H_{\Sigma_1}$, for some real number α, then Σ_1 and Σ_2 must coincide.*

From Proposition 1 we obtain the following consequence, which has a similar spirit to Theorem 3, but requiring neither the boundedness of the second fundamental form nor the integrability condition $|a^\top| \in \mathscr{L}^1(\Sigma)$.

COROLLARY 3.1. *Let $\psi\colon \Sigma^n \to \mathscr{H}^{n+1}$ be a complete spacelike hypersurface, which lies below a hyperplane $\mathscr{L}_\tau$ of $\mathscr{H}^{n+1}$. If, in a neighborhood of a common tangent point $p \in \Sigma^n \cap \mathscr{L}_\tau$, we have that $H \ge 1$, then Σ^n is the hyperplane $\mathscr{L}_\tau$.*

Proceeding, we present our last characterization of the spacelike hyperplanes of $\mathscr{H}^{n+1}$.

THEOREM 4. *Let $\psi\colon \Sigma^n \to \mathscr{H}^{n+1}$ be a complete spacelike hypersurface of $\mathscr{H}^{n+1}$ with bounded second fundamental form. Suppose that $\ell_a = \lambda f_a$, for some nonzero constant $\lambda \in \mathbb{R}$, and that, for some $1 \le r \le n-2$, the r-th mean curvature of Σ^n satisfies*

$$H_{r+2} \ge H_r > 0.$$

If $|a^\top| \in \mathscr{L}^1(\Sigma)$ then Σ^n is a hyperplane $\mathscr{L}_\tau$ for some $\tau > 0$.

PROOF. Since $\ell_a = \lambda f_a$, we observe that

$$|\nabla \ell_a|^2 = f_a^2 - \ell_a^2 = (\lambda^{-2} - 1)\ell_a^2. \tag{3.9}$$

In particular, from (3.9) we have that $\lambda^{-2} - 1 \geq 0$. Now, we define on Σ^n the following tangent vector field

$$X = \frac{\lambda^{-1}}{c_{r+1}} P_{r+1} \nabla \ell_a + \frac{1}{c_r} P_r \nabla \ell_a, \tag{3.10}$$

where $c_i = (i+1)\binom{n}{r}$. From (3.2), (3.9) and (3.10) we obtain

$$\begin{aligned} \operatorname{div} X &= \frac{\lambda^{-1}}{c_{r+1}} \operatorname{div}(P_{r+1}\nabla \ell_a) + \frac{1}{c_r} \operatorname{div}(P_r \nabla \ell_a) \\ &= \frac{\lambda^{-1}}{c_{r+1}} L_{r+1}\ell_a + \frac{1}{c_r} L_r \ell_a. \end{aligned}$$

Consequently, by using (3.2), we obtain from the above expression that

$$\begin{aligned} \operatorname{div} X &= \lambda^{-1} H_{r+2} f_a - \lambda^{-1} H_{r+1}\ell_a + H_{r+1} f_a - H_r \ell_a \\ &= (\lambda^{-2} H_{r+2} - H_r)\ell_a \\ &\geq (\lambda^{-2} - 1) H_r \ell_a \geq 0, \end{aligned} \tag{3.11}$$

on Σ^n. Moreover, since Σ^n has bounded second fundamental form and $|\nabla \ell_a| = |a^\top| \in \mathscr{L}^1(\Sigma)$, we get that $|X| \in \mathscr{L}^1(\Sigma)$. Thus, taking into account (3.11), we can apply Lemma 5 to obtain that $\operatorname{div} X = 0$ on Σ^n. Hence, returning to (3.11), we obtain that $\lambda^2 = 1$. Therefore, from (3.9) we conclude that the height function ℓ_a is constant on Σ^n and, consequently, Σ^n must be a hyperplane $\mathscr{L}_\tau$ for some $\tau > 0$.

REMARK 2. It is worth to make a brief discussion of the meaning of our assumption in the previous results concerning the integrability on the spacelike hypersurface Σ^n of $|a^\top|$ both from geometric and physical viewpoints. From the first viewpoint, [1, Lemma 1] asserts that if a complete spacelike hypersurface is contained into the closure of the interior domain enclosed by a spacelike hyperplane of $\mathscr{H}^{n+1}$, then it must be diffeomorphic to $\mathbb{R}^n$. In particular, it follows that there is no compact (without boundary) spacelike hypersurfaces in $\mathscr{H}^{n+1}$. In this sense, our assumption of $|a^\top| \in \mathscr{L}^1(\Sigma)$ in our previous results comes to supply the fact that Σ^n is non-compact. Moreover, this assumption can be also thought as a sort of asymptotic flatness like condition and, taking into account that our aim is to show that Σ^n is isometric to $\mathbb{R}^n$, it is a mild hypothesis.

On the other hand, some physical interpretation is now in order. In fact, assume $n = 3$ and use then the orthogonal decomposition $T_{\psi_p}\mathscr{H}^4 = \operatorname{Span}\{N_p\} \oplus N_p^\perp$, where $p \in \Sigma^3$, $\psi\colon \Sigma^3 \to \mathscr{H}^4$ is a spacelike hypersurface and $N_p^\perp =$

$d\psi_p(T_p\Sigma)$. Since from (3.1) it follows that $V^\top = a^\top$, we can write $V_p = e_pN_p + a_p^\top$, where $e_p = -\langle V_p, N_p\rangle > 0$ and $a_p^\top$ are, respectively, the energy and the 3-momentum that the instantaneous observer N_p measures for V_p.

Furthermore, the quantity $(1/e_p)a_p^\top$ is the relative velocity (and, hence, $(1/e_p)|a_p^\top|$ is the relative speed) of V_p with respect to N_p (for more details, we refer to [23, §2.1.3]). Note that $|a_p^\top| = \sqrt{-\langle V_p, V_p\rangle}\sinh\theta_p$, where θ_p is the hyperbolic angle between V_p and N_p. Thus, we get $|a_p^\top| = e_p\tanh\theta_p \leq e_p$. Consequently, the integrability of $|a^\top|$ on Σ^3 can be regarded as been the 3-momentum of N having integrable norm along Σ^3 and, in particular, such condition is satisfied when the total energy $\int_\Sigma e_p\, d\Sigma$ is finite.

We close our paper discussing the importance of the main hypothesis in our previous results. This is made through the following example:

Example 1. According to [21, §3] (see also [18, Example 2] or [19, §2]), the so-called hyperbolic cylinder of the de Sitter space $\mathbb{S}_1^{n+1}$, which is defined by

$$\{p \in \mathbb{S}_1^{n+1} : p_1^2 + \cdots + p_{k+1}^2 = \cosh^2\varrho\},$$

where $1 \leq k \leq n-1$ and $\varrho > 0$, has two connected components which are isometric to $\mathbb{S}^k(\cosh\varrho) \times \mathbb{H}^{n-k}(\sinh\varrho)$. Moreover, one of the components of the hyperbolic cylinder is contained in the steady state space $\mathscr{H}^{n+1}$, and it will be denote by $\mathscr{C}_{k,\varrho}$.

It is not difficult to verify that $\mathscr{C}_{k,\varrho}$ has the following future-pointing orientation

$$N(p) = \frac{1}{\cosh\varrho\sinh\varrho}(\cosh^2\varrho\, p - \xi(p)), \tag{3.12}$$

where $\xi\colon \mathscr{C}_{k,\varrho} \to \mathbb{L}^{n+2}$ is given by $\xi(p) = (p_1, \ldots, p_{k+1}, 0, \ldots, 0)$. Consequently, from (3.12) we conclude that $\mathscr{C}_{k,\varrho}$ is a isoparametric spacelike hypersurface of $\mathscr{H}^{n+1}$, whose principal curvatures with respect to N are given by

$$\lambda_1 = \cdots = \lambda_k = -\tanh\varrho \quad \text{and} \quad \lambda_{k+1} = \ldots = \lambda_n = -\coth\varrho.$$

Hence, considering $n \geq 3$ and $k = 1$, the r-th mean curvature of $\mathscr{C}_{1,\varrho}$ is

$$\binom{n}{r}H_r = \binom{n-1}{r-1}(\coth\varrho)^{r-1}\tanh\varrho + \binom{n-1}{r}(\coth\varrho)^r.$$

Thus, after a straightforward computation, it is not difficult to verify that

$$H_{r+1} - H_r = \frac{r}{n(\tanh\varrho)^{r+1}}(1-\tanh\varrho)\left((\tanh\varrho)^2 - \frac{\tanh\varrho}{r} + \frac{n-r-1}{r}\right). \tag{3.13}$$

Consequently, for either $r = 1$ or $r \geq 2$ and ϱ such that $\tanh \varrho > \frac{1}{2}$, we obtain from (3.13) that

$$0 < H_r < H_{r+1},$$

for all $1 \leq r \leq n - 1$. Therefore, in the context of Theorem 2, we see that the hypothesis $|a^\top| \in \mathscr{L}^1(\Sigma)$ is necessary to guarantee that the spacelike hypersurface Σ^n be a hyperplane.

Furthermore, from (3.12) we see that the support functions of $\mathscr{C}_{k,\varrho}$ satisfy the following relation

$$\ell_a = \tanh \varrho \, f_a.$$

Hence, considering once more the hyperbolic cylinder $\mathscr{C}_{1,\varrho}$, we see that $|a^\top| \in \mathscr{L}^1(\Sigma)$ is also a necessary hypothesis in Theorem 4.

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