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A STABILITY THEOREM FOR THE CHOQUET-ORDERING IN $C_{\mathbf{C}}(X)$

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1. Abstract.

Let X be a compact Hausdorff space, N a linear subspace of $C_{\mathbf{C}}(X)$, the Banach space of all continuous complex valued functions on X, R a sup-stable convex cone of continuous real valued functions on X, which contains the real parts of all functions in N. Assume N contains the constants and separates the points of X. Then for every complex valued Radon measure μ on X the Bishop–de Leeuw–Choquet theorem (cf. [1]) guarantees the existence of a boundary measure $\bar{\mu}$ such that $\bar{\mu}(f)=\mu(f)$ whenever $f \in N$. Hustad [5] showed that $\bar{\mu}$ can be chosen such that $\|\bar{\mu}\| \leqq \|\mu\|$. This paper generalizes his result to a certain type of 0-neighborhoods U in $C_{\mathbf{C}}(X)$: $\bar{\mu}$ may be chosen in U°, whenever $\mu \in U^\circ$, the (real) polar of U.

2. Introduction.

Let $M(X)$ be the dual of $C_{\mathbf{C}}(X)$, i.e. the space of all Radon measures on X, $M_{\mathbf{R}}$ its real subspace, M^+ the positive cone in $M_{\mathbf{R}}$. Denote by M_1 the unit ball in $M(X)$, $M_1^+ = M_1 \cap M^+$.

For $\mu, \nu \in M^+$ we say $\mu <_R \nu$ iff $\mu(f) \leqq \nu(f)$ for all $f \in R$. For $x \in X$ set

$$M_x^+ = \{\mu \in M^+ \mid \mu \underset{R}{>} \varepsilon_x\},$$

where ε_x denotes the Dirac measure at x. $\mu \in M(X)$ is said to be a boundary measure if $|\mu|$ is maximal in the ordering $<_R$ on M^+. For every real valued continuous function f on X define

$$\hat{f} = \inf\{g \in -R \mid g \geqq f\} \quad \text{and} \quad \check{f} = \sup\{g \in R \mid g \leqq f\}.$$

It is known that μ is a boundary measure if and only if $|\mu|(f) = |\mu|(\hat{f})$ for every $f \in C_{\mathbf{R}}(X)$ (cf. [1]).

The polar U° of a subset U of $C(X)$ (respectively $M(X)$) is the set of all elements $\mu \in M(X)$ (respectively $f \in C(X)$) such that $\operatorname{Re} \mu(f) \leqq 1$ for every $f \in U$ (respectively $\mu \in U$).

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DEFINITION 1. A subset U of $C(X)$ is said to be R-stable if for every measure $\mu \in U^\circ$ there is a boundary measure $\bar{\mu} \in U^\circ$ such that $\mu - \bar{\mu} \in N^\circ$.

DEFINITION 2. A function $\varphi: X \to \mathbf{R} \cup \{\infty\}$ is said to be R-superharmonic if it is lower semicontinuous and if $\varphi(x) \geqq \mu(\varphi)$ for all $\mu \in M_x^+$.

3. Statement and proof of the stability theorem.

THEOREM. *Let X be a compact Hausdorff space, N a linear subspace of $C(X)$ which contains the constant functions and separates the points of X, R a sup-stable convex cone $C_{\mathbf{R}}(X)$ such that* $\operatorname{Re} N \subset R$.

Suppose that $\varrho: X \times \gamma \to \mathbf{R} \cup \{\infty\}$, where $\gamma = \{z \in \mathbf{C} \mid |z| = 1\}$, is a strictly positive l.s.c. *function such that for every $z \in \gamma$ the function $\varrho_z: X \to \mathbf{R} \cup \{\infty\}$, $x \to \varrho(x, z)$, is R-superharmonic.*

Then

$$U = \{f \in C(X) \mid \operatorname{Re}(zf(x)) \leqq \varrho(x,z) \textit{ for all } z \in \gamma,\ x \in X\}$$

is an R-stable 0*-neighborhood in $C(X)$.*

The proof of this theorem follows the ideas of Hustad [5] and Hirsberg [4]:

(1) Set

$$S = \left\{\frac{z}{\varrho(x,z)}\varepsilon_x \mid z \in \gamma,\ x \in X\right\}.$$

Then $U^\circ = \overline{\operatorname{conv}}\, S$, i.e. the $\sigma(M(X), C(X))$-closed convex hull of S, since $U = S^\circ$. (Note that the bipolar of a set S coincides with the σ-closed convex hull of $S \cup \{0\}$). Obviously U° is σ-compact.

(2) Let K be the σ-closure of $S \cup \{0\}$. Then K contains all extreme points of U° and

$$K \subset \left\{ z\varepsilon_x \,\middle|\, x \in X,\ z \in \mathbf{C} \text{ such that } |z| \leqq \frac{1}{\varrho\left(x, \frac{z}{|z|}\right)} \right\}.$$

To prove this let $\{(z_\alpha/\varrho(x_\alpha, z_\alpha))\varepsilon_{x_\alpha}\}_{\alpha \in A}$ be a net in S converging to $\mu \in M(X)$. Because both γ and $\{\varepsilon_x \mid x \in X\}$ are compact there is a subnet $\{(z_\beta/\varrho(x_\beta, z_\beta))\varepsilon_{x_\beta}\}_{\beta \in B}$ such that $\varepsilon_{x_\beta} \to \varepsilon_{x_0}$, $z_\beta \to z_0$, and $\varrho(x_\beta, z_\beta) \to \varrho_0$, $x_0 \in X$, $z_0 \in \gamma$. Then

$$\varrho_0 = \lim \varrho(x_\beta, z_\beta) \geqq \varrho(x_0, z_0) > 0$$

since ϱ is l.s.c., $\mu = (z_0/\varrho_0)\varepsilon_{x_0}$, and

$$\left|\frac{z_0}{\varrho_0}\right| = \frac{1}{\varrho_0} \leqq \frac{1}{\varrho(x_0, z_0)}.$$

(3) Let N' be the dual of N and φ: $M(X) \to N'$ the restriction map. Then $\varphi(U^\circ)$ and $\varphi(K)$ both are $\sigma(N', N)$-compact, φ: $K \to \varphi(K)$ is a homeomorphism because N separates the points of K and clearly $\varphi(K)$ contains the extreme points of $\varphi(U^\circ)$.

(4) Let φ: $C(X) \to C(\varphi(K))$ be the canonical map

$$f \to \bar{f} \quad \text{such that} \quad \bar{f}(\varphi(z\varepsilon_x)) = zf(x).$$

Then $f \in U$ implies $\operatorname{Re}\bar{f} \leqq 1$ on $\varphi(K)$.

(5) Now suppose $\mu \in U^\circ$. We define the real linear functional μ_1 on $\varphi(C(X)) \subset C(\varphi(K))$ (regarded as real-linear spaces) by $\mu_1(\bar{f}) = \operatorname{Re}\mu(f)$ for every $f \in C(X)$. Let p be the seminorm on $C(\varphi(K))$

$$p(g) = \sup_{\varphi(K)} \operatorname{Re}(g) \quad \text{for every } g \in C(\varphi(K)).$$

Then

$$\mu_1(\bar{f}) = \operatorname{Re}\mu(f) \leq \sup_{\nu \in U^\circ} \operatorname{Re}\nu(f) = \sup_{\varphi(K)} \operatorname{Re}\bar{f} = p(\bar{f}) \quad \text{for all } f \in C(X).$$

By Hahn–Banach there is a real-linear extension μ_2 on $C(\varphi(K))$ of μ_1 such that

$$\mu_2(g) \leqq p(g) \quad \text{for all } g \in C(\varphi(K)).$$

Define the complex linear functional μ' on $C(\varphi(K))$ by

$$\mu'(g) = \mu_2(g) - i\mu_2(ig).$$

Then μ' is a measure on $\varphi(K)$ and for every $f \in C(X)$ we have

$$\mu'(\bar{f}) = \operatorname{Re}\mu(f) - i\operatorname{Re}\mu(if) = \operatorname{Re}\mu(f) + i\operatorname{Im}\mu(f) = \mu(f).$$

Furthermore μ' is a real measure because for every real-valued continuous function g on $\varphi(K)$, $p(ig) = 0$, hence $\mu_2(ig) = 0$. μ' is positive because $g \leqq 0$ implies $p(g) \leqq 0$, hence $\mu'(g) = \mu_2(g) \leqq 0$.

(6) $\varphi(U^\circ)$ is convex and compact, so by Choquet's theorem there is a positive measure μ'' on $\varphi(K)$ such that μ'' coincides with μ' on $A(\varphi(U^\circ))$, i.e. on every affine continuous function on $\varphi(U^\circ)$, hence particularly on $\varphi(N)$,

$$\|\mu''\| \leqq \|\mu'\| \leqq 1,$$

and μ'' is maximal with respect to the cone of continuous convex functions on $\varphi(U^\circ)$. μ'' finally defines a continuous linear functional $\bar{\mu}$ on $C(X)$, i.e. a measure on X by

$$\bar{\mu}(f) = \mu''(\bar{f}), \quad f \in C(X)\,.$$

What remains to show now is $\bar{\mu} \in U^\circ$ and $\bar{\mu}$ is a boundary measure on X.

(7) Let $f \in U$. Then

$$\operatorname{Re}\bar{\mu}(f) = \operatorname{Re}\mu''(\bar{f}) = \mu''(\operatorname{Re}\bar{f}) \leqq 1$$

because $\|\mu''\| \leqq 1$ and $\operatorname{Re}\bar{f} \leqq 1$ on $\varphi(K)$, hence $\bar{\mu} \in U^\circ$.

(8) For $f \in C_{\mathbf{R}}(X)$, $f \geqq 0$, we have

$$|\bar{\mu}|(f) = \sup\{|\bar{\mu}(g)| \mid g \in C(X),\ |g| \leqq f\}$$
$$= \sup\{|\mu''(\bar{g})| \mid g \in C(X),\ |g| \leqq f\} \leqq \mu''(|\bar{f}|)$$

because $|g| \leqq f$ on X implies $|\bar{g}| \leqq |\bar{f}|$ on $\varphi(K)$.

(9) To prove $|\bar{\mu}|$ is R-maximal it suffices to verify

$$|\bar{\mu}|(\hat{f}-f) = 0 \quad \text{for every } f \in C_{\mathbf{R}}(X),\ f \geqq 0\,.$$

We have

$$|\bar{\mu}|(\hat{f}-f) = \inf\{|\bar{\mu}|(g-f) \mid g \in -R,\ g \geqq f\}$$
$$\leqq \inf\{\mu''(|\bar{g}-\bar{f}|) \mid g \in -R,\ g \geqq f\} = \mu''(\inf\{|\bar{g}-\bar{f}|\} \mid g \in -R,\ g \geqq f\}$$

(note that the set $\{|\bar{g}-\bar{f}| \mid g \in -R,\ g \geqq f\}$ is directed downward on $\varphi(K)$, as $-R$ is infimum-stable on X). Because μ'' is maximal with respect to the convex functions on $\varphi(U^\circ)$ it suffices to show that pointwise on $\varphi(K)$:

(10) $\inf\{|\bar{g}-\bar{f}| \mid g \in -R,\ g \geqq f\} \leqq \widehat{|\bar{f}|} - |\bar{f}|$

where $\widehat{|\bar{f}|}$ denotes the upper (concave) envelope of $|\bar{f}|$ on $\varphi(K)$

$$\widehat{|\bar{f}|} = \inf\{h \in A(\varphi(U^\circ)) \mid h \geqq |\bar{f}| \text{ on } \varphi(K)\}\,.$$

To prove (10) consider that the real parts of the evaluations of the elements of N plus real constants form a dense subset of $A(\varphi(U^\circ))$ which coincides on $\varphi(K)$ with $\mathbf{R}+\varphi(N)$, because for elements of N φ means nothing but the evaluation map. So we have

$$\widehat{|\bar{f}|} = \inf\{\alpha_0+\operatorname{Re}\bar{j} \mid \alpha_0 \in \mathbf{R},\ j \in N \text{ and } \alpha_0+\operatorname{Re}\bar{j} \geqq |\bar{f}| \text{ on } \varphi(K)\}\,.$$

Now let $h = \alpha_0 + \operatorname{Re}\bar{j}$, such that $h \geqq |\bar{f}|$ on $\varphi(K)$. Clearly $\alpha_0 \geqq 0$ since h is positive on $\varphi(U^\circ)$, and for every $x \in X$ and $z_0 \in \gamma$

$$(\alpha_0+\operatorname{Re}\bar{j})\left(\varphi\left(\frac{z_0}{\varrho(x,z_0)}\varepsilon_x\right)\right) \geqq \left|\bar{f}\left(\varphi\left(\frac{z_0}{\varrho(x,z_0)}\varepsilon_x\right)\right)\right|,$$

that is,

$$\alpha_0 + \frac{1}{\varrho(x,z_0)} \operatorname{Re}(z_0 j(x)) \geqq \frac{1}{\varrho(x,z_0)} f(x),$$

that is,

$$f(x) \leqq \alpha_0 \varrho(x,z_0) + \operatorname{Re}(z_0 j(x))$$

for fixed z_0 then $z_0 j$ is an element of N as well, hence $\operatorname{Re}(z_0 j) \in -R$, and because $\varrho(x,z_0)$ is R-superharmonic we conclude (cf. [1], cor. I.3.6).

$$\text{(11)} \quad \hat{f}(x) = \sup\{\nu(f) \mid \nu \in M_x^+\} \leqq \alpha_0 \varrho(x,z_0) + \operatorname{Re}(z_0 j(x)) \quad \text{for all } x \in X,\ z_0 \in \gamma .$$

Now let $z\varepsilon_x \in K$, $\delta > 0$. (11) guarantees the existence of $g \in -R$ such that $g \geqq f$ and

$$g(x) \leqq \alpha_0 \varrho\left(x, \frac{z}{|z|}\right) + \operatorname{Re}\left(\frac{z}{|z|} j(x)\right) + \delta,$$

hence by (2) if $z \neq 0$ (the case $z = 0$ is trivial)

$$\begin{aligned} |z|\,|g(x)| &\leqq |z| \cdot \alpha_0 \varrho\left(x, \frac{z}{|z|}\right) + |z| \cdot \operatorname{Re}\left(\frac{z}{|z|} j(x)\right) + |z|\delta \\ &\leqq \frac{1}{\varrho\left(x, \frac{z}{|z|}\right)} \cdot \alpha_0 \varrho\left(x, \frac{z}{|z|}\right) + \operatorname{Re}(zj(x)) + |z|\delta \\ &\leqq \alpha_0 + \operatorname{Re}(zj(x)) + |z|\delta = h(\varphi(z\varepsilon_x)) + |z|\delta . \end{aligned}$$

Finally we conclude

$$\begin{aligned} |\bar{g} - \bar{f}|(\varphi(z\varepsilon_x)) &= |z|(g(x) - f(x)) \leqq h(\varphi(z\varepsilon_x)) - |\bar{f}|(\varphi(z\varepsilon_x)) + |z|\delta \\ &\leqq (h - |\bar{f}|)(\varphi(z\varepsilon_x)) + |z|\delta . \end{aligned}$$

Since δ was arbitrary (10) is verified, $\bar{\mu}$ is a boundary measure in U° such that $\mu - \bar{\mu} \in N^\circ$.

4. Applications.

4.1. Let K be a closed subset of the compact Hausdorff space X, N a linear subspace of $C(X)$ which contains the constant functions and separates the points of X, R a sup-stable convex cone in $C_{\mathbf{R}}(X)$ such that $\operatorname{Re} N \subset R$. Define $\varrho(x,z) = 1$ if $x \in K$ and $\varrho(x,z) = \infty$ elsewhere. Clearly ϱ is lower semicontinuous and R-superharmonic for fixed $z \in \gamma$ if $\nu >_R \varepsilon_x$ for all $x \in K$ implies μ is supported by K. In this case the above theorem guarantees that for every measure μ supported by K there is a boundary measure $\bar{\mu}$ supported by K such that $\mu - \bar{\mu} \in N^\circ$ and $\|\bar{\mu}\| \leqq \|\mu\|$.

4.2. Let μ be a complex measure on the compact Hausdorff space X, N and R as above, and write $\mu = h|\mu|$, where h is a Borel function of modulus one. Let γ_0 be a closed subset of γ such that range $h \subset \gamma_0$ and define $\varrho(x,z) = 1/\|\mu\|$ if $z \in \gamma_0$ and $\varrho(x,z) = \infty$ elsewhere. Clearly ϱ is lower semicontinuous and R-superharmonic and therefore defines an R-stable 0-neighborhood U in $C(X)$. $\mu \in U^\circ$, because for every $f \in U$, $x \in X$ we have $\operatorname{Re}(h(x)f(x)) \leqq 1/\|\mu\|$ since $h(x) \in \gamma_0$, hence

$$\operatorname{Re}\mu(f) = \operatorname{Re}|\mu|(hf) = |\mu|(\operatorname{Re}(hf)) \leqq \|\mu\| \cdot \frac{1}{\|\mu\|} = 1 .$$

Then by the above theorem there is a boundary measure $\bar{\mu} = \bar{h}|\bar{\mu}|$ such that $\mu - \bar{\mu} \in N^\circ$ and $\bar{\mu} \in U^\circ$. Clearly $\|\bar{\mu}\| \leqq \|\mu\|$, since U contains the ball with radius $1/\|\mu\|$ in $C(X)$.

If γ_0 is the intersection of a finite number of closed half circles in γ the unimodular function $\bar{h}$ can be chosen such that range $\bar{h} \subset \gamma_0$ as well. To prove this take any $\bar{h}$ such that $\bar{\mu} = \bar{h}|\bar{\mu}|$, denote

$$X_0 = \{x \in X \mid \bar{h}(x) \notin \gamma_0\}$$

and suppose $|\bar{\mu}|(X_0) > 0$. By assumption now there is at least one closed half circle β which contains γ_0 and for which $|\bar{\mu}|(X_\beta) > 0$, where

$$X_\beta = \{x \in X \mid \bar{h}(x) \notin \beta\} .$$

Because of the regularity of μ there is a compact subset K of X_β such that $|\bar{\mu}|(K) > 0$ and an open neighborhood V_ε of K such that $|\bar{\mu}|(V_\varepsilon \setminus K) < \varepsilon$. Let χ_ε be a continuous real valued function such that $0 \leqq \chi_\varepsilon \leqq 1$ and

$$\chi_\varepsilon|_K = 1, \quad \chi_\varepsilon|_{X \setminus V_\varepsilon} = 0$$

and denote $z_\beta \in \gamma$ the complex number which characterizes β by

$$\beta = \{z \in \gamma \mid \operatorname{Re} z z_\beta \leqq 0\} .$$

Then for each $\lambda > 0$ the function $\lambda z_\beta \chi_\varepsilon$ is an element of U, hence $\bar{\mu}(\lambda z_\beta \chi_\varepsilon) \leqq 1$ for each $\lambda \geqq 0$, that is,

$$\bar{\mu}(z_\beta \chi_\varepsilon) = |\bar{\mu}|(z_\beta \bar{h} \chi_\varepsilon) \leqq 0$$

which clearly contains a contradiction because $z_\beta \bar{h} > 0$ on K which implies $|\bar{\mu}|\,|_K(z_\beta \bar{h}) > 0$ and because ε can be chosen sufficiently small.

Therefore $|\bar{\mu}|(X_0)=0$ and $\bar{h}$ may be replaced by its restriction on $X \setminus X_0$ which only takes values in γ_0.

4.3. An extension theorem in [6] states the following:

Let X be a compact Hausdorff space, M a real linear subspaces in $C_{\mathsf{C}}(X)$ (respectively $C_{\mathsf{R}}(X)$), N a closed convex cone in M, which separates the points of X and contains the constant functions, R a sup-stable convex cone in $C_{\mathsf{R}}(X)$ which contains the real parts of all functions in $\operatorname{lin} N$ (the complex linear hull of N). Suppose Y is a compact subset of X such that

(1) for every measure μ supported by Y there is a boundary measure $\bar{\mu}$ supported by Y such that $\bar{\mu}-\mu \in (\operatorname{lin} N)^{\circ}$.
(2) For every complex boundary measure $\mu \in N^{\circ}$ implies $\mu|_Y \in M^{\circ}$.
(3) $\operatorname{lin} N|_Y$ is dense in $M|_Y$.

Suppose U is a 0-neighborhood in $C_{\mathsf{C}}(X)$ defined by a strictly positive bounded l.s.c. function $\varrho\colon X \times \gamma \to \mathsf{R}$ such that U is R-stable.

Then for every $f \in M$ such that $f|_Y \in U|_Y$ there is $g \in N \cap U$ such that $f|_Y = g|_Y$.

Applying our stability-theorem we know the neighborhood U to be R-stable if $\varrho_z\colon X \to \mathsf{R} \cup \{\infty\}$ is R-superharmonic for every $z \in \gamma$. Thus the above extension theorem contains generalizations of some well-known results, such as T. B. Andersen's split-face theorem [3] and Alfsen–Hirsberg's theorem about extensions of affine functions on compact convex sets [2].

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