JB ALGEBRAS WITH AN EXCEPTIONAL IDEAL

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Abstract.

Let \( \mathcal{A} \) be an arbitrary JB algebras with exceptional ideal \( J \). Then \( J \) is isomorphic to the set of all continuous cross sections vanishing at infinity of some fibre bundle \( \{ B, \Omega, \text{pr}, M^8_3, E_5 \} \), with locally compact base space \( \Omega \). Let \( J^0 = \{ a \in \mathcal{A} \mid a \cdot b = 0 \text{ for all } b \in J \} \) be the annihilator of \( J \). Then \( J^0 \) is a JB ideal and \( \mathcal{A}/(J + J^0) \) is a special JB algebra of type \( I_{\leq 3} \).

In [1] Alfsen, Shultz and Størmer have shown that every JB algebra \( \mathcal{A} \) with identity is, modulo its unique exceptional ideal \( J \), a JC algebra. These results extend also to the non-unital case, if one modifies the axioms for JB algebras slightly [7]. Thus we shall assume throughout that the JB algebras satisfy the axioms given in [7]. Then we may, if necessary, adjoin a unit and apply the methods of Alfsen, Shultz and Størmer.

If one interprets a JB algebra as the set of observables of a physical system, one would consider \( J \) as an unphysical part. Hence it seems desirable to know how the exceptional ideal \( J \) lies in the JB algebra \( \mathcal{A} \). Though one might expect \( J \) to be a direct summand, Alfsen, Shultz and Størmer have already constructed a counterexample for this conjecture [1, remark 9.8]. Here we shall show that in general \( J \) is almost a direct summand. More precisely we prove that \( \mathcal{A} \) is an extension of \( J \oplus J^0 \) by a special JB algebra of type \( I_{\leq 3} \). Here \( J^0 \) denotes the annihilator of \( J \) in \( \mathcal{A} \). This is done by describing \( J \) as a \( M^8_3 \)-fibre bundle over \( \hat{J} = \text{Prim} J \), its primitive ideal space. The proof for this follows the lines of the corresponding result for \( n \)-homogeneous C*-algebras [8].

Throughout \( \mathcal{A} \) will denote an arbitrary JB algebra satisfying the axioms of [7] and \( J \) the exceptional ideal of \( \mathcal{A} \). By an exceptional JB algebra \( J \) we understand a JB algebra, all of whose nontrivial factor representations are onto \( M^8_3 \). Equivalently an exceptional JB algebra can be defined via the s-identities [1, theorem 9.5].

**Lemma 1.** \( \hat{J} = \text{Prim} J \) with the Jacobson topology is a locally compact Hausdorff space and \( \mathcal{C}_0(\hat{J}) \), the algebra of all continuous real valued functions on \( \hat{J} \) vanishing at infinity, is the center of \( J \).

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Proof. The proof follows almost verbatim the proof of Kaplansky [5]. As an intermediate space one uses the set \( H \) of all \( J \)-homomorphisms of \( J \) into \( M_3^8 \) with the pointwise convergence topology. Using \( H \) we can define on \( \hat{J} \) the weak topology as the quotient topology from \( H \). With the aid of the functional calculus of JB algebras on shows then that \( J \) contains all weakly continuous real valued functions on \( \hat{J} \) vanishing at infinity. The remainder is then almost trivial.

Since \( \mathcal{C}_0(\hat{J}) \subseteq J \), \( J \) clearly has a bounded (bound 1) central approximate unit.

Our next aim is to show that locally \( J \) looks like an algebra of continuous functions with values in \( M_3^8 \). To do this we have to lift the matrix units of \( M_3^8 \) locally.

**Lemma 2.** Let \( a_1, a_2 \in \mathcal{C} \), the Cayley numbers, with

\[
a_i \circ a_j + a_j \circ a_i = -2 \delta_{ij}, \quad i, j = 1, 2.
\]

Then we have for every purely imaginary \( a \in \mathcal{C} \)

\[
a_1 \circ (a_2 \circ a) = a_2 \circ (a \circ a_1).
\]

This is shown by expanding \( a, a_1 \) and \( a_2 \) with respect to the canonical basis of \( \mathcal{C} \) [2, p. 221].

**Lemma 3.** For each \( P_0 \in \hat{J} \) there exists a neighborhood \( U \) of \( P_0 \) such that \( J(U) \cong \mathcal{C}(\bar{U}) \otimes M_3^8 \). Here \( J(U) \) is the quotient of \( J \) by the kernel of \( U \).

Proof. Let \( P_0 \in \hat{J} \) and let \( \{e_{ij}\}_{i,j=1}^3 \) be the usual matrix units of \( M_3(\mathcal{C}) \). Arguing as in [8, proof of the lemma in § 2] we can find elements \( e_1, e_2 \) and \( e_3 \) in \( J \) and a neighborhood \( U \) of \( P_0 \) such that \( e_i(P) = e_{ii} \) and such that the \( e_i(P) \) form a system of minimal orthogonal projections with sum 1 for all \( P \in U \). Choose now \( s \in J \) such that \( s(P_0) = e_{12} + e_{21} \) and

\[
[(e_1 + e_2) \circ s](P) = [s \circ (e_1 + e_2)](P) = s(P) \quad \text{for all } \ P \in U.
\]

Replacing \( s \) by \( s - U e_i s - U e_2 s \) if necessary we may even assume \( (U e_i s)(P) = 0 \) for all \( P \in U \).

Since \( s^2(P_0) = (e_{11} + e_{22}) \) we can replace \( s \) by an element \( f(s^2) \circ s = s_{12} \), for at suitable continuous function \( f \), such that

\[
s^2_{12}(P) = (e_1 + e_2)(P), \quad (U e_i s_{12})(P) = 0 \quad i = 1, 2, 3
\]

holds in some neighborhood of \( P_0 \), which we denote by \( U \) again. Similarly determine an element \( s_{13} \in J \) with analogous properties. Hence by the
coordinatization theorem [4] we may choose coordinates of $M^8_3$ at each $P \in U$ such that
\[ e_i(P) = e_{iib}, \quad s_{12}(P) = e_{12} + e_{21}, \quad s_{13}(P) = e_{13} + e_{31} \]
and
\[ s_{23}(P) = 2(s_{12} \circ s_{13})(P) = e_{23} + e_{32} \quad \text{for } i,j=1,2,3. \]

Now let $b_1, b_2, \ldots, b_8$ denote the canonical basis of $\mathcal{C}$. There exist elements $s^{(i)}_{ij} \in J$, $i=2,3$, such that $s^{(i)}_{12}(P_0) = b_i e_{12} - b_j e_{21}$. Replacing $s^{(2)}_{12}$ by $s^{(2)}_{12} - s_{12} \circ (s_{12} \circ s^{(2)}_{12})$ if necessary and using again the functional calculus in $J$, we can achieve
\[ s^{(i)}_{12} \circ s^{(j)}_{12}(P) = \delta_{ij}(e_1 + e_2)(P), \quad (U_{e_i} s^{(j)}_{12})(P) = 0, \quad i,j=1,2,3, \]
throughout a neighborhood of $P_0$, which we denote again by $U$. Above we have written $s^{(i)}_{ij}$ for $s_{ij}$. Arguing similarly with $s^{(3)}_{12} - \sum_{k=1}^2 s^{(k)}_{12} \circ (s^{(k)}_{12} \circ s^{(3)}_{12})$ we construct a local symmetry $s^{(3)}_{12}$ such that
\[ s^{(i)}_{12} \circ s^{(j)}_{12}(P) = \delta_{ij}(e_1 + e_2)(P), \quad U_{e_i} s^{(j)}_{12}(P) = 0 \]
holds for $P \in U$ and $i,j=1,2,3$.

Now let $s^{(4)}_{12} = 8(s_{13} \circ s^{(2)}_{12}) \circ (s_{23} \circ s^{(3)}_{12})$. Then the local symmetries $s^{(i)}_{12}$, $i=1,\ldots,4$ will satisfy (1) throughout $U$. Now choose $s^{(5)}_{12} \in J$ with
\[ s^{(5)}_{12}(P_0) = b_5 e_{12} - b_5 e_{21}. \]

Working with $s^{(5)}_{12} - \sum_{k=1}^4 s^{(k)}_{12} \circ (s^{(k)}_{12} \circ s^{(5)}_{12})$ as above we can determine a local symmetry $s^{(5)}_{12} \in J$ such (1) will hold also for the larger system. Define now
\[ s^{(5+i)}_{12} = 8(s_{13} \circ s^{(i)}_{12}) \circ (s_{23} \circ s^{(5)}_{12}) \quad i=1,2,3, \]
then the system $(s^{(i)}_{12})$, $i=1,\ldots,8$ satisfies (1) in $U_0$. For the proof of this one needs lemma 2. The remaining local symmetries $s^{(i)}_{13}$ and $s^{(i)}_{23}$, $i=2,\ldots,8$ are then defined by $2s^{(1)}_{23} s^{(i)}_{12} = s^{(i)}_{13}$ and $s^{(i)}_{23} = 2s^{(1)}_{12} s^{(i)}_{13}$. This proves the lemma.

An immediate consequence of this lemma is

**Theorem 1.** a) $J$ defines a fibre bundle $\mathcal{B}(J) = \{B, \text{pr}, \hat{J}, M^8_3, E_6\}$, the structure bundle, where $B = \bigcup_{P \in J} J(P)$ and where pr is defined by pr $(a(P)) = P \in \hat{J}$.

b) Conversely, the system $J(\mathcal{B})$ of all continuous cross sections vanishing at infinity of a fibre bundle $\mathcal{B} = \{B, \text{pr}, \Omega, M^8_3, E_6\}$ with $\Omega$ locally compact is a JB algebra.

c) $J$ is isomorphic to the Jordan algebra of all continuous cross sections of its structure bundle.

d) Two exceptional JB algebras are isomorphic iff their structure bundles are isomorphic.
This theorem is shown almost like the corresponding results for $n$-homogeneous C*-algebras in [8, § 2] once one knows that $E_6 = \text{Aut} \, M_3^2$ [3]. Obviously the norm of the JB algebra in (b) is given by the sup norm of the fibres. In order to prove (c) one defines for each $a \in J$ the map $f_a : J \ni P \rightarrow a(P)$. Clearly $f_a$ is a continuous cross section vanishing at infinity and $a \rightarrow f_a$ is a $J$-isomorphism of $J$ onto $f(J)$. It remains to show that $f(J) = J(\mathfrak{A}(J))$, i.e. that $f$ is onto. For this it suffices that every continuous cross section with compact support is in $f(J)$. However since $\mathcal{C}_0(\tilde{J}) \subset J$ and since $J$ is locally trivial a partition of unity argument reduces this problem to the case of a trivial bundle.

Now let $\mathfrak{A}$ be an arbitrary JB algebra with exceptional ideal $J$. For each $y \in \mathfrak{A}$ let $J(y)$ denote the JB subalgebra of $\mathfrak{A}$ generated by $y$. If $\mathfrak{A}$ has a unit we choose the subalgebra generated by $y$ and 1. Then we can define the centralizer $\mathcal{I}(y)$ by

$$ \mathcal{I}(y) = \{ a \in \mathfrak{A} \mid a \circ (y^n \circ y^m) = (a \circ y^n) \circ y^m \text{ for all } n,m \geq 0 \} . $$

Clearly

$$ \mathcal{I}(y) = \{ a \in \mathfrak{A} \mid a \circ (y_1 \circ y_2) = (a \circ y_1) \circ y_2 \text{ for all } y_i \in J(y) \} . $$

It is not too hard to see that this definition coincides with that of [1]. For $S \subset \mathfrak{A}$ let $\mathcal{I}(S) = \bigcap_{y \in S} \mathcal{I}(y)$ denote the centralizer of $S$.

**Lemma 4.** $J \cap \mathcal{I}(J) = \mathcal{C}_0(\tilde{J}) \subset \mathcal{I}(\mathfrak{A})$

**Proof.** Let $a \in \mathcal{I}(J) \cap J$ and $b \in \mathfrak{A}$. Consider $c = a \circ (b^n \circ b^m) - (a \circ b^n) \circ b^m$ and let $\pi$ be an arbitrary factor representation of $\mathfrak{A}$. Then we have $\pi(c) = 0$ if $J$ is annihilated and $\pi(a) = x \cdot 1$ or $\pi(c) = 0$ if $J$ is not annihilated. Since all factor representations are faithful, we get $c = 0$.

**Lemma 5.** The annihilator $J^0$ of $J$, defined by

$$ J^0 = \{ a \in \mathfrak{A} \mid a \circ b = 0 \text{ for all } b \in J \} $$

satisfies

$$ J^0 = \{ a \in \mathfrak{A} \mid a \circ b = 0 \text{ for all } b \in \mathcal{C}_0(\tilde{J}) \} . $$

$J^0$ is a closed special JB ideal of $\mathfrak{A}$.

This lemma follows immediately from the fact that $J$ contains a bounded central approximate unit.

**Definition.** A JB algebra $\mathfrak{A}$ is called essentially exceptional if $J^0 = (0)$. 
THEOREM 2. Let $\mathcal{A}$ be an arbitrary JB algebra with exceptional ideal $J$. Then $J \oplus J^0$ is a closed JB ideal of $\mathcal{A}$ and $\mathcal{A}/(J \oplus J^0)$ is a special JB algebra of type $I_{\leq 3}$, i.e. all factors representations of $\mathcal{A}/(J \oplus J^0)$ are special Jordan factors of type $I_{\leq 3}$.

PROOF. Since the set of factor representations of type $I_{\leq 3}$ is closed in $\mathcal{A}$ and since $\hat{J}$ is dense in $(\mathcal{A}/J^0)^\wedge$, $(\mathcal{A}/J^0)$ is a JB algebra of type $I_{\leq 3}$. Hence $\mathcal{A}/(J \oplus J^0)$ is a special JB algebra of type $I_{\leq 3}$.

LEMMA 6. Let $\mathcal{A}$ be an arbitrary JB algebra and let $\hat{\mathcal{A}} = \text{Prim} \mathcal{A}$ denote its primitive spectrum with the Jacobson topology. Then the sets $\mathcal{A}$ corresponding to factor representations of type $I_{\leq n}$ are closed.

PROOF. Since in JB algebra one has the functional calculus of continuous functions the usual C*-proof shows that for each $x \in \mathcal{A}$ the function $f_x : \mathcal{A} \ni P \to |x(P)|$ is lower semicontinuous. Now define for each $x \in \mathcal{A}^+ = \{ y^2 : y \in \mathcal{A} \}$ the function

$$g_x : \mathcal{A} \ni P \to \sup \sum |x_i(P)|,$$

where the supremum is taken over all finite sets $\{ x_i \} \subset \mathcal{A}^+$ with $0 \leq \sum x_i \leq x$. Then $\sum |x_i(P)| \leq \sum \text{Tr} x_i(P) \leq \text{Tr} x(P)$ and thus $g_x(P) \leq \text{Tr} x(P)$. Conversely we can always choose $\{ x_i \} \subset \mathcal{A}^+$ such that $\sum x_i \leq x$ and such that $\sum |x_i(P)| = \text{Tr} x(P)$. Hence the function $g_x(\cdot) = \text{Tr} x(\cdot)$ is lower semicontinuous on $\mathcal{A}$. Clearly

$$\mathcal{A}_n = \{ P \in \mathcal{A} \mid \text{Tr} 1(P) \leq n \}$$

is closed.

This result can probably be shown also be using polynomial identities. However we remark that the usual polynomial identities for associative rings do not work. As an example consider

$$\mathcal{N} = \{ x \circ (y \circ z) - (x \circ y) \circ z \mid x, y, z \in \mathcal{A} \}$$

the associator set of $\mathcal{A}$ then every factor representation of type $I_{\leq 2}$ maps $\mathcal{N}^2 = \{ a \circ b \mid a, b \in \mathcal{N} \}$ onto a central element. In other words $a \circ (b \circ c) - (a \circ b) \circ c$, $a \in \mathcal{N}^2$, $b, c \in \mathcal{A}$ is annihilated by all elements of $2 \mathcal{A}$. Every factor representation $\pi$ of type $I_2$ maps $\mathcal{A}$ onto a spin factor and we say $\pi$ is of type $I_{2,n}$ if $\dim \pi(\mathcal{A}) = n + 1$.

Using $\mathcal{N}$ and a refined argument, one can even show that the set of all factor representations of type $I_{\leq 2,n}$ are closed.

Theorem 2 implies in particular that any essentially exceptional JB algebra is of type $I_{\leq 3}$. There is a different way to view theorem 2. Let $\mathcal{A}$ be an essentially
exceptional JB algebra. Let $a \in \mathcal{A}$ and $P \in \mathcal{J}$. Then $a(P)$ is defined. By using the center of $J$ and a suitable coordinate bundle for $\mathcal{B}(J)$, it is easy to see that $\mathcal{J} \ni P \rightarrow a(P)$ defines a bounded continuous cross section of $\mathcal{B}(J)$. Clearly the map $\mathcal{A} \ni a \rightarrow (a(P))$ is a $J$-isomorphism of $\mathcal{A}$ into the JB algebra $M(J)$ of all bounded continuous cross sections of $\mathcal{B}(J)$. In fact $M(J)$ can be considered as the multiplier algebra of $J$. Let $a \in M(J)$ be positive and let $(u_x)$ be an bounded increasing central approximate unit of $J$. Then $au_x$ is a bounded increasing family of elements of $J$. Hence $(a \circ u_x)$ converges strongly in $\mathcal{J}$, the enveloping algebra of $J$. Here $\mathcal{J}$ denotes the strong closure of $J$ in $\mathcal{A}$. Thus $M(J)$ can also be viewed as the idealizer of $J$ in $\mathcal{J}$. In particular we see, $\mathcal{A} \subset M(J) \subset \mathcal{J} = \mathcal{C}(\Omega) \otimes M_3^8$ [6]. This can be used to improve theorem 2 slightly. (The authors thank the referee for providing a proof of this result).

**Corollary.** Let $\mathcal{A}$ be an essentially exceptional JB algebra. Then $\mathcal{A}$ is of type $I_{\leq 3}$ and every factor representation maps $\mathcal{A}$ onto a subfactor of $M_3^8$.

**Proof.** We may consider $\mathcal{A}$ a subalgebra of $\mathcal{B} = \mathcal{J} = \mathcal{C}(\Omega) \otimes M_3^8$. Let $\varrho$ be any pure state of $\mathcal{A}$ and let $\tilde{\varrho}$ be an arbitrary pure state extension of $\varrho$ to $\mathcal{B}$. By $c(\varrho)$ respective $c(\tilde{\varrho})$ denote the central support $\varrho$, $\tilde{\varrho}$ in $\mathcal{A}^{**}$ respectively $\mathcal{B}^{**}$. Considering $\mathcal{A}^{**}$ as a subalgebra of $\mathcal{B}^{**}$ the map $x \rightarrow xc(\varrho)$ defined an embedding of $\mathcal{A}^{**}c(\varrho)$ into $\mathcal{B}^{**}c(\tilde{\varrho}) \cong M_3^8$.

Conversely any subfactor $\mathcal{D}$ of $M_3^8$ can be obtained from a factor representation of a suitable essentially exceptional JB algebra. To see this choose a canonical basis of $M_3^8 b_1, \ldots, b_{27}$ such that $b_1, \ldots, b_r$ is a basis of $\mathcal{D}$. Now let $\mathcal{A}$ be the algebra of all sequence $(a_n)$ with $a_n \in M_3^8$ and $a_n = \sum \alpha_{n,k} b_k$ where we assume $\alpha_{n,k} \to 0$ for $27 \geq k \geq r+1$ and $\alpha_{n,k} \to \alpha_k \in \mathbb{R}$ for $1 \leq k \leq r$. Then $J = c_0 \otimes M_3^8$, $J^0 = (0)$ and $\mathcal{A}/J \cong \mathcal{D}$.

**References**
