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PICARD SCHEMES OF QUOTIENTS BY FINITE COMMUTATIVE GROUP SCHEMES

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0. Introduction.

Let N be a finite commutative group scheme acting freely on a variety X, proper over a field k. Assume the quotient Y exists. This paper mainly deals with relating the Picard scheme of Y with the Picard scheme of X and the Cartier dual $D(N)$ of N.

We have divided the material in 2 sections:

1. Linearized invertible sheaves and Picard groups
2. Picard schemes of quotients.

In Section 1 we discuss the G-linearized Picard group $\mathrm{Pic}^G(X)$ and the natural map

$$p\colon \mathrm{Pic}^G(X) \to \mathrm{Pic}(X) .$$

G is an arbitrary k-group scheme, acting on a variety X, proper over k. We obtain an exact sequence

$$(1.3) \qquad 0 \to \chi(G) \to \mathrm{Pic}^G(X) \underset{p}{\to} (\mathrm{Pic}(X))^G \underset{q}{\to} H^2(G, G_m) ,$$

where $\chi(G)$ is the character group of G, $(\mathrm{Pic}(X))^G$ is the group of universally G-fixed invertible sheaves on X and $H^2(G, G_m)$ is the second Hochschild cohomology group. The above map q and the cohomology group is studied.

Section 2 is devoted to a functorial study of the results from Section 1, especially the sequence (1.3). For some finite group schemes N, acting freely on X, we obtain the desirable exact sequence of commutative group schemes

$$(2.1) \qquad 0 \to D(N) \to \mathbf{Pic}_{Y/k} \underset{p}{\to} (\mathbf{Pic}_{X/k})^N \to 0 ,$$

where the latter scheme is the fixed point scheme of the action of N on $\mathbf{Pic}_{X/k}$. The sequence (2.1) determines the Picard schemes of the counter examples by Igusa [6] and Serre [11] to the "completeness of the characteristic linear system of curves on an algebraic surface". Our investigations moreover show that if N acts with a fixed point on X, then the "N-linearized Picard functor of X over k" is representable (2.13).

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For definition and properties of the Picard scheme we refer to [3, V, VI]. For the notion of Cartier dual and other concepts from the theory of commutative group schemes, we refer to Oort's notes [10]. Finally we refer to [1] for definition and properties of Hochschild cohomology.

A word about words. We use "variety X proper over k" synonymous with "scheme X, proper over k and $\Gamma(\mathcal{O}_X)=k$". We mainly choose the latter expression, because it is functorial: it base extends to "scheme $X\times S$, proper over S and $\Gamma(\mathcal{O}_{X\times S})=\Gamma(\mathcal{O}_S)$".

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1. Linearized invertible sheaves and Picard groups.

Let k be a field. Throughout this section X is a scheme, proper over k and $\Gamma(\mathcal{O}_X)=H^0(X,\mathcal{O}_X)=k$. Assume X is equipped with an action of a k-group scheme G, $\sigma\colon G\times X\to X$.

Recall the following definition:

Definition 1.1. [7, § 3]. An invertible sheaf $\mathcal{L}\in\operatorname{Pic}(X)$ is called G-linearized if

1.1.1: There exists an isomorphism $\varphi\colon\sigma^*\mathcal{L}\to p_2{}^*\mathcal{L}$, where $p_2\colon G\times X\to X$ is the projection onto X.

1.1.2: φ satisfies the following cocycle condition: If $m\colon G\times G\to G$ is the group law, p_{23}, $m\times 1_X$, $1_G\times\sigma$ all map $G\times G\times X\to G\times X$, then the following diagram of morphisms between invertible sheaves on $G\times G\times X$ is commutative:

$$\begin{tikzcd}
(\sigma\circ(1_G\times\sigma))^*\mathcal{L} \arrow[r, "(1_G\times\sigma)^*\varphi"] \arrow[dd, equal] & (p_2\circ(1_G\times\sigma))^*\mathcal{L} \arrow[d, equal] & \\
& (\sigma\times p_{23})^*\mathcal{L} \arrow[r, "p_{23}{}^*\varphi"] & (p_2\circ p_{23})^*\mathcal{L} \arrow[d, equal] \\
(\sigma\circ(m\times 1_X))^*\mathcal{L} \arrow[rr, "(m\times 1_X)^*\varphi"] & & (p_2\circ(m\times 1_X))^*\mathcal{L}\,.
\end{tikzcd}$$

The isomorphism φ of 1.1.1, satisfying 1.1.2, is called a G-linearization of $\mathcal{L}$.

Remark. The G-linearized invertible sheaves on X, modulo isomorphisms, form an abelian group, which we denote $\operatorname{Pic}^G(X)$. Note that we have a canonical group homomorphism $p\colon\operatorname{Pic}^G(X)\to\operatorname{Pic}(X)$, namely the one that to an invertible sheaf $\mathcal{L}$, equipped with a G-linearization φ, simply assigns the invertible sheaf $\mathcal{L}\in\operatorname{Pic}(X)$.

PROPOSITION 1.2. *The kernel of* p: $\mathrm{Pic}^G(X) \to \mathrm{Pic}(X)$ *is naturally isomorphic to*

$$\chi(G) = \mathrm{Hom}_{k\text{-grp}}(G, G_m),$$

the character group of G.

PROOF. Clearly ker (p) may be identified with the group of G-linearizations of $\mathcal{O}_X$. Since the line bundle associated to $\mathcal{O}_X$ is $X \times A_1$, we have to study morphisms $\bar{\sigma}$: $G \times X \times A_1 \to X \times A_1$, such that $p_1 \circ \bar{\sigma} = \sigma \circ p_{12}$, where p_{12}: $G \times X \times A_1 \to G \times X$ (respectively p_1: $X \times A_1 \to X$) is the projection onto $G \times X$ (respectively X). In terms of S-valued points $g \in G(S)$, $x \in X(S)$, $a \in A_1(S)$ we then have

$$\bar{\sigma}(g, x, a) = (\sigma(g, x), \chi(g, x, a)).$$

Since the action is linear on A_1, we know $\chi(g, x, a) = a\chi(g, x, 1)$. Hence we have a morphism $\chi = \chi(-, -, 1)$: $G \times X \to A_1$, which factors through a morphism χ: $G \to A_1$, since $\Gamma(\mathcal{O}_{G \times X}) = \Gamma(\mathcal{O}_G)$. The cocycle condition 1.1.2 implies that χ is in fact a group homomorphism from G to G_m.

REMARK. The image of p is contained in

$$\{\mathcal{L} \in \mathrm{Pic}(X) \mid \sigma^*\mathcal{L} \otimes p_2^*\mathcal{L}^{-1} \text{ is isomorphic to } \mathcal{O}_{G \times X}\}.$$

This group is denoted $(\mathrm{Pic}(X))^G$.

PROPOSITION 1.3. *There exists a homomorphism* q: $(\mathrm{Pic}(X))^G \to H^2(G, G_m)$, *such that*

$$\mathcal{O} \to \chi(G) \to \mathrm{Pic}^G(X) \xrightarrow[p]{} (\mathrm{Pic}(X))^G \xrightarrow[q]{} H^2(G, G_m)$$

is an exact sequence of abelian groups.

PROOF. $H^2(G, G_m)$ is the 2nd Hochschild cohomology group with trivial action of G on G_m.

If $\mathcal{L} \in (\mathrm{Pic}(X))^G$ there exists an isomorphism φ: $\sigma^*\mathcal{L} \to p_2^*\mathcal{L}$. φ induces two isomorphisms between the same invertible sheaves on $G \times G \times X$:

$$(\sigma \circ (1_G \times \sigma))^*\mathcal{L} \underset{(m \times 1_X)^*\varphi}{\overset{(p_{23}^*\varphi) \circ (1_G \times \sigma)^*\varphi}{\rightrightarrows}} (p_2 \circ (m \times 1_X))^*\mathcal{L},$$

cf. 1.1.2.

These isomorphisms differ only by a unit $d_\varphi \in \Gamma(\mathcal{O}_{G \times G \times X}) = \Gamma(\mathcal{O}_{G \times G})$. Therefore we may consider d_φ as a morphism of schemes d_φ: $G \times G \to G_m$.

We claim that d_φ is a 2-cocycle [1, II, § 3]. In other words, for any k-scheme S, and any $g, g', g'' \in G(S)$, we have

$$d(g',g'')\cdot d(g+g',g'')^{-1}\cdot d(g,g'+g'')\cdot d(g,g')^{-1} = 1 \in G_m(S),$$

writing $d = d_\varphi$ for brevity.

First some notation: Let S be any k-scheme. Base extend the action σ of G on X to an action σ_S of $G \times S$ on $X \times S$ over S:

$$(\sigma_S : (G \times S) \underset{S}{\times} (X \times S) \to X \times S) = (\sigma \times 1_S : G \times X \times S \to X \times S).$$

Define $\mathscr{L}_S = p_1^* \mathscr{L}$ where $p_1 \colon X \times S \to X$ is the projection onto X. The isomorphism φ base extends to an isomorphism $\varphi_S \colon \sigma_S^* \mathscr{L}_S \to p_2^* \mathscr{L}_S$, where p_2 denotes the projection

$$(G \times S) \underset{S}{\times} (X \times S) \to X \times S.$$

The isomorphism φ_S induces two isomorphisms between the same invertible sheaves on $G \times X \times X \times S$:

$$(1.3.1) \qquad (\sigma_S \circ (1_{G \times S} \underset{S}{\times} \sigma_S))^* \mathscr{L}_S \underset{(m \times 1_{X \times S})^* \varphi_S}{\overset{(p_{23}^* \varphi_S) \circ (1_{G \times S} \times \sigma_S)^* \varphi_S}{\rightrightarrows}} (p_2 \circ (m \times 1_{X \times S}))^* \mathscr{L}_S.$$

These isomorphisms must differ by the unit

$$d_{\varphi_S} = d_\varphi \circ p_{12} \in \Gamma(\mathscr{O}_{G \times G \times S}) = \Gamma(\mathscr{O}_{(G \times S) \underset{S}{\times} (G \times S)}),$$

where $p_{12} \colon G \times G \times S \to G \times G$ is the projection onto $G \times G$.

For any $g \in G(S)$, let g_S denote the corresponding S-morphism. $(g, 1_S)$: $S \to G \times S$. The composite

$$\sigma_S \circ (g_S \underset{S}{\times} 1_{X \times S}) \colon X \times S = S \underset{S}{\times} (X \times S) \to (G \times S) \underset{S}{\times} (X \times S) \to (X \times S)$$

is denoted by T_g and the pull back of φ_S along $(g_S \underset{S}{\times} 1_{X \times S})$ is denoted by $\varphi_g \colon T_g^* \mathscr{L}_S \to \mathscr{L}_S$.

Now let $g, g' \in G(S)$. If we pull back the sheaves and morphisms of (1.3.1) along

$$\begin{aligned}(g_S, g'_S) \underset{S}{\times} 1_{X \times S} \colon X \times S &= S \underset{S}{\times} (X \times S) \to (G \times S) \underset{S}{\times} (G \times S) \underset{S}{\times} (X \times S) \\ &= G \times G \times X \times S,\end{aligned}$$

we obtain that the two isomorphisms

$$(1.3.2) \qquad T_{g+g'}^* \mathscr{L}_S \underset{\varphi_{g+g'}}{\overset{\varphi_{g'} \circ T_{g'}^* \varphi_g}{\rightrightarrows}} \mathscr{L}_S$$

only differ by $d_{\varphi_S} \circ (g_S, g'_S) = d_\varphi(g, g') \in G_m(S)$.

Let $g, g', g'' \in G_m(S)$. We are now able to prove that $d = d_\varphi$ is a 2-cocycle:

$$\varphi_{g+(g'+g'')} = \varphi_{(g+g')+g''} = d(g+g',g'')\varphi_{g''} \circ T_{g''}^* \varphi_{g+g'}$$

$$= d(g+g',g'')\cdot d(g,g')\varphi_{g''}\circ T^*_{g''}(\varphi_{g'}\circ T^*_{g'}\varphi_g)$$

$$= d(g+g',g'')\cdot d(g,g')\varphi_{g''}\circ T^*_{g''}\varphi_{g'}\circ T^*_{g'+g''}\varphi_g$$

$$= d(g+g',g'')\cdot d(g,g')\cdot d(g',g'')^{-1}\varphi_{g'+g'}{}'\circ T^*_{g'+g''}\varphi_g$$

$$= d(g+g',g'')\cdot d(g,g')\cdot d(g',g'')^{-1}\cdot d(g,g'+g'')^{-1}\varphi_{g+(g'+g'')}\,.$$

Hence $d=d_\varphi$ is a 2-cocycle.

If φ': $\sigma^*\mathscr{L} \to p_2^*\mathscr{L}$ is another isomorphism, then $\varphi'=c\varphi$, where c is a unit in $\Gamma(\mathcal{O}_{G\times X})=\Gamma(\mathcal{O}_G)$. Consider c as a morphism $G \to G_m$. If S is any k-scheme and $g,g' \in G(S)$, then $\varphi'_{g+g'}=c(g+g')\varphi_{g+g'}$ differs from

$$\varphi'_{g'}\circ T^*_{g'}\varphi'_g = c(g')c(g)\varphi_{g'}\circ T^*_{g'}\varphi_g$$

by $d_\varphi(g,g')\cdot c(g')^{-1}\cdot c(g+g')\cdot c(g)^{-1} \in G_m(S)$, that is, d_φ and $d_{\varphi'}$ are cohomologous. Hence to any $\mathscr{L} \in (\mathrm{Pic}\,(X))^G$ we may assign a class $q(\mathscr{L}) \in H^2(G,G_m)$. q is obviously a group homomorphism.

To prove exactness of the sequence, we only need to see that $\mathscr{L}$ admits a G-linearization if and only if $q(\mathscr{L})=0$. Only if follows immediately from the construction of q. Conversely, if $q(\mathscr{L})=0$, then the 2-cocycle d_φ associated with any isomorphism $\varphi\colon \sigma^*\mathscr{L} \to p_2^*\mathscr{L}$ is a coboundary. Hence there is a morphism c_φ: $G \to G_m$ such that

$$d_\varphi(g,g') = c_\varphi(g')\cdot c_\varphi(g+g')^{-1}\cdot c_\varphi(g) \quad \text{for } g,g' \in G_m(S)\,.$$

Change φ to $\varphi'=c_\varphi^{-1}\varphi$ and check that φ' is a G-linearization of $\mathscr{L}$.

We next raise the following question: Is the above 2-cocycle d_φ in fact a symmetric cocycle?

Recall that a cocycle d is symmetric if for any k-scheme S and any $g,g' \in G(S)$ then $d(g,g')=d(g',g)$. The cohomology classes of symmetric cocycles form a subgroup $H^2_s(G,G_m)\subseteq H^2(G,G_m)$.

Now let $d \in Z^2(G,G_m)$ be any 2-cocycle. Consider

$$e(g,g') = d(g,g')\cdot d(g',g)^{-1} \in G_m(S)\,.$$

e defines a morphism of schemes $G\times G \to G_m$ and it is easily checked that e is a skew-symmetric bihomomorphism:

(i) $e(g,g'+g'')=e(g,g')\cdot e(g,g'')$
(ii) $e(g+g',g'')=e(g,g'')\cdot e(g',g'')$
(iii) $e(g,g)=1$.

LEMMA 1.4. *Assume the k-group scheme G is commutative. If G acts on X with a fixed point $x \in X$, then q: $(\mathrm{Pic}\,(X))^G \to H^2(G,G_m)$ factors through $H^2_s(G,G_m)$.*

PROOF. Let $\mathscr{L} \in (\mathrm{Pic}(X))^G$ and let $d = d_\varphi$ be an associated cocycle. Notice that if d is a non-symmetric cocycle, then after base extension to $k(x)$, d is still non-symmetric. We may therefore assume x is a k-rational point.

To d we have associated a skew-symmetric bihomomorphism e. We prove that for any k-scheme S and any $g, g' \in G(S)$ then $e(g,g') = 1$. In the notation from (1.3):

$$e(g,g')\varphi_{g'} \circ T_{g'}^* \varphi_g = \varphi_g \circ T_g^* \varphi_{g'} \colon T_{g+g'}^* \mathscr{L}_S \to \mathscr{L}_S .$$

$G \times S$ acts on $X \times S$ with fixed point $x = x \times 1_S \colon S \to X \times S$. Therefore $x^* T_g^* \mathscr{L}_S = x^* \mathscr{L}_S$, and hence

$$e(g,g') x^* \varphi_{g'} \circ x^* T_{g'}^* \varphi_g = x^* \varphi_g \circ x^* T_g^* \varphi_{g'} \colon x^* \mathscr{L}_S \to x^* \mathscr{L}_S \tag{1.4.1}$$

is an automorphism of the sheaf $x^*\mathscr{L}_S$ on S. Moreover $x^*\varphi_g$ and $x^* T_g^* \varphi_{g'} = x^* \varphi_{g'}$ are automorphisms of the same sheaf $x^* \mathscr{L}_S$. Hence they are multiplications by units from $\Gamma(\mathcal{O}_S)$. We therefore may rewrite (1.4.1) as multiplication in $G_m(S)$:

$$e(g,g') \cdot (x^* \varphi_{g'}) \cdot (x^* \varphi_g) = (x^* \varphi_g) \cdot (x^* \varphi_{g'}) .$$

Consequently $e(g,g') = 1$.

LEMMA 1.5. *Assume $G = N$ is a finite commutative k-group scheme. Let $\mathscr{L} \in (\mathrm{Pic}(X))^N$. Then $q(\mathscr{L}^{|N|}) \in H_s^2(N, G_m)$ for some power $\mathscr{L}^{|N|}$ of $\mathscr{L}$.*

PROOF. Set $|N| = \operatorname{rank} N = \dim_k(E)$, where E is the affine coordinate ring of N. If $d = d_\varphi$ is a cocycle associated to $\mathscr{L}$, then $d^{|N|}$ is associated to $\mathscr{L}^{|N|}$. We may assume $d(0,0) = 1$, whence

$$d(0,n) = d(n,0) = 1 .$$

In fact,

$$\begin{aligned} & d(n,0) \cdot d(-n+n,0)^{-1} \cdot d(-n,n+0) \cdot d(-n,n)^{-1} \\ &\quad = 1 \\ &\quad = d(n,-n) \cdot d(0+n,-n)^{-1} \cdot d(0,n-n) \cdot d(0,n)^{-1} . \end{aligned}$$

It now follows that $d^{|N|}$ is a symmetric cocycle:

$$\begin{aligned} d^{|N|}(n,n') &= (d(n,n'))^{|N|} = (e(n,n') d(n',n))^{|N|} \\ &= e(|N|n,n') \cdot (d(n',n))^{|N|} = d^{|N|}(n',n) , \end{aligned}$$

since $e(|N|n, n') = e(0,n') = 1$.

Let N be a finite commutative k-group scheme. Let $d \in Z^2(N, G_m)$ be any 2-cocycle. The skew-symmetric bihomomorphism $e\colon N \times N \to G_m$ gives rise to a homomorphism of k-group schemes $\gamma\colon N \to D(N)$ such that $e(n,n') = \langle n, \gamma(n')\rangle$, cf. [8, p. 223]. $D(N)$ is the Cartier dual of N and $\langle \cdot, \cdot \rangle\colon N \times D(N) \to G_m$ is the universal pairing.

For the remaining part of this section we assume k is an algebraically closed field.

LEMMA 1.6. *Let N be a finite commutative k-group scheme. Then*

1.6.1. $H^2_s(N, G_m) = 0$
1.6.2. *If N is*

(a) *reduced and $N(k)$ is a finite cyclic group, or*
(b) *reduced, but $D(N)$ is a local group scheme and vice versa, or*
(c) *a local group scheme of height* $\leqq 1$, (see [1 II, § 7 No. 4]),

then $H^2(N, G_m) = 0$.

PROOF. In case 1.6.2 (b), $\mathrm{Hom}_{k\text{-grp}}(N, D(N)) = 0$ whence $Z^2(N, G_m) = Z^2_s(N, G_m)$. The remaining statements are straightforward consequence of (the proof of) [8, Lemma 1, p. 223] and the fact that $H^2_s(N, G_m) = \mathrm{Ext}^1(N, G_m)$, [1, II, § 3, No. 2].

LEMMA 1.7. *Let G be a smooth, connected algebraic group. Then* $H^2(G, G_m) = 0$.

PROOF. Let $d \in Z^2(G, G_m)$. We may assume $d(0,0) = 1$, whence $d(g, 0) = d(0, g) = 1$, too. By a result of Rosenlicht (e.g. [2, 2.2]) d is a group homomorphism from $G \times G$ to G_m. Therefore

$$d(g+g', g'') = d(g, g'') \cdot d(g', 0) = d(g, g'')$$

is independent of g'. Especially

$$d(g, g') = d(g-g, g') = d(0, g') = 1,$$

i.e. d is a coboundary.

Let us collect the results obtained until now:

THEOREM 1.8. *Let a smooth connected linear algebraic group G act on a normal algebraic variety $\bar{X}$, proper over k. Then there exists an exact sequence*

$$0 \to \chi(G) \to \mathrm{Pic}^G(\bar{X}) \xrightarrow[p]{} \mathrm{Pic}(\bar{X}) \to \mathrm{Pic}(G).$$

PROOF. (1.3) implies

$$0 \to \chi(G) \to \mathrm{Pic}^G(\bar{X}) \to (\mathrm{Pic}(\bar{X}))^G \to 0$$

is exact. As a consequence of $\bar{X}$ being normal, G acts trivially on Pic $(\bar{X})$. Hence the Seesaw principle [8, p. 54] applied to $\sigma^*\mathscr{L}\otimes p_2^*\mathscr{L}^{-1}$ proves the following: The pull back $\mathscr{L}_x$ of $\mathscr{L}$ along $\sigma\circ(1_G,x)\colon G=G\times\mathrm{Spec}(k)\to G\times\bar{X}\to\bar{X}$ is independent of $x\in\bar{X}$. Moreover

$$\sigma^*\mathscr{L}\otimes p_2^*\mathscr{L}^{-1} \cong \mathcal{O}_{G\times\bar{X}}$$

if and only if $\mathscr{L}_x$ is trivial in Pic (G).

THEOREM 1.9. *Let a finite commutative group scheme N act on an algebraic variety $\bar{X}$, proper over k.*

1.9.1. *If N acts with fixed points, then we have an exact sequence*

$$0 \to \chi(N) \to \mathrm{Pic}^N(\bar{X}) \xrightarrow[p]{} (\mathrm{Pic}(\bar{X}))^N \to 0\,.$$

1.9.2. *If N acts freely on $\bar{X}$ such that the quotient $\bar{Y}$ exists, then we have an exact sequence*

$$0 \to \chi(N) \to \mathrm{Pic}(\bar{Y}) \xrightarrow[p]{} (\mathrm{Pic}(\bar{X}))^N\,.$$

If $H^2(N,G_m)=0$, then p is surjective.

PROOF. Follows from (1.3), (1.4), (1.6) and the following facts about quotients (see [8, p. 113] or [1, III, § 2, No. 6]):

Let a finite group scheme G act on a k-scheme Z by $\sigma\colon G\times Z\to Z$. Assume that the orbit of any point is contained in an open affine subset of Z.

i) There exist a k-scheme Z' and a morphism $\pi\colon Z\to Z'$ such that (Z',π) is a categorial quotient [7, definition 0.5] and π is integral and surjective.

ii) If moreover G acts freely on Z [7, definition 0.8], then π is finite and flat and the natural morphism $(\sigma,p_2)\colon G\times Z\to Z\underset{Z'}{\times}Z$ is an isomorphism, i.e. $\pi\colon Z\to Z'$ is a principal G-bundle.

iii) If $\pi\colon Z\to Z'$ is a principal G-bundle, then π induces an isomorphism $\pi^*\colon \mathrm{Pic}(Z')\to\mathrm{Pic}^G(Z)$. (This follows from descent theory [4, VIII], cf. [8, p. 115 ff.]).

2. Picard schemes of quotients.

In this section k denotes a field, not necessarily algebraically closed, X is a scheme, proper over k, and $\Gamma(\mathcal{O}_X)=k$. N is a finite commutative group scheme acting on X by $\sigma: N \times X \to X$, such that the orbit of any point is contained in an open affine subset of X. Let (Y, π) denote the quotient.

We want to compare the Picard scheme of Y with that of X, and the main result in this section is:

THEOREM 2.1. *If N acts freely on X, and if the group scheme N is*
(a) *reduced and $N(\bar{k})$ is a cyclic group ($\bar{k}$ is an algebraic closure of k), or*
(b) *reduced, but $D(N)$ is local and vice versa, or*
(c) *local of height $\leqq 1$,*
then π^*: $\mathbf{Pic}_{Y/k} \to \mathbf{Pic}_{X/k}$ induces an exact sequence of commutative group schemes:

$$0 \to D(N) \to \mathbf{Pic}_{Y/k} \to (\mathbf{Pic}_{X/k})^N \to 0\,,$$

where $(\mathbf{Pic}_{X/k})^N$ is the fixed point scheme under the action of N on $\mathbf{Pic}_{X/k}$.

The above theorem enables us to compute the Picard scheme of the Igusa example [6] and of the Serre example [11].

EXAMPLE 2.2 (Igusa). Let k be an algebraically closed field of characteristic 2. E is an ordinary elliptic curve (t denotes the non-zero point of order 2), and F is any elliptic curve. The constant group scheme $\mathbf{Z}/2\mathbf{Z}$ acts freely on $E \times F$ by $(x, y) \to (x+t, -y)$. Let Y be the quotient.

Y is a non-singular surface and its Albanese variety is $\mathrm{Alb}\,(Y) = E/\langle t \rangle$, which is the quotient of E under the free $\mathbf{Z}/2\mathbf{Z}$ action $x \to x+t$. If we look on the torsion part of the Picard scheme of Y, then

$$0 \to D(\mathbf{Z}/2\mathbf{Z}) \to \mathbf{Pic}^{\tau}_{Y/k} \to ((E \times F)\hat{\ })^{\mathbf{Z}/2\mathbf{Z}} = \hat{E} \times \ker\,(2_{\hat{F}}) \to 0$$

is exact. $\hat{A}$ denotes the dual of an abelian variety A, and 2_A is multiplication by 2 on A.

We therefore have determined $\mathbf{Pic}^{\tau}_{Y/k}$:

$$\mathbf{Pic}^{\tau}_{Y/k} = \begin{cases} (E/\langle t \rangle)\hat{\ } \times \mu_2 \times \mathbf{Z}/2\mathbf{Z} & \text{if } F \text{ is ordinary} \\ (E/\langle t \rangle)\hat{\ } \times M_2 & \text{if } F \text{ is supersingular}\,. \end{cases}$$

$\mu_2 = D(\mathbf{Z}/2\mathbf{Z}) = \mathrm{Spec}\,(k[T]/T^2 - 1)$ with comultiplication $T \to T \otimes T$, and $M_2 = \mathrm{Spec}\,(k[T]/T^4)$ with coaddition $T \to 1 \otimes T + T \otimes 1 + T^2 \otimes T^2$, cf. Oort [10].

EXAMPLE 2.3 (Serre). There exists a non-singular surface X in $\boldsymbol{P}_3$, defined over a field of characteristic p (assume the field to be algebraically closed), such that the group scheme $\mathbf{Z}/p\mathbf{Z}$ ($p \geqq 5$) acts freely on X. Let Y be the quotient.

We first prove that $\mathbf{Pic}^{\tau}_{X/k} = 0$. Remark that X is simply connected and that $H^0(X, \Omega^1_X) = 0$, where Ω^1_X is the sheaf of 1-forms on X. For any prime number l, the first etale cohomology group with coefficients in the constant sheaf $\mathbf{Z}/l\mathbf{Z}$ is 0. Take cohomology of the Kummer sequence

$$0 \to \mathbf{Z}/l\mathbf{Z} \to G_m \to G_m \to 0$$

(l is prime to p) to see that there is no l-torsion in $\operatorname{Pic}(X)$. But there is no p-torsion either, because the p-torsion in $\operatorname{Pic}(X)$ is a subgroup of $H^0(X, \Omega_X^1) = 0$ (cf. [11, 11]). We next claim that $H^1(X, \mathcal{O}_X) = 0$. Consider the Frobenius endomorphism

$$F: H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X) .$$

By [11, 10], the kernel of F can be injected into $H^0(X, \Omega_X^1) = 0$. Hence F is injective. On the other hand $0 = H^1_{\text{et}}(X, \mathbf{Z}/p\mathbf{Z}) =$ the semi simple part of $H^1(X, \mathcal{O}_X)$. Therefore F is nilpotent. Since F is both nilpotent and injective, $H^1(X, \mathcal{O}_X)$ must be 0.

Now Theorem 2.1 determines $\mathbf{Pic}^{\tau}_{Y/k} = D(\mathbf{Z}/p\mathbf{Z}) = \mu_p$.

The proof of 2.1 is based on a functorial study of the results from Section 1. Let X, N, σ be as in the beginning of this section (but we do not assume N acts freely on X).

If S is any k-scheme, we let $N \times S$ act on $X \times S$ via

$$\sigma_S = \sigma \times 1_S: (N \times S) \underset{S}{\times} (X \times S) = N \times X \times S \to X \times S .$$

Define the following two contravariant functors from the category of k-schemes $(\mathbf{Sch}/k)$ to the category of abelian groups $(\mathbf{Ab})$, $\underline{\mathbf{Pic}}_X^N$ and $[\underline{\mathbf{Pic}}]_X^N$:

Definition 2.4. $\underline{\mathbf{Pic}}_X^N(S) = \operatorname{Pic}^{N \times S}(X \times S) =$ the group of $N \times S$-linearized invertible sheaves on $X \times S$.

$$\begin{aligned} [\underline{\mathbf{Pic}}]_X^N(S) &= (\operatorname{Pic}(X \times S))^{N \times S} \\ &= \{\mathscr{L} \in \operatorname{Pic}(X \times S) \mid \sigma_S^* \mathscr{L} \otimes p_2^* \mathscr{L}^{-1} \text{ is isomorphic to } \mathcal{O}_{X \times S}\} . \end{aligned}$$

Proposition 2.5. *There is a natural morphism of functors p: $\underline{\mathbf{Pic}}_X^N \to [\underline{\mathbf{Pic}}]_X^N$, such that the kernel of p is the functor $S \to$* $\operatorname{Hom}_{S\text{-grp}}(N \times S, G_m \times S)$.

Proof. For any $S \in \mathbf{Sch}/k$, $p(S)$ is simply the map that to an $N \times S$-linearized invertible sheaf $\mathscr{L}$ on $X \times S$ assigns $\mathscr{L} \in \operatorname{Pic}(X \times S)$. The proof is now a straightforward consequence of (1.2).

Definition 2.6. For any $S \in \mathbf{Sch}/k$, let the commutative S-group scheme $N \times S$ act trivially on $G_m \times S$. Define $H^i_N(S) = H^i(N \times S, G_m \times S) =$ the ith Hochschild cohomology group relative to S, (see [1, II, § 3]). H^i_N is obviously a contravariant abelian group valued functor on $\mathbf{Sch}/k$.

Note that $H^1{}_N(S) = \mathrm{Hom}_{S\text{-grp}}(N \times S, G_m \times S)$.

PROPOSITON 2.7. *There is a natural morphism* q: $\underline{[\mathbf{Pic}]}_X^N \to H_N^2$ *such that*

$$0 \to H_N^1 \to \underline{\mathbf{Pic}}_X^N \xrightarrow[p]{} \underline{[\mathbf{Pic}]}_X^N \xrightarrow[q]{} H_N^2$$

is an exact sequence of functors.

The proof is an easy generalization of (1.3).

We next want to sheafify the above functors with respect to the faithfully flat, finite presentation (fppf) topology on **Sch**/k. If F is a functor from **Sch**/k to **Ab**, we let $\mathscr{F}$ denote the associatéd fppf sheaf.

PROPOSITION 2.8 *The functor* H_N^1 *is representable.* H_N^1 *is represented by the Cartier dual of* N, $D(N)$. (Cf. [10, I.2–5, III.16–1].)

PROPOSITION 2.9. *If* N *acts freely on* X *with quotient* Y, *then* $\mathscr{P}ic_X^N$ *is represented by* $\mathbf{Pic}_{Y/k}$.

PROOF. π: $X \to Y = X/N$ is a principal N-bundle, whence $\pi \times 1_S : X \times S \to Y \times S$ is a principal $N \times S$-bundle. By descent theory [4, VIII],

$$\underline{\mathbf{Pic}}_X^N(S) = \mathrm{Pic}^{N \times S}(X \times S) \cong \mathrm{Pic}(Y \times S)\,.$$

Hence $\mathscr{P}ic_X^N$ is the Picard functor on Y relative to Spec(k), cf. [9]. Since Y is proper over k, this functor is representable by [9].

PROPOSITION 2.10. *The* fppf *sheaf* $[\mathscr{P}ic]_X^N$ *is represented by* $(\mathbf{Pic}_{X/k})^N$.

PROOF. It suffices to prove that $\underline{[\mathbf{Pic}]}_X^N(S)$ is isomorphic to

$$\{\mathscr{L} \in \mathrm{Pic}\,(X \times S) \mid \forall S' \to S,\ \forall n \in N(S'),\ T_n^*\mathscr{L}_{S'} \otimes \mathscr{L}_{S'}^{-1} \cong \mathcal{O}_{S'}\}\,,$$

since the fppf sheaf associated the latter functor is represented by $(\mathbf{Pic}_{X/k})^N$.

If $\mathscr{L} \in \underline{[\mathbf{Pic}]}_X^N(S)$ we know that there is an isomorphism φ_S: $\sigma_S^*\mathscr{L} \to p_2^*\mathscr{L}$, which for any base extension $S' \to S$ and any $n \in N(S')$ induces an isomorphism φ_n: $T_n^*\mathscr{L}_{S'} \to \mathscr{L}_{S'}$. Conversely, let $\mathscr{L}$ be fixed by all $n \in N(S')$, for each S'. Especially for $S' = N \times S$ and n: $N \times S \to N$ the projection onto N we have an isomorphism $T_n^*\mathscr{L}_{N \times S} \to \mathscr{L}_{N \times S}$. Such an isomorphism is just an isomorphism φ_S: $\sigma_S^*\mathscr{L} \to p_2\mathscr{L}$. Namely, under the identification $N \times X \times S = X \times N \times S$ we have $\mathscr{L}_{N \times S} = p_2^*\mathscr{L}$ and $T_n^*\mathscr{L}_{N \times S} = \sigma_S^*\mathscr{L}$.

We have an interesting subfunctor of H_N^2, namely

$$H_{N,s}^2 : S \to H_s^2(N \times S, G_m \times S)\,,$$

the classes of symmetric cocycles.

LEMMA 2.11. *The natural map* q: $\underline{[\mathbf{Pic}]}_X^N \to H_N^2$ *factors through* $H_{N,s}^2$ *in the following cases*:

a) N is a reduced group scheme and $N(\bar{k})$ is a cyclic group,
b) N is reduced, but $D(N)$ is a local group scheme, and vice versa,
c) N is a local group scheme of height $\leqq 1$.
d) The action of N on X has a fixed point $x \in X$.

PROOF. For any k-scheme S, $q(S)$ factors through $H_s^2(N \times S, G_m \times S)$: If N is of type (a), (b) or (c), then

$$H^2(N \times S, G_m \times S) = H_s^2(N \times S, G_m \times S)\,,$$

as seen by generalizing the proof of [8, lemma 1 (ii), p. 223] to group schemes over an arbitrary base.

Let $\mathscr{L} \in \underline{[\mathbf{Pic}]}_X^N(S)$. Note that if $q(S)(\mathscr{L}) \in H_N^2(S)$ is not in $H_{N,s}^2(S)$, then, after a faithfully flat extension t: $S' \to S$,

$$H_N^2(t)(q(S)(\mathscr{L})) = q(S')(\mathscr{L}_{S'})$$

is not in $H_{N,s}^2(S')$. Therefore, to prove (d), we may assume $N \times S$ acts on $X \times S$ with a S-rational fixed point $x \in X \times S$. A proof similar to that of (1.4) carries over.

Finally remark, that if $\mathscr{L} \in \underline{[\mathbf{Pic}]}_X^N(S)$, then $q(S)(\mathscr{L}^{|N|}) \in H_{N,s}^2(S)$, where $|N| = \operatorname{rank} N$, cf. (1.5).

In Section 1 we insisted that the map q: $(\operatorname{Pic}(X))^N \to H^2(N, G_m)$ should factor through $H_s^2(N, G_m)$, simply because that group is 0. In this section, too, we want q: $\underline{[\mathbf{Pic}]}_X^N \to H_N^2$ to factor through $H_{N,s}^2$, because of

LEMMA 2.12. *The* fppf *sheaf associated to* $H_{N,s}^2$ *is* 0, *that is*,

$$\mathscr{H}_{N,s}^2(S) = 0, \qquad \textit{all } S \in \mathbf{Sch}/k\,.$$

PROOF. [5, VIII, 3.3.1], together with the fact that

$$H_s^2(N \times S, G_m \times S) = \operatorname{Ext}_S^1(N \times S, G_m \times S)\,.$$

We now give a proof of Theorem 2.1:

Combining (2.7) and (2.11) we obtain an exact sequence of functors

$$0 \to H_N^1 \to \underline{\mathbf{Pic}}_X^N \to \underline{[\mathbf{Pic}]}_X^N \to H_{N,s}^2\,.$$

Noticing that "taking associated fppf sheaf" is an exact functor, we get an exact sequence of fppf sheaves

$$0 \to \mathscr{H}_N^1 \to \mathscr{P}\!\mathit{ic}_X^N \to [\mathscr{P}\!\mathit{ic}]_X^N \to 0\,.$$

Since the three sheaves in question are representable ((2.8), (2.9), (2.10)), the theorem follows from [10, III 17–7].

Moreover, our investigations show the following result:

THEOREM 2.13. *If N acts on X with a fixed point $x \in X$, then the "N-linearized Picard functor of X relative to k", $\mathscr{P}ic_X^N$* (cf. 2.4) *is representable by a group scheme, locally of finite type over k. The representing object, denoted $\mathbf{Pic}_{X/k}^N$, fits into an exact sequence of commutative group schemes*

$$0 \to D(N) \to \mathbf{Pic}_{X/k}^N \to (\mathbf{Pic}_{X/k})^N \to 0 .$$

PROOF. The assumption on the action ensures

$$0 \to \mathscr{H}_N^1 \to \mathscr{P}ic_X^N \to [\mathscr{P}ic]_X^N \to 0$$

is an exact sequence of fppf sheaves. The first and the last sheaf is representable, hence $\mathscr{P}ic_X^N$ is representable, [10, III 17–4].

REMARK 2.14. We may of course generalize some of the results to schemes over an arbitrary (locally noetherian) base scheme S. E.g. a reformulation of

Theorem 2.1: Let X be a S-scheme, such that (i) the structural morphism f: $X \to S$ is proper and flat, (ii) $f_*\mathcal{O}_X = \mathcal{O}_S$, (iii) X/S has a Picard scheme $\mathbf{Pic}_{X/S}$. Let π: $X/S \to Y/S$ be a principal N/S-bundle, where N is a finite commutative S-group scheme. If we further assume

$$H^2(N \underset{S}{\times} T, G_{m,S} \underset{S}{\times} T) = H_s^2(N \underset{S}{\times} T, G_{m,S} \underset{S}{\times} T) ,$$

then Y/S has a Picard scheme $\mathbf{Pic}_{Y/S}$, fitting into an exact sequence of S-group schemes:

$$0 \to D(N) \to \mathbf{Pic}_{Y/S} \to (\mathbf{Pic}_{X/S})^N \to 0 .$$

The reformulation of 2.13: Assume N/S acts on X/S (satisfying (i), (ii), (iii) above) with a S'-rational fixedpoint, where $S' \to S$ is a faithfully flat morphism. Then "the N-linearized Picard functor of X/S relative to S" is representable, and the representing object $\mathbf{Pic}_{X/S}^N$ fits into an exact sequence

$$0 \to D(N) \to \mathbf{Pic}_{X/S}^N \to (\mathbf{Pic}_{X/S})^N \to 0 .$$

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