MATH. SCAND. 42 (1978), 169–179

THE SOLUTION OF A MINIMAX PROBLEM CONNECTED TO THE IRREDUCIBILITY OF POLYNOMIALS

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1. Introduction and statement of results.

In [2] H. Tverberg poses the following problem: Find the value of

$$R = \inf_{\varphi \in T_c} \left[\max_{x \in [\alpha, \beta]} H(x) + \max\,(G(-1), G(1)) \right] \tag{1.1}$$

where T_c is the set of continuous functions φ on $[-1,1]$ such that

$$\varphi(-1) = \varphi(1) = 0, \quad 0 \leqq \varphi \leqq 1 \quad \text{and} \quad \int_{-1}^{1} \varphi(t)\,dt = 1 .$$

$[\alpha, \beta]$ is the convex hull of $\operatorname{supp} \varphi$ and

$$H(x) = \int_{-1}^{1} (1-\varphi(t)) \log|x-t|\,dt, \qquad G(x) = \int_{-1}^{1} \varphi(t) \log|x-t|\,dt .$$

The determination of R is related to the following irreducibility theorem: If a polynomial f of degree n has integral coefficients and there are n integers a_i so that

$$0 < |f(a_i)| < P(n) \quad \text{for } 1 \leqq i \leqq n ,$$

then f is irreducible over the rationals. The largest such number $P(n)$ is proved in [2] to have the form

$$P(n) = (\lambda_0 + o(1))^m m! \quad \left(m = \left[\frac{n+1}{2} \right] \right)$$

where the constant λ_0 equals $\exp(1+\frac{1}{2}R)$.

We shall prove that there exists a unique function φ_0 which is optimal for the problem (1.1). The function φ_0 is given by

Received June 3, 1977

$$\varphi_0(t) = \begin{cases} \frac{2}{\pi}\operatorname{Arctan}\sqrt{3-4t^2}; & |t| \leqq \frac{\sqrt{3}}{2} \\ 0 \quad ; & \frac{\sqrt{3}}{2} \leqq |t| \leqq 1 \end{cases}$$

The corresponding minimal value is $R = -2+4\log 2 - \frac{3}{2}\log 3$. (This solution was anticipated in [2]). Hence the value of λ_0 is $\frac{4}{3}\cdot 3^{\frac{1}{4}} = 1.754\ldots$.

We will proceed as follows. After reformulating the problem slightly, we first prove that there exists at least one optimal function in $L^\infty[-1,1]$ for the problem (Lemma 1). We then prove that for such an optimal function φ the function

$$H_\varphi(x) = H(x) = \int_{-1}^{1} (1-\varphi(t))\log|x-t|\,dt$$

must be constant on the convex hull $[\alpha,\beta]$ of $\operatorname{supp}\varphi$ (Theorem 1). This reduces the problem to solving an integral equation $H_\varphi(x) = \text{constant}$ on $[\alpha,\beta]$. Using known inversion formulas for the Hilbert transform it is finally proved that the only bounded solution φ of this integral equation is the function φ_0 above (Theorem 2).

The author is greatly indebted to H. Tverberg for many valuable conversations. Theorem 2 is due to him.

2. Existence of optimal functions.

Put

$$T = \left\{\varphi \in L^\infty[-1,1]\ ;\ 0 \leqq \varphi \leqq 1,\ \int_{-1}^{1} \varphi(t)\,dt = 1\right\}.$$

Define

$$G_\varphi(x) = \int_{-1}^{1} \varphi(t)\log|x-t|\,dt$$

and

$$R_\varphi = \max_{x\in[\alpha,\beta]} G_{1-\varphi}(x) + \max\left(G_\varphi(-1), G_\varphi(1)\right), \qquad \text{for } \varphi \in T,$$

where $[\alpha,\beta]$ is the convex hull of $\operatorname{supp}\varphi$.

We will replace the family T_c by the larger family T and consider

$$R_1 = \inf\{R_\varphi\ ;\ \varphi \in T\}.$$

It turns out that the optimal function for this problem is the function φ_0 above, which is a member of T_c. Therefore $R = R_1$, so that

$$R = \inf\{R_\varphi \ ; \ \varphi \in T\} . \tag{2.1}$$

However, (2.1) can also be deduced directly as follows:

If $\varphi \in T$ we can find a sequence $\{\varphi_n\} \subset T_c$ converging to φ weakstar in $L^\infty[-1,1]$ such that $\operatorname{supp}\varphi_n \subseteqq [\alpha,\beta]$, the convex hull of $\operatorname{supp}\varphi$. Then $G_{\varphi_n}(x) \to G_\varphi(x)$ for all $x \in [-1,1]$. Since $\{G_{\varphi_n}\}$ is equicontinuous on $[-1,1]$ (because $0 \leqq \varphi_n \leqq 1$), we get by the Ascoli theorem that $G_{\varphi_n} \to G_\varphi$ *uniformly* on $[-1,1]$. Therefore

$$\begin{aligned} R_\varphi &= \max_{x\in[\alpha,\beta]} G_{1-\varphi}(x) + \max\,(G_\varphi(-1), G_\varphi(1)) \\ &= \lim_{n\to\infty}\left[\max_{x\in[\alpha,\beta]} G_{1-\varphi_n}(x) + \max\,(G_{\varphi_n}(-1), G_{\varphi_n}(1))\right] \geqq \lim_{n\to\infty} R_{\varphi_n} , \end{aligned}$$

which proves (2.1).

Next, observe that if we put $\psi = 1-\varphi$ we have $\psi \in T$ if $\varphi \in T$ and

$$\begin{aligned} G_\psi(x) &= \int_{-1}^{1} \log|x-t|\,dt - G_\varphi(x) \\ &= (1-x)\log(1+x) + (1+x)\log(1+x) - 2 - G_\varphi(x) . \end{aligned}$$

Hence

$$\begin{aligned} G_\psi(-1) &= 2\log 2 - 2 - G_\varphi(-1) \\ G_\psi(1) &= 2\log 2 - 2 - G_\varphi(1) . \end{aligned}$$

Therefore, if we put

$$N_\psi = \max_{[\alpha,\beta]} G_\psi(x) - \min\,(G_\psi(-1), G_\psi(1)) \ ; \ [\alpha,\beta] = \operatorname{conv}(\operatorname{supp}(1-\psi)) \tag{2.2}$$

$$N = \inf\{N_\psi \ ; \ \psi \in T\} \tag{2.3}$$

we have

$$N_\psi = R_\varphi + 2 - 2\log 2 \quad \text{and} \quad N = R + 2 - 2\log 2 .$$

Therefore we proceed to work with the problem (2.3). A result similar to the following lemma is mentioned in Tverberg [2, p. 14]. Since it is so crucial for our approach to the problem we state it again and include a proof:

LEMMA 1. *There exists a function $\psi_0 \in T$ which is optimal for the problem (2.3). that is, $N_{\psi_0} = N$.*

PROOF. Let $\{\psi_n\}_{n=1}^{\infty} \subset T$ be a sequence such that $N_{\psi_n} \to N$. Since the unit ball in $L^\infty[-1,1]$ is compact in the weak-star topology, there exists a subnet $\{\psi_i\}_{i\in I}$ converging weak-star to a function $\psi_0 \in L^\infty[-1,1]$. Since the weak-star topology on the unit ball of $L^\infty[-1,1]$ is metrizable, we can replace the subnet by a subsequence which we again denote by $\{\psi_n\}_{n=1}^{\infty}$.

We have

$$\int_{-1}^{1} f(t)\psi_n(t)\,dt \to \int_{-1}^{1} f(t)\psi_0(t)\,dt \qquad \text{as } n\to\infty$$

for all $f \in L^1[-1,1]$. This implies that

$$0 \leqq \psi_0 \leqq 1, \qquad \int_{-1}^{1} \psi_0(t)\,dt = 1\,,$$

so that $\psi_0 \in T$. Moreover, turning to a subsequence if necessary we can assume that $\alpha_n \to \alpha_0$, $\beta_n \to \beta_0$, where $[\alpha_n,\beta_n]$ is the convex hull of supp $(1-\psi_n)$. Then supp $(1-\psi_0) \subset [\alpha_0,\beta_0]$. Since $G_{\psi_n}(x) \to G_{\psi_0}(x)$ for all $x \in [-1,1]$, we get using the Ascoli theorem again that $G_{\psi_n} \to G_{\psi_0}$ uniformly on $[-1,1]$. Hence

$$\begin{aligned} N \leqq N_{\psi_0} &= \max_{\text{conv (supp } (1-\psi_0))} G_{\psi_0}(x) - \min\left(G_{\psi_0}(-1), G_{\psi_0}(1)\right) \\ &\leqq \max_{[\alpha_0,\beta_0]} G_{\psi_0}(x) - \min\left(G_{\psi_0}(-1), G_{\psi_0}(1)\right) \\ &= \lim_{n\to\infty} N_{\psi_n} = N\,. \end{aligned}$$

We conclude that $N_{\psi_0} = N$ and the proof is complete.

3. Proof that if ψ is optimal then G_ψ must be constant on $[\alpha,\beta]$.

We will prove this using a variational technique. The idea is simple: If G_ψ is not constant on $[\alpha,\beta]$, we modify ψ slightly — by adding and subtracting suitably — to obtain a function $\psi_1 \in T$ such that $N_{\psi_1} < N_\psi$. The idea is to add a little to ψ at points where G_ψ is big (thereby reducing G_ψ near these points) and subtract from ψ accordingly at a point where G_ψ is small.

The next three lemmas enables us to carry out these modifications on ψ such that the modified function ψ_1 still belongs to T.

LEMMA 2. *Let $\psi \in T$. If $\psi = 1$ a.e. in a neighbourhood of $c \in (-1,1)$, then G_ψ is twice continuously differentiable at c and $G''_\psi(c) > 0$.*

PROOF. Tverberg [2, page 19].

LEMMA 3. *Let $\psi \in T$. If $\psi = 0$ a.e. in a neighbourhood of $c \in (-1,1)$, then G_ψ is twice continuously differentiable at c and $G''_\psi(c) < 0$.*

PROOF. As in Tverberg [2, page 19], we get

$$G''_\psi(c) = -\left(\int_{-1}^{c-\varepsilon} + \int_{c+\varepsilon}^{1}\right)\psi(t)\frac{dt}{(c-t)^2} < 0 .$$

Immediate consequences of lemmas 2 and 3 are:

LEMMA 4. (i) *Let* x_0 *be a local minimum point for* G_ψ *in* $[\alpha,\beta] \subset (-1,1)$. *Then*

$$\int_J \psi(t)\,dt > 0 \text{ for every open interval } J \ni x_0 .$$

(ii) *Let* x_1 *be a local maximum point for* G_ψ *in* $[\alpha,\beta] \subset (-1,1)$. *Then*

$$\int_J (1-\psi(t))\,dt > 0 \quad \text{for every open interval } J \ni x_1 .$$

Hence if G_ψ is not constant on $[\alpha,\beta]$ it is possible to add something to ψ near maximum points and subtract near minimum points such that the modified function still belongs to T. However, it is not clear that this can be done such that N_ψ is reduced. For this we need some technical lemmas:

LEMMA 5. *Let* $0 \leqq a < b \leqq 1$. *Then*

$$\frac{\log(1-a)}{\log(1-b)} < \frac{a}{b} < \frac{\log(1+a)}{\log(1+b)} .$$

PROOF. Using Cauchy's mean value theorem for a quotient we get

$$\frac{\log(1+xa)-\log(1+0\cdot a)}{\log(1+xb)-\log(1+0\cdot b)} = \frac{\dfrac{a}{1+ya}}{\dfrac{b}{1+yb}} = \frac{a}{b}\cdot\frac{1+yb}{1+ya}$$

with y strictly between 0 and x. Putting $x = \pm 1$ we get the lemma.

LEMMA 6. *Assume* $-1 < x_1 < x_0 < x_2 < 1$. *Define*

$$f_\lambda(x) = \lambda\log|x-x_1| + (1-\lambda)\log|x-x_2| - \log|x-x_0|$$

for $x \in [-1,1]$, $\lambda \in [0,1]$. *Choose* $\lambda_0 \in (0,1)$ *such that* $x_0 = \lambda_0 x_1 + (1-\lambda_0)x_2$. *Then we have*:

(I) $f'_\lambda(x) < 0$ *for all* $x \in [-1, x_1) \Leftrightarrow \lambda_0 \dfrac{1+x_1}{1+x_0} < \lambda$.

(II) $f'_\lambda(x) > 0$ *for all* $x \in (x_2, 1] \Leftrightarrow \lambda < \lambda_0 \dfrac{1-x_1}{1-x_0}$.

(III)

$$f_\lambda(-1) = f_\lambda(1) \Leftrightarrow \lambda = \bar{\lambda} = \log\frac{(1-x_2)(1+x_0)}{(1+x_2)(1-x_0)}\bigg/\log\frac{(1-x_2)(1+x_1)}{(1+x_2)(1-x_1)}.$$

(IV) $\lambda_0 \dfrac{1+x_1}{1+x_0} < \bar{\lambda} < \lambda_0 \dfrac{1-x_1}{1-x_0}$.

PROOF. (I): Let $-1 \leqq x < x_1$. Then

$$f'_\lambda(x) = \frac{\lambda}{x-x_1} + \frac{1-\lambda}{x-x_2} - \frac{1}{x-x_0} = \frac{(x_1-x)(x_2-x)-(x_0-x)(x_3-x)}{(x_0-x)(x_1-x)(x_2-x)},$$

where $x_3 = \lambda x_2 + (1-\lambda)x_1$. Put

$$g(x) = (x_1-x)(x_2-x)-(x_0-x)(x_3-x).$$

Then we see that

$$\begin{aligned} f'_\lambda(x) < 0 &\Leftrightarrow g(x) < 0 \\ &\Leftrightarrow x_3 > x + \frac{(x_1-x)(x_2-x)}{x_0-x} \\ &\Leftrightarrow \lambda > \frac{(x_2-x_0)(x_1-x)}{(x_2-x_1)(x_0-x)} = \lambda_0 \frac{x_1-x}{x_0-x}. \end{aligned}$$

The last inequality holds for all $x \in [-1, x_1]$ if and only if $\lambda > \lambda_0(1+x_1)/(1+x_0)$.

(II): If we apply (I) to $f_{1-\lambda}$ with x_1, x_0, x_2, λ_0 replaced by $-x_2, -x_0, -x_1, 1-\lambda_0$, we obtain (II).

(III): This is straightforward.

(IV): If we substitute $x_0 = \lambda_0 x_1 + (1-\lambda_0)x_2$ we see that

$$\frac{(1-x_2)(1+x_0)}{(1+x_2)(1-x_0)} = 1 - \frac{2\lambda_0(x_2-x_1)}{(1+x_2)(1-x_0)}.$$

Similarly

$$\frac{(1-x_2)(1+x_1)}{(1+x_2)(1-x_1)} = 1 - \frac{2(x_2-x_1)}{(1+x_2)(1-x_1)}.$$

Hence by lemma 5

$$\bar{\lambda} < \lambda_0 \frac{1-x_1}{1-x_0}.$$

To obtain the other inequality, we rewrite $\bar{\lambda}$ as

$$\bar{\lambda} = \frac{\log\dfrac{(1+x_2)(1-x_0)}{(1-x_2)(1+x_0)}}{\log\dfrac{(1+x_2)(1-x_1)}{(1-x_2)(1+x_1)}} = \frac{\log\left(1+\dfrac{2\lambda_0(x_2-x_1)}{(1-x_2)(1+x_0)}\right)}{\log\left(1+\dfrac{2(x_2-x_1)}{(1-x_2)(1+x_1)}\right)} > \lambda_0\frac{1+x_1}{1+x_0}$$

again by lemma 5.

We are now ready for the main result in this section:

THEOREM 1. *Let ψ be a function in T such that $N_\psi = N$. Then*

$$G_\psi(x) \text{ is constant on } [\alpha,\beta]\,,$$

where as before $[\alpha,\beta]$ is the convex hull of supp $(1-\psi)$.

PROOF. In [2] it is proved that the function φ_0 mentioned in the introduction gives

$$N_{1-\varphi_0} = 2\log 2 - \tfrac{2}{3}\log 3 < 0\,.$$

Hence if ψ is optimal we have $N_\psi < 0$ and therefore $[\alpha,\beta] \subset (-1,1)$.
Assume $G_\psi(x)$ is not constant on $[\alpha,\beta]$. Then

$$m = \min_{[\alpha,\beta]} G_\psi(x) < \max_{[\alpha,\beta]} G_\psi(x) = M\,.$$

The easier case, when $G_\psi(\alpha)=m$, or $G_\psi(\beta)=m$, will be dealt with afterwards. For the moment we choose an $x_0 \in (\alpha,\beta)$ so that $G_\psi(x_0)=m$. Put

$$M_1 = \max\{G_\psi(x)\ ;\ x \in [\alpha, x_0]\}$$

$$M_2 = \max\{G_\psi(x)\ ;\ x \in [x_0, \beta]\}\,.$$

Then $M_1 > m$, $M_2 > m$. Let furthermore

$$x_1 = \max\{x \in [\alpha, x_0]\ ;\ G_\psi(x) = M_1\}$$

$$x_2 = \min\{x \in [x_0, \beta]\ ;\ G_\psi(x) = M_2\}\,.$$

Then $-1 < \alpha \leqq x_1 < x_0 < x_2 \leqq \beta < 1$.

Let $\delta > 0$ and let J_0, J_1, J_2 be disjoint relatively open intervals in $[\alpha,\beta]$ of length 2δ centered at x_0, x_1, x_2, respectively. Then by lemma 4 we can find positive numbers $u, v, w \leqq 1$ such that

$$u\int_{J_1}(1-\psi(t))\,dt = v\int_{J_2}(1-\psi(t))\,dt = w\int_{J_0}\psi(t)\,dt = \varepsilon > 0\,.$$

Now put

$$\Delta(t) = \begin{cases} \bar{\lambda}u(1-\psi(t)) & ; \quad t \in J_1 \\ -w\psi(t) & ; \quad t \in J_0 \\ (1-\bar{\lambda})v(1-\psi(t)); & \quad t \in J_2 \\ 0 & \quad \text{otherwise} \end{cases}$$

where $\bar{\lambda}$ is the quantity of lemma 6. Then clearly, as $0<\bar{\lambda}<1$,

$$0 \leqq \psi(t)+\Delta(t) \leqq 1 \quad \text{for all } t \in [-1,1] .$$

Since

$$\int_{-1}^{1} \Delta(t)\,dt = \bar{\lambda}\varepsilon-\varepsilon+(1-\bar{\lambda})\varepsilon = 0 ,$$

we have $\psi+\Delta \in T$. Moreover $\Delta=0$ outside $[\alpha,\beta]$.

Now consider

$$\begin{aligned} G_\Delta(x) &= \left(\int_{J_1}+\int_{J_2}+\int_{J_0}\right)\Delta(t)\log|x-t|\,dt \\ &= \bar{\lambda}\int_{J_1} u(1-\psi(t))\log|x-t|\,dt+(1-\bar{\lambda})\int_{J_2} v(1-\psi(t))\log|x-t|\,dt \\ &\quad -\int_{J_0} w\psi(t)\log|x-t|\,dt \\ &= \varepsilon(\bar{\lambda}\log|x-t_1|+(1-\bar{\lambda})\log|x-t_2|-\log|x-t_0|) , \end{aligned}$$

where $t_i \in J_i$ depend on x.

Let U_0, U_1, U_2 be disjoint open intervals centered at x_0, x_1, x_2 respectively. Then as $\delta \to 0$,

$$\frac{1}{\varepsilon}G_\Delta(x) \to f_{\bar{\lambda}}(x) \quad \text{uniformly on } [-1,1]\setminus\bigcup_{i=0}^{2} U_i ,$$

where $f_{\bar{\lambda}}$ is the function from lemma 6. Therefore, if we put

$$\Lambda = \min(G_\Delta(-1), G_\Delta(1)), \qquad U_i = (r_i, s_i); \qquad 0\leqq i\leqq 2 ,$$

we have for sufficiently small δ that

(i) $\max\{G_\Delta(x)\ ;\ x \in [\alpha,r_1] \cup [s_2,\beta]\} < \Lambda$.

Choose positive numbers σ_1,σ_2 such that $x_1+\sigma_1<x_2-\sigma_2$ and

$$x_1 < x < x_1+\sigma_1 \Rightarrow f_{\bar{\lambda}}(x) < f_{\bar{\lambda}}(-1)$$

$$x_2-\sigma_2 < x < x_2 \Rightarrow f_{\bar{\lambda}}(x) < f_{\bar{\lambda}}(1) .$$

Then if δ is small enough, we obtain

$$\text{(ii)}\quad \max\left\{G_\Delta(x)\ ;\ x \in [x_1, x_1+\sigma_1] \cup [x_2-\sigma_2, x_2] \setminus \bigcup_{i=0}^{2} U_1\right\} < \Lambda .$$

(Note that σ_1, σ_2 are independent of δ).

Finally if $x \in U_1$ we have

$$\text{(iii)}\quad G_\Delta(x) < \varepsilon(\bar\lambda \log(s_1-r_1)+\log 2-\log(r_0-s_1)) < \Lambda,$$

$$\text{if } s_r - r_1 \text{ is chosen small enough} .$$

Similarly for $x \in U_2$.

Summing up, we conclude from (i), (ii) and (iii) that

(a) $G_\Delta(x) < \Lambda$ for all $x \in [\alpha, x_1+\sigma_1] \cup [x_2-\sigma_2, \beta]$, for sufficiently small δ.

Furthermore, choosing δ small enough, we can obtain

(b) $\bar m + \max\{|G_\Delta(x)|\ ;\ x \in [x_1+\sigma_1, x_2-\sigma_2]\}$
$< M - \max\{|G_\Delta(x)|\ ;\ x \in [-1,1]\}$,

where $\bar m = \max\{G_\psi(x)\ ;\ x \in [x_1+\sigma_1, x_2-\sigma_2]\}$.

Now consider $G_{\psi+\Delta}(x) = G_\psi(x) + G_\Delta(x)$. From (a) and (b) we have

$$\max_{[\alpha,\beta]} G_{\psi+\Delta}(x) = \max_{[\alpha, x_1+\sigma_1] \cup [x_2-\sigma_2, \beta]} (G_\psi(x)+G_\Delta(x)\}$$
$$< \max_{[\alpha,\beta]} G_\psi(x)+\Lambda .$$

Therefore,

$$N_{\psi+\Delta} = \max_{[\alpha,\beta]}(G_{\psi+\Delta}(x)) - \min(G_{\psi+\Delta}(-1), G_{\psi+\Delta}(1))$$
$$< \max_{[\alpha,\beta]} G_\psi(x)+\Lambda-\min(G_\psi(-1), G_\psi(1))-\min(G_\Delta(-1), G_\Delta(1)) = N_\psi .$$

This contradiction proves the theorem for the case that $G_\psi(\alpha) \neq m$, $G_\psi(\beta) \neq m$.

In the case when, say, $G_\psi(\beta) = m$, we put $x_0 = \beta$ and proceed as above except we use the value $\bar\lambda = 1$. So in this case we define

$$\Delta(t) = \begin{cases} u(1-\psi(t)); & t \in J_1 \\ -w\psi(t) \quad ; & t \in J_0 \\ 0 & \text{otherwise} . \end{cases}$$

The proof of the necessary inequalities proceed similarly, except they are easier to establish in this case. This completes the proof of theorem 1.

4. The solution of the integral equation.

The last step in the solution of our minimax problem consists of solving the integral equation that an optimal ψ must satisfy, according to theorem 1.

THEOREM 2 (Tverberg). *Let* $\varphi \in T$ *satisfy*

$$\int_{-1}^{1} (1-\varphi(t))\log|x-t|\,dt = K \quad \textit{for } x \in [\alpha,\beta] \tag{4.1}$$

where $[\alpha,\beta]$ *is the convex hull of* $\operatorname{supp}\varphi$ *and* K *is a constant.*

Then $[\alpha,\beta]=[-\frac{1}{2}\sqrt{3},\frac{1}{2}\sqrt{3}\}$ *and*

$$\varphi(t) = \varphi_0(t) = \begin{cases} \dfrac{2}{\pi}\operatorname{Arc\,tan}\sqrt{3-4t^2} & ; \quad |t| \leqq \frac{1}{2}\sqrt{3} \\ 0 & ; \quad \frac{1}{2}\sqrt{3} \leqq |t| \leqq 1 \end{cases}$$

PROOF. Differentiating (4.1) we get

$$\int_{\alpha}^{\beta} \frac{\varphi(t)}{x-t}\,dt = \log(1+x)-\log(1-x), \qquad x \in [\alpha,\beta]\,, \tag{4.2}$$

where we take the Cauchy principal value of the integral, that is,

$$\lim_{\varepsilon\to 0}\left(\int_{\alpha}^{x-\varepsilon} + \int_{x+\varepsilon}^{\beta}\right).$$

The general solution of the equation (4.2) is given by

$$\pi^2\sqrt{(x-\alpha)(\beta-x)}\,\varphi(x) = \int_{\alpha}^{\beta} \frac{\sqrt{(t-\alpha)(\beta-t)}}{t-x}\log\frac{1+t}{1-t}\,dt + \pi C\,, \tag{4.3}$$

where $C=\int_{\alpha}^{\beta}\varphi(t)\,dt$ (see [1, p. 178]). Thus in our case $C=1$. As $\varphi \leqq 1$, the limit of the right hand side of (4.3) is 0 as x goes to α or β. Thus, as is easy to see,

$$\int_{\alpha}^{\beta} \sqrt{\frac{\beta-t}{t-\alpha}}\log\frac{1+t}{1-t}\,dt = -\int_{\alpha}^{\beta}\sqrt{\frac{t-\alpha}{\beta-t}}\log\frac{1+t}{1-t}\,dt = -\pi \tag{4.4}$$

Replacing t by $\alpha+\beta-t$ in the second integral of (4.4), we conclude

$$\int_{\alpha}^{\beta}\sqrt{\frac{\beta-t}{t-\alpha}}\log\frac{(1+t)(1+\alpha+\beta-t)}{(1-t)(1-\alpha-\beta+t)}\,dt = 0\,.$$

Thus $\alpha+\beta=0$, since the integrand has the same sign as $\alpha+\beta$. Therefore we have from (4.4)

$$\beta\int_0^1\left(\sqrt{\frac{1+t}{1-t}}-\sqrt{\frac{1-t}{1+t}}\right)\log\frac{1+\beta t}{1-\beta t}\,dt = \pi\,.$$

Since the integrand increases with β, we conclude that β, and hence φ, is unique. The function φ_0 defined in the introduction satisfies

$$0 \leqq \varphi_0 \leqq 1, \qquad \int_{-1}^{1} \varphi(t)\,dt = 1 .$$

And it was shown in [2, pp. 15–16], that φ_0 satisfies the equation (4.1), with $K = -1+\frac{3}{2}\log 3 - \log 4$. So we must have $\varphi = \varphi_0$ and the proof is complete.

Thus the function φ_0 is the solution to our problem. It follows that the minimum value $N = N_{1-\varphi_0}$ is $2\log 2 - \frac{3}{2}\log 3$. This gives

$$R = -2+4\log 2-\tfrac{3}{2}\log 3 ,$$

and

$$\lambda_0 = \exp(1+\tfrac{1}{2}R) = \tfrac{4}{3}\cdot 3^{\frac{1}{4}}$$

as mentioned in the introduction.

REFERENCES

1. F. G. Tricomi, *Integral equations*, Interscience Publishers, London, New York, 1957.
2. H. Tverberg, *On the irreducibikity of polynomials taking small values*, Math. Scand. 32 (1971), 5–21.

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