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A MAXIMAL ABELIAN SUBALGEBRA OF THE CALKIN ALGEBRA WITH THE EXTENSION PROPERTY

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Introduction.

A C*-subalgebra $\mathscr{A}$ of a C*-algebra $\mathscr{B}$ has the extension property relative to $\mathscr{B}$ if each pure state on $\mathscr{A}$ has a unique (hence pure) state extension to $\mathscr{B}$. In [9] Kadison and Singer showed that if $\mathscr{M}$ is a masa (a maximal abelian self-adjoint subalgebra) in $\mathscr{B}(\mathscr{H})$ (the bounded linear operators on a complex separable Hilbert space $\mathscr{H}$) that is nonatomic in the sense that it is isomorphic to $L^\infty(0,1)$, then $\mathscr{M}$ does not have the extension property relative to $\mathscr{B}(\mathscr{H})$. Kadison and Singer inclined to the view that if $\mathscr{D}$ is a masa in $\mathscr{B}(\mathscr{H})$ that is atomic in the sense that it is isomorphic to l^∞, then $\mathscr{D}$ also does not have the extension property relative to $\mathscr{B}(\mathscr{H})$. However, evidence has been accumulating [3, 4, 11] that suggests that perhaps in fact atomic masas do have the extension property relative to $\mathscr{B}(\mathscr{H})$. Our purpose here is to present further evidence along these lines.

It is proved in [3, (3.6)] (although not stated in this generality) that if $\mathscr{A}$ is a masa in a C*-algebra $\mathscr{B}$ and $\mathscr{A}$ is generated (as a C*-algebra) by its projections, then $\mathscr{A}$ has the extension property relative to $\mathscr{B}$ if and only if

$$\mathscr{A} \dotplus [\mathscr{A}^+, \mathscr{B}]^- = \mathscr{B},$$

where $\dotplus$ denotes direct sum and $[\mathscr{A}^+, \mathscr{B}]^-$ = the norm closure of

$$\{AX - XA \,:\, A \in \mathscr{A},\ A \geq 0,\ X \in \mathscr{B}\}.$$

(Because $\mathscr{A}$ is generated by its projections $[\mathscr{A}^+, \mathscr{B}]^-$ is a subspace of $\mathscr{B}$.) This observation motivates the following definition. We say that a C*-subalgebra $\mathscr{A}$ of a C*-algebra $\mathscr{B}$ *splits* $\mathscr{B}$ if

$$\mathscr{A}' \dotplus [\mathscr{A}^+, \mathscr{B}]^- = \mathscr{B},$$

where $\mathscr{A}'$ is the commutant of $\mathscr{A}$ in $\mathscr{B}$ and $[\mathscr{A}^+, \mathscr{B}]^-$ is as before. In general $[\mathscr{A}^+, \mathscr{B}]^-$ is no longer a subspace but the sum is still direct in the sense that $\mathscr{A}' \cap [\mathscr{A}^+, \mathscr{B}]^- = \{0\}$. This can be seen as follows. Fix $a = a^*$ in $\mathscr{A}'$, $a_1 \geq 0$

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in $\mathscr{A}$ and x in $\mathscr{B}$. Choose a state f on $\mathscr{B}$ such that $f(a)=\|a\|$ and f is a homomorphism on $C^*(a,a_1)$, the C*-algebra generated by a, a_1 and the identity. (We shall only consider C*-algebras with identity.) Then by the Cauchy–Schwartz inequality f is $C^*(a,a_1)$-*multiplicative on* $\mathscr{B}$ in the sense that $f(a_2y)=f(a_2)f(y)=f(ya_2)$ for a_2 in $C^*(a,a_1)$ and y in $\mathscr{B}$. Hence

$$\|a\| = f(a) = |f(a-[a_1,x])| \leqq \|a-[a_1,x]\| .$$

Thus, the atomic masa $\mathscr{D}$ has the extension property relative to $\mathscr{B}(\mathscr{H})$ if and only if $\mathscr{D}$ splits $\mathscr{B}(\mathscr{H})$ and by Kadison and Singer's result nonatomic masas do not split $\mathscr{B}(\mathscr{H})$. On the other hand, $\mathscr{B}(\mathscr{H})$ splits itself [1, 5] and it is trivial that $\mathbf{C}I$ splits $\mathscr{B}(\mathscr{H})$. The first result to be proved here is that $\sum\oplus\mathscr{B}(\mathscr{H})$ acting on $\mathscr{H}\oplus\mathscr{H}\oplus\ldots$ splits $\mathscr{B}(\mathscr{H}\oplus\mathscr{H}\oplus\ldots)$. This fact is then used to construct (with the aid of the continuum hypothesis) a masa $\mathscr{A}$ in the Calkin algebra $\mathscr{E}$ ($\mathscr{B}(\mathscr{H})$ modulo the compact operators $\mathscr{K}=\mathscr{K}(\mathscr{H})$) that is generated by its projections, splits $\mathscr{E}$ and so has the extension property relative to $\mathscr{E}$. Furthermore, it is shown that if $\mathscr{E}$ has "enough" inner automorphisms that preserve $\mathscr{A}$, then each atomic masa $\mathscr{D}$ splits $\mathscr{B}(\mathscr{H})$. Finally, the technique used to construct $\mathscr{A}$ is employed to construct an infinite compact subset of homomorphisms on $\mathscr{D}$ that have unique state extensions to $\mathscr{B}(\mathscr{H})$.

The results.

It is convenient to set some notation. If $\{e_n\}$ is an orthonormal sequence in a Hilbert space and σ is a subset of $\omega=\{1,2,\ldots\}$, P_σ shall denote the projection onto the span of $\{e_n : n\in\sigma\}$. The canonical map of $\mathscr{B}(\mathscr{H})$ onto $\mathscr{E}$ shall be denoted by π.

LEMMA 1. *If $T\in\mathscr{B}(\mathscr{H})$, $\{e_n\}$ is an orthonormal basis for $\mathscr{H}$ and $\{\sigma_k\}_{k=1}^\infty$ are infinite disjoint subsets of ω, then there are infinite subsets τ_k of σ_k, $k=1,2,\ldots$ and a bounded complex sequence $\{b_k\}$ such that $P_\tau(T-B)P_\tau\in\mathscr{K}$, where $\tau=\bigcup\tau_k$ and $B=\sum b_kP_{\sigma_k}$.*

PROOF. Fix a map φ of ω onto ω such that $\varphi^{-1}(n)$ is infinite for all n. Choose a subsequence of ω as follows. Pick n_1 in $\sigma_{\varphi(1)}$. Suppose $n_1,\ldots,n_j$ have been selected. Choose $n_{j+1}\in\sigma_{\varphi(j+1)}-\{n_1,\ldots,n_j\}$ so that

$$\sum_{i=1}^{j} |(Te_{n_{j+1}},e_{n_i})|^2+\sum_{i=1}^{j} |(Te_{n_i},e_{n_{j+1}})|^2 < 2^{-(j+1)} .$$

This choice is possible because $\sigma_{(j+1)}-\{n_1,\ldots,n_j\}$ is infinite and the Fourier coefficients of a vector with respect to an orthonormal sequence converge to zero. This procedure determines a subsequence $\{n_j\}$ of ω such that $\{j : n_j\in\sigma_k\}$

is infinite for $k=1,2,\ldots$ and

$$(1) \qquad \sum_{i \neq j} |(Te_{n_i}, e_{n_j})|^2 < \infty ,$$

Relabel the sequence as n_{jk} so that for each fixed k, $n_{jk} \in \sigma_k$, $j=1,2,\ldots$. By passing to subsequences of each $\{n_{jk}\}_{j=1}^{\infty}$ if necessary, we may assume that

$$(2) \qquad \lim_j \, (Te_{n_{jk}}, e_{n_{jk}}) = b_k \quad \text{and} \quad \sum_j |(Te_{n_{jk}}, e_{n_{jk}}) - b_k|^2 < 2^{-k} .$$

Write $B=\sum b_k P_{\sigma_k}$, $\tau_k=\{n_{jk}\}_{j=1}^{\infty}$, and $\tau=\bigcup \tau_k$. Then by (1) and (2) $P_\tau(T-B)P_\tau$ is a compact operator (in fact it is even Hilbert–Schmidt).

THEOREM 2. *If $\mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \ldots$ is a decomposition of the Hilbert space $\mathscr{H}$ into orthogonal infinite dimensional subspaces, then $\mathscr{A}=\sum \oplus \mathscr{B}(\mathscr{H}_n)$ splits $\mathscr{B}(\mathscr{H})$.*

PROOF. Let P_n denote the projection of $\mathscr{H}$ onto $\mathscr{H}_n$, $n=1,2,\ldots$. Then $\mathscr{A} =\{P_n\}'$ and

$$\mathscr{A}' = \{\sum a_n P_n : \{a_n\} \text{ is a bounded complex sequence}\} .$$

Fix T in $\mathscr{B}(\mathscr{H})$ and apply Lemma 1 to obtain projections $Q_n \leq P_n$ and a bounded complex sequence $\{b_n\}$ such that $Q(T-B)Q$ is compact, where $Q =\sum Q_n$ and $B=\sum b_n P_n$. Choose isometries V_n in $\mathscr{B}(\mathscr{H}_n)$ such that $V_n\mathscr{H}_n =Q_n\mathscr{H}_n$ and write

$$A_n = V_n + V_n^* + 2P_n, \quad A = \sum \oplus A_n \quad \text{and} \quad V = \sum \oplus V_n .$$

Then $A \geq 0$ and

$$[A,V] = \sum \oplus [A_n, V_n] = \sum (P_n - Q_n) = I - Q = R .$$

Since $A \geq 0$, A is a d-symmetric operator in the sense of [6] and so by [6, (3.2)]

$$[A, \mathscr{B}(\mathscr{H})]^- \supseteqq R\mathscr{B}(\mathscr{H}) + \mathscr{B}(\mathscr{H})R .$$

Also, since A has no reducing eigenvalues $[A, \mathscr{B}(\mathscr{H})]^-$ contains the compact operators [6, (2.6)]. Therefore

$$\begin{aligned} T - B &= R(T-B)R + R(T-B)Q + Q(T-B)R + Q(T-B)Q \\ &\subseteqq [A, \mathscr{B}(\mathscr{H})]^- + \mathscr{K} = [A, \mathscr{B}(\mathscr{H})]^- \end{aligned}$$

and $T \in \mathscr{A}' \dotplus [\mathscr{A}^+, \mathscr{B}(\mathscr{A})]^-$.

Note that we have actually proved the stronger fact that $T=B+\lim\,[A, X_n]$, $A \in \mathscr{A}^+$, $B \in \mathscr{A}'$, $X_n \in \mathscr{B}(\mathscr{H})$.

THEOREM 3. *Assuming the continuum hypothesis, there is a masa $\mathscr{A}$ in the Calkin algebra $\mathscr{E}$ that is generated by its projections and splits $\mathscr{E}$ (and so has the extension property relative to $\mathscr{E}$).*

PROOF. Well-order $\mathscr{B}(\mathscr{H})$ as $\{T_\alpha\}_{\alpha<\aleph_1=2^{\aleph_0}}$ and suppose $T_1=I$. Choose C*-algebras in $\mathscr{B}(\mathscr{H})$ by transfinite induction as follows. Write $\mathscr{B}_1=\mathbf{C}I+\mathscr{K}$ and suppose that for some ordinal $\alpha<\aleph_1$ and all ordinals $\beta<\alpha$, C*-algebras $\mathscr{B}_\beta$ have been selected such that

(i) If $\gamma<\beta$, $\mathscr{B}_\gamma\subseteqq\mathscr{B}_\beta$.
(ii) $\pi(\mathscr{B}_\beta)$ is generated by a countable family of commuting projections.
(iii) There are A_β, B_β in $\mathscr{B}_\beta$ and $X_{n\beta}$ in $\mathscr{B}(\mathscr{H})$ such that $\lim_n \|T_\beta-B_\beta-[A_\beta, X_{n\beta}]\|=0$.

We use Theorem 2 and an argument of Brown, Douglas and Fillmore [7, (5.3)] to choose $\mathscr{B}_\alpha$. Write $\mathscr{B}=\mathrm{C}^*(\{B_\beta\}_{\beta<\alpha})$. Since α is a countable ordinal, $\pi(\mathscr{B})$ is generated by a countable family of commuting projections and hence $\pi(\mathscr{B})$ is singly generated [12, p. 293] by $\pi(A)$ for some $A=A^*$ in $\mathscr{B}$. (If $\pi(\mathscr{B})$ is generated by $\{p_n\}$, $\pi(A)=\sum 3^{-n}(2p_n-1)$.) By the Weyl–von Neumann theorem we may perturb A by a compact operator and assume that A is diagonal. In fact we may assume that $A=\sum a_nP_n$, where $\{P_n\}$ is a sequence of mutually orthogonal *infinite dimensional* projections with $\sum P_n=I$ [2, p. 249]. By Theorem 2 there is an operator $B_\alpha=\sum b_nP_n$ and a positive operator A_α in $\{A\}'$ such that $T_\alpha-B_\alpha\in[A_\alpha,\mathscr{B}(\mathscr{H})]^-$. Choose sequences $\{E_n\}$ and $\{F_n\}$ of spectral projections of B_α and A_α respectively so that

$$\{A_\alpha, B_\alpha\}\subseteqq \mathrm{C}^*(\{E_n\},\{F_n\}).$$

Then $\{E_n\}$, $\{F_n\}$, and A pairwise commute and $\mathscr{B}_\alpha=\mathrm{C}^*(A,\{E_n\},\{F_n\})+\mathscr{K}$ has the desired properties. This completes the induction.

Clearly, the C*-algebra $\mathscr{A}=\bigcup\{\pi(\mathscr{B}_\alpha):\alpha<\aleph_1\}$ is generated by its projections and $\mathscr{A}\dotplus[\mathscr{A}^+,\mathscr{E}]^-=\mathscr{E}$. To complete the proof, we must show that $\mathscr{A}$ is a masa in $\mathscr{E}$. To this end we show $\mathscr{A}$ has the extension property relative to $\mathscr{E}$. Fix a complex homomorphism h on $\mathscr{A}$ and states f and g on $\mathscr{E}$ that extend h. Then as noted in the introduction f and g are $\mathscr{A}$-multiplicative on $\mathscr{E}$ and so

$$\begin{aligned} f(\pi(T_\alpha)) &= f(\pi(B_\alpha))+\lim_n f([\pi(A_\alpha),\pi(X_{n\alpha})]) \\ &= h(\pi(B_\alpha)) = g(\pi(B_\alpha)) = g(\pi(T_\alpha)).\end{aligned}$$

Thus $f=g$ and it now follows from the Stone–Weierstrass theorem that $\mathscr{A}$ is maximal abelian.

REMARKS. (1) If $\mathscr{D}$ is an atomic masa in $\mathscr{B}(\mathscr{H})$ and $\mathscr{D}$ has the extension property relative to $\mathscr{B}(\mathscr{H})$ then since $\pi(\mathscr{D})$ is maximal abelian in $\mathscr{E}$ [8], $\pi(\mathscr{D})$

splits $\mathscr{E}$. Conversely, if $\pi(\mathscr{D})$ splits $\mathscr{E}$, then $\pi(\mathscr{D})$ has the extension property relative to $\mathscr{E}$ and since pure vector states on $\mathscr{D}$ have unique state extensions to $\mathscr{B}(\mathscr{H})$, it would follow that $\mathscr{D}$ has the extension property relative to $\mathscr{B}(\mathscr{H})$. In fact this is the case if the algebra $\mathscr{A}$ constructed in Theorem 3 *lifts* to an abelian C*-algebra in $\mathscr{B}(\mathscr{H})$, i.e., if $\pi^{-1}(\mathscr{A})=\mathscr{B}+\mathscr{K}$, where $\mathscr{B}$ is an abelian C*-algebra. For then $\mathscr{B}$ can be taken to be maximal abelian and so $\mathscr{B}$ has the form $\mathscr{D}\oplus\mathscr{M}$, where $\mathscr{D}$ is an atomic masa and $\mathscr{M}$ is a nonatomic masa. But since π is injective on $\mathscr{M}$, $\pi(\mathscr{M})$ does not have the extension property relative to $\mathscr{E}$ and so this summand could not occur, i.e., $\mathscr{B}=\mathscr{D}$.

(2) The question "Does $\mathscr{A}$ lift to an abelian C*-algebra in $\mathscr{B}(\mathscr{H})$?" is obviously a nonseparable version of a problem considered by Brown, Douglas and Fillmore [7]. However, the techniques used by Brown, Douglas and Fillmore to solve the problem in the separable case do not generalize. In particular, if $\mathscr{D}$ is an atomic masa in $\mathscr{B}(\mathscr{H})$, there is *no* trivial extension of $\mathscr{K}(\mathscr{H})$ by $\pi(\mathscr{D})\cong C(\beta\omega\setminus\omega)$, where $\beta\omega$ denotes the Stone–Čech compactification of the integers [7, (5.12)].

(3) In addition to the fact that $\mathscr{A}$ and $\pi(\mathscr{D})$ are each generated by their projections, each of these algebras is the range of a unique norm one projection from $\mathscr{E}$. Indeed, there is a unique norm one projection from $\mathscr{B}(\mathscr{H})$ onto $\mathscr{D}$ that maps $\mathscr{K}$ onto $\mathscr{D}\cap\mathscr{K}$ [9] and so induces a norm one projection of $\mathscr{E}$ onto $\pi(\mathscr{D})$ that is easily seen to be unique. The projection for $\mathscr{A}$ is given by $\pi(T_\alpha)\mapsto\pi(B_\alpha)$ (notation as in the proof of Theorem 3). This map is linear because $[\mathscr{A}^+,\mathscr{E}]^-$ is a subspace of $\mathscr{E}$ and it is the unique norm one projection because all such maps vanish on $[\mathscr{A}^+,\mathscr{E}]^-$ [14].

We now show that if $\mathscr{E}$ has "enough" inner automorphisms which preserve $\mathscr{A}$, then each atomic masa $\mathscr{D}$ splits $\mathscr{B}(\mathscr{H})$. Let us say that a C*-subalgebra $\mathscr{A}$ of a C*-algebra $\mathscr{B}$ is *permutable in* $\mathscr{B}$ if there are mutually orthogonal projections $p_0, p_1, \ldots$ in $\mathscr{A}$ and unitaries u_n in $\mathscr{B}$ such that $u_n\mathscr{A}u_n^*=\mathscr{A}$ and $u_np_0u_n^*=p_n$, $n=1,2,\ldots$.

THEOREM 4. *If there is a permutable masa $\mathscr{A}$ in $\mathscr{E}$ that splits $\mathscr{E}$, then each atomic masa $\mathscr{D}$ in $\mathscr{B}(\mathscr{H})$ splits $\mathscr{B}(\mathscr{H})$.*

PROOF. Fix an atomic masa $\mathscr{D}$ in $\mathscr{B}(\mathscr{H})$. Then for an orthonormal basis $\{e_n\}$ in $\mathscr{H}$, an operator D is in $\mathscr{D}$ if and only if its associated matrix $\{(De_n, e_m)\}$ is diagonal. If T is any operator, then [3, (8.2)] $T=T_1+M+K$, where $T_1\in[\mathscr{D},\mathscr{B}(\mathscr{H})]$, K is compact and $M=\sum\oplus M_k$ is block diagonal with respect to $\{e_n\}$ in the sense that the matrix associated with M is the direct sum of m_k by m_k matrices ($m_k<\infty$). Since $\mathscr{D}\dot{+}[\mathscr{D},\mathscr{B}(\mathscr{H})]^-$ contains $\mathscr{K}$, to show $\mathscr{D}$ splits $\mathscr{B}(\mathscr{H})$ it suffices to show that $\mathscr{D}\dot{+}[\mathscr{D},\mathscr{B}(\mathscr{H})]^-$ contains all operators that are block diagonal with respect to the basis $\{e_n\}$.

Fix a block diagonal operator $M=\sum\oplus M_k$ and suppose $\mathscr{A}$ is a permutable masa in $\mathscr{E}$ that splits $\mathscr{E}$. Let $p_0, p_1, \ldots$ and $u_1, u_2, \ldots$ denote orthogonal projections in $\mathscr{A}$ and unitaries in $\mathscr{E}$ such that $u_n\mathscr{A}u_n^*=\mathscr{A}$ and $u_n p_0 u_n^*=p_n$, $n=1,2,\ldots$. We shall construct a representation ϱ of $\mathscr{E}$ and diagonal operators D and D_n in a compression of $\varrho(\mathscr{E})$ such that

$$M = D+\lim_n [D_n, X_n]$$

for some operators X_n. Choose a partition of ω into finite subsets σ_k such that each σ_k contains $m_k=\dim M_k$ elements and write $q_k=\sum_{n\in\sigma_k} p_n$. Fix a pure state f on $\mathscr{E}$ such that the restriction of f to $\mathscr{A}$ is a homomorphism and $f(p_0)=1$. Let $\varrho\colon \mathscr{E} \to \mathscr{B}(\mathscr{H}_\varrho)$ denote the GNS representation of $\mathscr{E}$ induced by f and let x_0 denote the canonical cyclic vector in $\mathscr{H}_\varrho$. Since f is a homomorphism on $\mathscr{A}$, $\varrho(a)x_0=f(a)x_0$ for each $a\in\mathscr{A}$. In particular, $\varrho(p_0)x_0=x_0$. Hence if $x_n=\varrho(u_n)x_0$,

$$\varrho(a)x_n = \varrho(u_n a_1 u_n^*)x_n = f(a_1)x_n$$

for $a=u_n a_1 u_n^*$ in $\mathscr{A}$ and $n=1,2,\ldots$. In particular $\varrho(p_n)x_n=\varrho(u_n p_0 u_n^*)x_n=x_n$, $n=1,2,\ldots$. Thus, the operators in $\varrho(\mathscr{A})$ are diagonal with respect to the orthonormal sequence $\{x_0, x_1,\ldots\}$ in $\mathscr{H}_\varrho$. (Of course, $\{x_0,x_1,\ldots\}$ does not form a basis since $\mathscr{H}_\varrho$ is nonseparable.) Write E_k for the projection of $\mathscr{H}_\varrho$ onto the span of $\{x_n : n\in\sigma_k\}$ and put $E=\sum E_k$. Note that $\dim E_k=m_k=\dim M_k$ and

$$E_k \le \varrho(q_k) = \varrho\left(\sum_{n\in\sigma_k} p_n\right).$$

Now choose a sequence $\{Q_k\}$ of mutually orthogonal projections in $\mathscr{B}(\mathscr{H})$ such that $\pi(Q_k)=q_k$, $k=1,2,\ldots$. (This may be accomplished by arguing as in the second paragraph of the proof of Theorem 3.) Let ϱ_k denote the representation of $\mathscr{B}(Q_k\mathscr{H})$ obtained by restricting $\varrho\circ\pi$. That is,

$$\varrho_k = \varrho(q_k)\varrho\circ\pi\,|_{\varrho(q_k)\mathscr{H}_\varrho}.$$

Since f is a pure state, $\varrho\circ\pi$ is an irreducible representation of $\mathscr{B}(\mathscr{H})$ and so each ϱ_k is irreducible. Therefore, by the Kadison transitivity theorem, there are operators T_k in $\mathscr{B}(Q_k\mathscr{H})$ such that $\varrho_k(T_k)$ has the matrix M_k with respect to the sequence $\{x_n : n\in\sigma_k\}$. If $T=\sum\oplus T_k$, then

$$E_k\varrho(\pi(T)E_k) = E_k\varrho_k(T_k)E_k$$

and so

$$\sum E_k\varrho(\pi(T))E_k\,|_{E\mathscr{H}_\varrho} = \sum\oplus M_k = M .$$

Since $\mathscr{A}$ splits $\mathscr{E}$, $\varrho(\mathscr{A})$ splits $\varrho(\mathscr{E})$ and so

$$\varrho(\pi(T)) = \varrho(b)+\lim_n [\varrho(a_n), \varrho(c_n)]$$

for a_n, b in $\mathscr{A}$ and c_n in $\mathscr{E}$. Now

$$D = \sum E_k\varrho(b)E_k|_{E\mathscr{H}_\varrho} \quad \text{and} \quad D_n = \sum E_k\varrho(a_n)E_k|_{E\mathscr{H}_\varrho}$$

are diagonal with respect to the basis $\{x_0, x_1, \ldots\}$ for $E\mathscr{H}_\varrho$. Therefore

$$M = \sum \oplus M_k = \sum E_k\varrho(\pi(T)E_k|_{E\mathscr{H}_\varrho} = D + \lim_n [D_n, \sum E_k\varrho(c_n)E_k|_{E\mathscr{H}_\varrho}]$$

and the theorem is proved.

COROLLARY 5. *If the algebra $\mathscr{A}$ constructed in Theorem 3 assuming the continuum hypothesis is permutable, then each atomic masa $\mathscr{D}$ in $\mathscr{B}(\mathscr{H})$ has the extension property relative to $\mathscr{B}(\mathscr{H})$ even if the continuum hypothesis is false.*

PROOF. By Theorems 3 and 4, each atomic masa $\mathscr{D}$ splits $\mathscr{B}(\mathscr{H})$ if the continuum hypothesis is true. However, the statement "$\mathscr{D}$ splits $\mathscr{B}(\mathscr{H})$" is a π_1^2 statement and so by Platek's absoluteness theorem [10], if a proof of the statement exists which uses the continuum hypothesis, then there is a proof that does not require the continuum hypothesis. As noted in the introduction, $\mathscr{D}$ splits $\mathscr{B}(\mathscr{H})$ if and only if $\mathscr{D}$ has the extension property relative to $\mathscr{B}(\mathscr{H})$.

REMARKS. (1) The author is grateful to A. Kechris and S. Simpson for helpful discussions concerning absoluteness arguments.

(2) It seems to be generally felt by logicians that Platek's absoluteness theorem should be interpreted to mean that the continuum hypothesis is usually of little use in proving π_1^2 statements. Since the continuum hypothesis seems to play an essential role in the proof of Theorem 3, perhaps Corollary 5 should be interpreted to mean that a proof that $\mathscr{A}$ is permutable would itself imply that $\mathscr{D}$ splits $\mathscr{B}(\mathscr{H})$. This point of view is supported by the fact that there does not appear to be any obvious way to alter the construction in Theorem 3 so as to make $\mathscr{A}$ permutable in $\mathscr{E}$. On the other hand, absoluteness arguments have been used occasionally in proving π_1^2 statements and it is tempting to try to show that atomic masas have the extension property via this method.

In [11] Reid showed that certain homomorphisms on $\mathscr{D}$ do have unique state extensions to $\mathscr{B}(\mathscr{H})$. The final result to be presented here is similar and in fact may viewed as an extension of Reid's work.

THEOREM 6. *Let $\mathscr{D}$ denote an atomic masa in $\mathscr{B}(\mathscr{H})$. Assuming the continuum hypothesis, there is an infinite compact subset C in the maximal ideal space of $\mathscr{D}$ such that each homomorphism h in C has a unique state extension to $\mathscr{B}(\mathscr{H})$.*

PROOF. Fix an orthonormal basis $\{e_n\}$ for $\mathscr{H}$ such that an operator D is in $\mathscr{D}$ if and only if its associated matrix $\{(De_n, e_n)\}$ is diagonal. We first define collections $\{\sigma_{\alpha n}\}$ of subsets of ω by transfinite induction. Well-order $\mathscr{B}(\mathscr{H})$ as $\{T_\alpha\}_{\alpha<\aleph_1=2^{\aleph_0}}$. Apply Lemma 1 to T_1 to obtain a sequence $\sigma_{11}, \sigma_{12}, \ldots$ of infinite disjoint subsets of ω and an operator D_1 in $\mathscr{D}$ such that $P_1(T_1-D_1)P_1$ is compact, where $P_1=\sum_n P_{\sigma_{1n}}$ (As in Lemma 1, P_σ denotes the projection of $\mathscr{H}$ onto the span of $\{e_n : n \in \sigma\}$). Suppose that for some ordinal $\alpha<\aleph_1$ and all ordinals $\beta<\alpha$ infinite disjoint subsets $\{\sigma_{\beta n}\}_{n=1}^{\infty}$ of ω and operators D_β in $\mathscr{D}$ have been chosen such that

(i) For each n the collection $\{\sigma_{\beta n}\}_{\beta<\alpha}$ has the property that finite intersections of sets in the collection are infinite.

(ii) $P_\beta(T_\beta-D_\beta)P_\beta$ is compact, where $P_\beta=\sum_n P_{\beta n}$.

Choose $\sigma_{\alpha n}$, $n=1,2,\ldots$ as follows. Since α is a countable ordinal, the collections $\{\sigma_{\beta n}\}_{\beta<\alpha}$ are countable for $n=1,2,\ldots$ and so by (i) we may choose infinite disjoint subsets $\sigma'_{\alpha n}$ such that all but a finite number of elements of $\sigma'_{\alpha n}$ lie in $\sigma_{\beta n}$, $\beta<\alpha$, $n=1,2,\ldots$. Apply Lemma 1 to T_α and $\{\sigma'_{\alpha n}\}$ to obtain infinite subsets $\sigma_{\alpha n}$ of $\sigma'_{\alpha n}$ and a diagonal operator D_α such that $P_\alpha(T_\alpha-D_\alpha)P_\alpha$ is a compact operator, where $P_\alpha=\sum_n P_{\sigma_{\alpha n}}$. The induction is complete. We now define a sequence $\{h_n\}$ of homomorphisms on $\mathscr{D}$ by using the correspondence between the maximal ideal space of $\mathscr{D}$ and $\beta\omega$, the Stone–Čech compactification of ω. Recall that $\beta\omega$ may be thought of as the ultrafilters on ω with the hull-kernel topology and that each subset σ of ω determines a clopen subset

$$W(\sigma) = \{\mathscr{U} \in \beta\omega : \sigma \in \mathscr{U}\}$$

in $\beta\omega$ [13]. Further, if h is a complex homomorphism on $\mathscr{D}$, then h is determined by its action on the projections P_σ in $\mathscr{D}$ and

$$\mathscr{U}_h = \{\sigma \subseteqq \omega : h(P_\sigma)=1\}$$

is an ultrafilter on ω. Using these facts it is straightforward to show that the map $h \to \mathscr{U}_h$ is a homeomorphism of the maximal ideal space of $\mathscr{D}$ onto $\beta\omega$. So, choosing a complex homomorphism on $\mathscr{D}$ amounts to selecting an ultrafilter on ω. Also note that a homomorphism h vanishes on $\mathscr{D} \cap \mathscr{K}$ if and only if the corresponding $\mathscr{U}_h$ is a free ultrafilter. Now the compact subsets $\{W(\sigma_{\alpha n})\}_{\alpha<\aleph_1}$ corresponding to the $\sigma_{\alpha n}$'s chosen above have the finite intersection property for each fixed n. Therefore $\bigcap_{\alpha<\aleph_1} W(\sigma_{\alpha n}) \neq \varnothing$ for $n=1,2,\ldots$ and we may choose ultrafilters $\mathscr{U}_n$ in $\beta\omega$ such that $\sigma_{\alpha n} \in \mathscr{U}_n$, $\alpha<\aleph_1$, $n=1,2,\ldots$. Furthermore, since finite intersections of $\{\sigma_{\alpha n}\}$ are infinite, we may assume each $\mathscr{U}_n$ is a free ultrafilter. Let h_n denote the homomorphism on $\mathscr{D}$ corresponding to $\mathscr{U}_n$. Note that $h_n(P_{\sigma_{\alpha n}})=1$ for $\alpha<\aleph_1$, $n=1,2,\ldots$. To complete the proof, we show that each homomorphism in C, the weak*-closure of $\{h_n\}$, has a unique state extension to $\mathscr{B}(\mathscr{H})$. Fix h in C and a state f on $\mathscr{B}(\mathscr{H})$ that extends h. Since each h_n vanishes on $\mathscr{D} \cap \mathscr{K}$, h vanishes on $\mathscr{D} \cap \mathscr{K}$ and f vanishes on $\mathscr{K}$. If $T = T_\alpha \in \mathscr{B}(\mathscr{H})$, then $P_\alpha T_\alpha P_\alpha = P_\alpha D_\alpha P_\alpha + K_\alpha$, where $K_\alpha \in \mathscr{K}$, $D_\alpha \in \mathscr{D}$ and $P_\alpha = \sum P_{\sigma_{\alpha n}}$. Since $P_{\sigma_{\alpha n}} \leq P_\alpha$, $n = 1, 2, \ldots$, $h_n(P_\alpha) = 1$, $n = 1, 2, \ldots$ and $h(P_\alpha) = 1 = f(P_\alpha)$. Furthermore, f is $\mathscr{D}$-multiplicative on $\mathscr{B}(\mathscr{H})$ so $f(T_\alpha) = f(D_\alpha) = h(D_\alpha)$. Therefore all state extensions of h to $\mathscr{B}(\mathscr{H})$ agree.

REMARK. Recall that an ultrafilter $\mathscr{U}$ on ω is called *selective* if for each partition $\{\sigma_k\}$ of ω either $\sigma_k \in \mathscr{U}$ for some k, or else there is a set σ in $\mathscr{U}$ such that $\sigma \cap \sigma_k$ contains at most one point for $k = 1, 2, \ldots$. In [11] Reid showed that (among others) homomorphisms of $\mathscr{D}$ corresponding to selective ultrafilters have unique state extensions to $\mathscr{B}(\mathscr{H})$. We remark that it can be shown using arguments similar to those given in Lemma 1 and Theorem 6 that if $\{h_n\}$ is a sequence of complex homomorphisms on $\mathscr{D}$ corresponding to selective ultrafilters on ω and the (unique) state extensions of the h_n's to $\mathscr{B}(\mathscr{H})$ are not unitarily equivalent, then each h in the weak*-closure of $\{h_n\}$ has a unique state extension to $\mathscr{B}(\mathscr{H})$.

Concluding remarks and questions.

In order to make further progress on the extension problem for atomic masas in $\mathscr{B}(\mathscr{H})$ it appears that a clearer understanding of the structure of the masas in the Calkin algebra would be useful.

If $\mathscr{B}$ is a masa in $\mathscr{B}(\mathscr{H})$, then $\pi(\mathscr{B})$ is a masa in $\mathscr{E}$ [8] and $\pi(\mathscr{B})$ is generated by its projections. Brown, Douglas and Fillmore observed [7, p. 126] that there are masas in the Calkin algebra that are not generated by their projections (because they contain normal elements that do not lift to normal operators in $\mathscr{B}(\mathscr{H})$). If $\mathscr{A}$ is a masa in $\mathscr{E}$ that is generated by its projections, does $\mathscr{A}$ lift to a masa in $\mathscr{B}(\mathscr{H})$? If so, then each atomic masa $\mathscr{D}$ in $\mathscr{B}(\mathscr{H})$ has the extension property relative to $\mathscr{B}(\mathscr{H})$.

If $\mathscr{B}$ is a masa in $\mathscr{B}(\mathscr{H})$, then $\pi(\mathscr{B})$ is permutable in $\mathscr{E}$. If $\mathscr{A}$ is a masa in $\mathscr{E}$ that is generated by its projections, is $\mathscr{A}$ permutable in $\mathscr{E}$? A positive answer would again imply a positive answer to the extension problem for atomic masas in $\mathscr{B}(\mathscr{H})$.

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