THE IVERSEN THEOREM IN A POLYDISC

C. L. CHILDRESS and G. L. CSORDAS

1. Introduction.

One of the fundamental results in the study of the boundary behavior of meromorphic functions of a single complex variable is the Iversen Theorem [5]. This result, which may be viewed as an extended form of a local maximum principle, states that the boundary of the cluster set is contained in the radial boundary cluster set modulo a set of measure zero. An excellent account of the extensions of the Iversen Theorem and its significance in the theory of functions may be found in Collingwood and Lohwater [2, Chapter 5] (cf. also Lohwater [6, Chapter 5] and the references contained therein).


The purpose of this paper is to provide a direct and geometric proof of the Iversen Theorem for meromorphic functions defined on a polydisc. Our proof is based on a result which is of independent interest in itself. Theorem 2 is an extension of a result of Carathéodory [1] to the polydisc. This theorem asserts that the cluster set of a bounded analytic function $f$ at a point $P$ on the distinguished boundary of the polydisc is contained in the closure of the convex hull of the set of radial limit values of $f$ in a vicinity of $P$.

For the sake of simplicity, we shall present our theorems in the unit polydisc in $\mathbb{C}^2$. We remark, however, that our results remain valid for a polydisc in $\mathbb{C}^n$, where $n \geq 2$.

2.

Throughout this paper, $U^2$ will denote the unit polydisc in $\mathbb{C}^2$, $T^2$ will denote the distinguished boundary of $U^2$, and $m_2$ will denote the normalized Lebesgue measure on $T^2$, i.e., $m_2(T^2) = 1$.

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We begin with a lemma.

**Lemma 1.** Let \( f(z, w) \) be analytic and bounded, \( |f| < M \), in \( U^2 \), and let \( \varepsilon > 0 \). Let \( G \) be a subset of \( T^2 \), with \( m_2(G) < \varepsilon \), such that the radial limit
\[
f^*(e^{i\theta}, e^{i\phi}) = \lim_{r \to 1} f(re^{i\theta}, re^{i\phi})
\]
exists for \((e^{i\theta}, e^{i\phi}) \in G^c = T^2 - G\). Then \( f(0, 0) = \xi_1 + \xi_2 \), where \( \xi_1 \) lies in \( \bar{S} \), the closure of the convex hull of
\[
S = \{ f^*(e^{i\theta}, e^{i\phi}) \mid (e^{i\theta}, e^{i\phi}) \in G^c \},
\]
and where \( |\xi_2| < 2\varepsilon M \).

**Proof.** We shall prove the lemma with the aid of an auxiliary function \( g \) on \( T^2 \) defined by
\[
g = \begin{cases} 
    f^* & \text{on } G^c, \\
    f^*(1, 1) & \text{on } G,
\end{cases}
\]
where we have tacitly assumed that the point \((1, 1)\) lies in \( G^c \). Since \( f(z, w) \) is analytic and bounded in \( U^2 \), it follows that
\[
f(0, 0) = \int_{T^2} f^* \, dm_2
\]
\[
= \int_{T^2} g \, dm_2 + \int_{T^2} (f^* - g) \, dm_2
\]
\[
= \int_{T^2} g \, dm_2 + \int_{G} (f^* - g) \, dm_2.
\]
Next we set \( f(0, 0) = \xi_1 + \xi_2 \), where
\[
\xi_1 = \int_{T^2} g \, dm_2 \quad \text{and} \quad \xi_2 = \int_{G} (f^* - g) \, dm_2.
\]
Then the assumptions \( |f| < M \) and \( m_2(G) < \varepsilon \) imply that \( |\xi_2| < 2\varepsilon M \). Thus, in order to complete the proof of the lemma, it suffices to show that \( \xi_1 \) lies in \( \bar{S} \), the closure of the convex hull of \( S \).

Consider a closed half-plane \( H \) which contains \( \bar{S} \). It suffices to show that \( \xi_1 \) lies in \( H \). We may choose real numbers \( \theta_0 \) and \( \alpha \) such that \( e^{i\theta_0}H + i\alpha \) is the closed upper half-plane. Since the range of \( g \) lies in \( S \subseteq H \), \( h = e^{i\theta_0}g + i\alpha \) satisfies \( \text{Im} \, h \geq 0 \). Since \( m_2 \) is a probability measure,
\[ 0 \leq \int_{T^2} (\text{Im } h) \, dm_2 \]
\[ = \text{Im} \int_{T^2} h \, dm_2 \]
\[ = \text{Im} \left( e^{i\theta_0} \int_{T^2} g \, dm_2 + i\alpha \right). \]

It follows that \( \xi_1 = \int_{T^2} g \, dm_2 \) lies in \( H \), and so the lemma is proved.

Let \( \zeta = f(z, w) \) be an extended complex-valued function defined on the polydisc \( U^2 \), and let \( P \) be a point on \( \partial U^2 \), the boundary of \( U^2 \). The \textit{cluster set} \( C(f, P) \) of \( f \) at the point \( P \) is the set of all values \( \zeta \) in the extended complex plane such that there exists a sequence \( \{(z_n, w_n)\} \) in \( U^2 \) with the properties that \( (z_n, w_n) \to P \) and \( f(z_n, w_n) \to \zeta \).

Preliminaries aside, we shall now apply our lemma to prove the following extension of a theorem of Carathéodory [1] (cf. also Collingwood and Lohwater [2, p. 96]).

**Theorem 2.** Let \( f(z, w) \) be analytic and bounded, \(|f| < M\), in \( U^2 \). Let \( A \) denote the set

\[ \{(e^{i\theta}, e^{i\varphi}) \mid \theta_1 < \theta < \theta_2, \varphi_1 < \varphi < \varphi_2\} . \]

If for every point \((e^{i\theta}, e^{i\varphi}) \in A - E\), where \( m_2(E) = 0\), the radial limit

\[ f^*(e^{i\theta}, e^{i\varphi}) = \lim_{r \to 1} f(re^{i\theta}, re^{i\varphi}) \]

exists and lies in a set \( V \), then, for every point \( P \in A \), the cluster set \( C(f, P) \) is contained in \( \bar{V} \), the closure of the convex hull of \( V \).

**Proof.** Since \( f \) is analytic and bounded in \( U^2 \), it follows from a theorem of Marcinkiewicz and Zygmund [8, p. 316] that there is a set \( E_1 \subseteq T^2 \) of measure zero such that \( f(z, w) \to f^*(e^{i\theta}, e^{i\varphi}) \) uniformly, whenever \((z, w) \to (e^{i\theta}, e^{i\varphi})\) from inside any fixed Stolz domain whose vertex is at the point \((e^{i\theta}, e^{i\varphi}) \in E_1\). We shall assume, without any circumlocution, that the negligible set \( E, m_2(E) = 0 \), mentioned in the theorem includes the set \( E_1 \).

For a suitable choice of the point \((z_0, w_0) \) in \( U^2 \), the map

\[ (\zeta, \eta) = \Phi(z, w) = \begin{pmatrix} z_0 - z \\ w_0 - w \\ 1 - z_0 \bar{z} \\ 1 - \bar{w}_0 w \end{pmatrix} \]

transforms the set \( A \) into a product set \( B \subseteq T^2 \), whose area we may write as
1 - \varepsilon/2. Moreover, \( \Phi \) takes the set \( E, m_2(E) = 0 \), into a set \( \tilde{F} \subseteq T^2 \) whose measure is also zero.

We shall next cover the set \( F \cup (T^2 - B) \) with an open set \( G \) such that \( m_2(G) < \varepsilon \), and consider the function

\[
g(\zeta, \eta) = f \circ \Phi^{-1}(\zeta, \eta) = f \left( \frac{z_0 - \zeta}{1 - \overline{z}_0 \zeta}, \frac{w_0 - \eta}{1 - \overline{w}_0 \eta} \right).
\]

Lemma 1 allows us to write

\[ f(z_0, w_0) = g(0, 0) = \xi_1 + \xi_2, \]

where \(|\xi_2| < 2\varepsilon M\), and where \( \xi_1 \) lies in the closure of the convex hull of the set

\[ S = \{ g^*(e^{i\alpha}, e^{i\beta}) \mid (e^{i\alpha}, e^{i\beta}) \in G^c \} . \]

We will show that \( S \subseteq V \). Then \( \xi_1 \in \tilde{V} \). The proof will then be complete, because \( \epsilon \) tends to zero with \(|P - (z_0, w_0)|\).

If \((e^{i\alpha}, e^{i\beta}) \in G^c\), then the point \((e^{i\theta}, e^{i\varphi}) = \Phi^{-1}(e^{i\alpha}, e^{i\beta})\) is an element of \( A - E\). Moreover, \( g(re^{i\alpha}, re^{i\beta}) = f(r_1 e^{i\theta}, r_2 e^{i\varphi}) \), where \( r_1 \) and \( r_2 \) tend to 1 as \( r \) tends to 1. Now it follows from the result of Marcinkiewicz and Zygmund, cited at the beginning of this proof, that

\[
g^*(e^{i\alpha}, e^{i\beta}) = \lim_{r \to 1} g(re^{i\alpha}, re^{i\beta}) = f^*(e^{i\theta}, e^{i\varphi}).
\]

Since \( f^*(e^{i\theta}, e^{i\varphi}) \) lies in \( V \) by hypothesis, \( S \subseteq V \), and the proof of the theorem is complete.

**Theorem 3.** Let \( f(z, w) \) be analytic and bounded in \( U^2 \), let \((e^{i\alpha_0}, e^{i\varphi_0}) \) be an arbitrary point on \( T^2 \), and let \( E \) be an any subset of \( T^2 \) with \( m_2(E) = 0 \). Then

\[
\lim_{(z, w) \to (e^{i\alpha_0}, e^{i\varphi_0})} |f(z, w)| \leq \lim_{(\theta, \varphi) \to (\theta_0, \varphi_0)} \lim_{(z, w) \to (e^{i\theta}, e^{i\varphi}) \notin E} |f(z, w)|.
\]

**Proof.** Let \( E_1 \) denote the set of measure zero on \( T^2 \) for which

\[
f^*(e^{i\theta}, e^{i\varphi}) = \lim_{r \to 1} f(re^{i\theta}, re^{i\varphi})
\]

fails to exist. If \( E_2 = E \cup E_1 \), then

\[
\lim_{(\theta, \varphi) \to (\theta_0, \varphi_0)} \left( \lim_{(z, w) \to (e^{i\theta}, e^{i\varphi}) \notin E} |f(z, w)| \right) \geq \lim_{(\theta, \varphi) \to (\theta_0, \varphi_0)} \left( \lim_{(z, w) \to (e^{i\theta}, e^{i\varphi}) \notin E_2} |f(z, w)| \right) \geq \lim_{(\theta, \varphi) \to (\theta_0, \varphi_0)} \left| f^*(e^{i\theta}, e^{i\varphi}) \right| .
\]
Let \( \delta > 0 \), and let \( A_\delta \) denote the set
\[
\{ (e^{i\theta}, e^{i\varphi}) \mid \theta_0 - \delta < \theta < \theta_0 + \delta, \varphi_0 - \delta < \varphi < \varphi_0 + \delta \}.
\]

By Theorem 2, the cluster set \( C(f, (e^{i\theta_0}, e^{i\varphi_0})) \) is contained in the closure of the convex hull of the set
\[
V_\delta = \{ f^*(e^{i\theta}, e^{i\varphi}) \mid (e^{i\theta}, e^{i\varphi}) \in A_\delta - E_2 \}.
\]

Since \( \delta > 0 \) is arbitrary, it follows that
\[
\lim_{(\theta, \varphi) \to (\theta_0, \varphi_0), (e^{i\theta}, e^{i\varphi}) \notin E_2} |f^*(e^{i\theta}, e^{i\varphi})| \geq \lim_{(z, w) \to (e^{i\theta_0}, e^{i\varphi_0})} |f(z, w)|,
\]
and thus the proof of Theorem 3 is complete.

Alternatively, Theorem 3 may be stated as

**Theorem 4.** Let \( f(z, w) \) be analytic and bounded in \( U^2 \), let \((e^{i\theta_0}, e^{i\varphi_0})\) be any point on \( T^2 \), and let \( E \) be an arbitrary set of measure zero on \( T^2 \). Then
\[
\lim_{(z, w) \to (e^{i\theta_0}, e^{i\varphi_0})} |f(z, w)| \leq \lim_{(\theta, \varphi) \to (\theta_0, \varphi_0), (e^{i\theta}, e^{i\varphi}) \notin E} |f^*(e^{i\theta}, e^{i\varphi})|.
\]

In order to prove our main result, the Iversen Theorem for meromorphic functions in the polydisc \( U^2 \), we introduce some additional terminology and definitions.

Let \( \zeta = f(z, w) \) be an extended complex-valued function defined on the polydisc \( U^2 \). The **radial cluster set of \( f \) at the point** \((e^{i\theta}, e^{i\varphi})\) on \( T^2 \) is denoted by \( C_{\text{rad}}(f, (e^{i\theta}, e^{i\varphi})) \), and is defined as the set of all limiting values of \( f(z, w) \) when the defining sequences \( \{ (z_n, w_n) \} \) satisfy \( (z_n, w_n)^{1/n} = (e^{i\theta}, e^{i\varphi}) \) for all \( n \). We define the **radial boundary cluster set modulo \( E \)**, \( E \subseteq T^2 \), of \( f \) at the point \((e^{i\theta_0}, e^{i\varphi_0})\) to be the set
\[
C_{R \setminus \bar{E}}(f, (e^{i\theta_0}, e^{i\varphi_0})) = \bigcap_{n > 0} \left\{ \bigcup_{|((\theta, \varphi) - (\theta_0, \varphi_0)| < \eta} C_{\text{rad}}(f, (e^{i\theta}, e^{i\varphi})) \right\},
\]
where the union is taken over all \((e^{i\theta}, e^{i\varphi})\) such that \( 0 < |(\theta, \varphi) - (\theta_0, \varphi_0)| < \eta \) and \((e^{i\theta}, e^{i\varphi}) \notin E \), and where the bar denotes the closure operator.

We are now in a position to state and prove the principal result of this paper.

**Theorem 5.** If \( f(z, w) \) is a meromorphic function in \( U^2 \), and if \( E \subseteq T^2 \) is a set of measure zero, then at every point \((e^{i\theta_0}, e^{i\varphi_0}) \in T^2\),
\[
\partial C(f, (e^{i\theta_0}, e^{i\varphi_0})) \subseteq C_{R \setminus \bar{E}}(f, (e^{i\theta_0}, e^{i\varphi_0})),
\]
where \( \partial C \) denotes the boundary of \( C \).
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PROOF. If Theorem 5 were not true, there would be a point \( \zeta_0 \in \partial C(f, (e^{i\theta_0}, e^{i\phi_0})) \) not in \( C_{R-E}(f, (e^{i\theta_0}, e^{i\phi_0})) \). Thus, we would be able to find a \( \delta > 0 \) such that \( \zeta_0 \) is at a distance greater than \( \delta \) from all points of \( C_{R-E}(f, (e^{i\theta_0}, e^{i\phi_0})) \). Since \( \zeta_0 \) is a boundary point of \( C(f, (e^{i\theta_0}, e^{i\phi_0})) \), we can find a second point \( \zeta_1 \) not in \( C(f, (e^{i\theta_0}, e^{i\phi_0})) \) such that \( |\zeta_1 - \zeta_0| < \frac{1}{2}\delta \); this implies that the distance between \( \zeta_1 \) and any point of \( C_{R-E}(f, (e^{i\theta_0}, e^{i\phi_0})) \) is greater than \( \frac{1}{4}\delta \). Since the function \( g = (f - \zeta_1)^{-1} \) is bounded in a neighbourhood of \( (e^{i\theta_0}, e^{i\phi_0}) \), we can apply Theorem 4 to \( g \) and obtain

\[
\frac{2}{\delta} < \frac{1}{|\zeta_0 - \zeta_1|} \leq \lim_{(z, w) \to (e^{i\theta_0}, e^{i\phi_0})} |g(z, w)| \leq \lim_{(\theta, \phi) \to (\theta_0, \phi_0)} |g^*(e^{i\theta}, e^{i\phi})| \leq \frac{2}{\delta}.
\]

This contradiction establishes the theorem.

Since \( C_{R-E}(f, (e^{i\theta}, e^{i\phi})) \) is a subset of \( C(f, (e^{i\theta}, e^{i\phi})) \), the inclusion relation in Theorem 5 is equivalent to the assertion that the difference set \( C(f, (e^{i\theta}, e^{i\phi})) - C_{R-E}(f, (e^{i\theta}, e^{i\phi})) \) is open.

We remark, in conclusion, that our proof of Theorem 5 yields the following stronger result:

\[
\partial C(f, (e^{i\theta_0}, e^{i\phi_0})) \subseteq \bigcap_{n > 0} \left\{ \bigcup_{0 < |(\theta, \phi) - (\theta_0, \phi_0)| < \frac{\delta}{n}} \bigcup_{(e^{i\theta}, e^{i\phi}) \notin E} C^{*}_{rad}(f, (e^{i\theta}, e^{i\phi})) \right\},
\]

where \( C^{*}_{rad}(f, (e^{i\theta}, e^{i\phi})) \) denotes the set of all points \( \zeta \) such that there is a sequence \( \{(z_n, w_n)\} \) in \( U^2 \) with the properties that \( (z_n, w_n) = (r_n e^{i\theta}, r_n e^{i\phi}) \) for all \( n \), \( r_n \to 1 \), and \( f(z_n, w_n) \to \zeta \).

A more detailed treatment of the relations between the various cluster sets of a function defined on the polydisc will be provided by the authors in a forthcoming paper.

REFERENCES


CASE WESTERN UNIVERSITY
CLEVELAND, OHIO 44106, U.S.A.

AND

UNIVERSITY OF HAWAII
HONOLULU, HAWAII 96822, U.S.A.