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SOME COMBINATORIAL IDENTITIES

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The purpose of this note is to state and prove, in an elementary way, some formulas for evaluating certain finite sums of quotients between products of binomial coefficients, classical as well as q-generalised. The following notations will be used throughout in the following:

$$a \geqq b \geqq 0, \quad k,l,m > 0, \quad 0 < a_1 \leqq a_2 \leqq \ldots \leqq a_m ,$$

$$0 < b_1 \leqq b_2 \leqq \ldots \leqq b_{m-1} ,$$

$$a_2 \leqq b_1,\ a_3 \leqq b_2, \ldots, a_m \leqq b_{m-1} ,$$

$$s = a_1 + a_2 + \ldots + a_m - b_1 - b_2 - \ldots - b_{m-1}$$

are integral numbers.

$$\binom{a}{b} = \frac{a!}{b!\,(a-b)!}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{(1-q)(1-q^2)\ldots(1-q^a)}{(1-q)(1-q^2)\ldots(1-q^b)(1-q)(1-q^2)\ldots(1-q^{a-b})}$$

$$A = \frac{a_1!a_2!\ldots a_m!}{b_1!b_2!\ldots b_{m-1}!} .$$

$$B = \sum_{n=1}^{m-1} \frac{b_n(b_n+1)}{2} - \sum_{n=1}^{m} \frac{a_n(a_n+1)}{2} .$$

$$C = \sum_{n=1}^{m-1} \left(1+\tfrac{1}{2}+\ldots+\frac{1}{b_n}\right) - \sum_{n=1}^{m} \left(1+\tfrac{1}{2}+\ldots+\frac{1}{a_n}\right) .$$

$$D = \frac{(1-q)(1-q^2)\ldots(1-q^{a_1})\ldots(1-q)(1-q^2)\ldots(1-q^{a_m})}{(1-q)(1-q^2)\ldots(1-q^{b_1})\ldots(1-q)(1-q^2)\ldots(1-q^{b_{m-1}})} .$$

$$E = \sum_{n=1}^{m} \frac{1-q^{a_n+1}}{1-q} - \sum_{n=1}^{m-1} \frac{1-q^{b_n+1}}{1-q} .$$

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$$F = \sum_{n=1}^{m} \frac{1-q^{a_n+1}}{q^{a_n}(1-q)} - \sum_{n=1}^{m-1} \frac{1-q^{b_n+1}}{q^{b_n}(1-q)}.$$

$$S = \sum_{k=1}^{a_1} \frac{\binom{a_1}{k}\binom{a_2}{k}\cdots\binom{a_m}{k}}{\binom{b_1}{k}\binom{b_2}{k}\cdots\binom{b_{m-1}}{k}} (-1)^k k^l .$$

$$T = \sum_{k=0}^{a_1} \frac{\begin{bmatrix} a_1 \\ k \end{bmatrix}\begin{bmatrix} a_2 \\ k \end{bmatrix}\cdots\begin{bmatrix} a_m \\ k \end{bmatrix}}{\begin{bmatrix} b_1 \\ k \end{bmatrix}\begin{bmatrix} b_2 \\ k \end{bmatrix}\cdots\begin{bmatrix} b_{m-1} \\ k \end{bmatrix}} (-1)^k q^{\frac{k(k-1)}{2}} q^{k(l-s+1)} .$$

The following formulas will then be valid

(1)	$S = 0$	for $1 \leqq l \leqq s-1$
(2)	$S = (-1)^s A$	for $1 \leqq l = s$
(3)	$S = (-1)^{s+1} AB$	for $1 \leqq l = s+1$
(4)	$S = -1$	for $0 = l \leqq s-1$
(5)	$S = A-1$	for $0 = l = s$
(6)	$S = AB-1$	for $0 = l = s+1$
(7)	$S = C$	for $-1 = l \leqq s-1$
(8)	$S = C-A$	for $-1 = l = s$
(9)	$S = C-AB$	for $-1 = l = s+1$
(10)	$T = 0$	for $0 \leqq l \leqq s-1$
(11)	$T = D$	for $0 \leqq l = s$
(12)	$T = DE$	for $0 \leqq l = s+1$
(13)	$T = (-1)^s D q^B$	for $-1 = l \leqq s-1$
(14)	$T = D(1-q^B)$	for $-1 = l = s$
(15)	$T = D(E+q^B)$	for $-1 = l = s+1$
(16)	$T = (-1)^s DF q^B$	for $-2 = l \leqq s-1$
(17)	$T = D(1+Fq^B)$	for $-2 = l = s$
(18)	$T = D(E-Fq^B)$	for $-2 = l = s+1$

Most of the formulas (1)–(9) are well known for $m=1$, see e.g. [2, (1.13), (1.45) etc.]. For (7) in the case $m=2$, see [4, p. 44 and 92]. For (10) in the case $m=1$, $l=s-1$, see [3, (2.14) p. 92] (with $x=n$). Of course, the formulas could be stated in terms of generalised hypergeometric series.

PROOFS OF THE FORMULAS. For the formulas (1)–(9), setting

$$f(x) = \frac{x(x-1)\ldots(x-b_1)x(x-1)\ldots(x-b_2)\ldots x(x-1)\ldots(x-b_{m-1})}{x(x-1)\ldots(x-a_1)x(x-1)\ldots(x-a_2)\ldots x(x-1)\cdots(x-a_m)}$$

we have by development in partial fractions, remembering the stated inequalities between $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_{m-1}$,

$$(19)\quad Af(x)\cdot x^l = \sum_{k=1}^{a_1} \frac{\binom{a_1}{k}\binom{a_2}{k}\cdots\binom{a_m}{k}}{\binom{b_1}{k}\binom{b_2}{k}\binom{b_{m-1}}{k}}(-1)^k(-1)^s k^l \frac{1}{x-k}+\alpha+\frac{\beta}{x}+\frac{\gamma}{x^2}\,.$$

Here $\alpha=0$ in all cases with the exception of (3), (6) and (9), where we have $\alpha=A$.

In the cases (1)–(6) we have $\gamma=0$, for (1)–(3), $\beta=0$, whereas for (4)–(6) we have $\beta=(-1)^s$.

In the cases (7)–(9), the value of γ will not be used (actually $\gamma=(-1)^s$), whereas, with $g(x)=Axf(x)$,

$$\beta = g'(0) = g(0)\frac{g'(0)}{g(0)} = g(0)\left[\frac{d}{dx}\log|g(x)|\right]_{x=0} = (-1)^{s+1}C\,.$$

After multiplication on both sides of (19) with $x^2(x-1)(x-2)\ldots(x-a_1)$, the formulas follow by comparing coefficients for the highest power of x in the resulting polynomial identity.

For the formulas (10)–(18) we use a similar procedure, setting

$$h(x) = \frac{(x-1)(x-q)\ldots(x-q^{b_1})\ldots(x-1)(x-q)\ldots(x-q^{b_{m-1}})}{(x-1)(x-q)\ldots(x-q^{a_1})\ldots(x-1)(x-q)\ldots(x-q^{a_m})}\times$$
$$\times\frac{x^{a_1-1}x^{a_2-1}\ldots x^{a_m-1}}{x^{b_1-1}x^{b_2-1}\ldots x^{b_{m-1}-1}}$$

and

$$(20)\qquad Dh(x)x^{l-s+1}$$
$$= \sum_{k=0}^{a_1}\frac{\begin{bmatrix}a_1\\k\end{bmatrix}\begin{bmatrix}a_2\\k\end{bmatrix}\cdots\begin{bmatrix}a_m\\k\end{bmatrix}}{\begin{bmatrix}b_1\\k\end{bmatrix}\begin{bmatrix}b_2\\k\end{bmatrix}\cdots\begin{bmatrix}b_{m-1}\\k\end{bmatrix}}(-1)^k q^{\frac{k(k-1)}{2}}\, q^{k(l-s+1)}\frac{1}{x-q^k}+\alpha+\frac{\beta}{x}+\frac{\gamma}{x^2}\,.$$

Here $\alpha=0$ with the exception of (12), (15) and (18), where $\alpha=D$. In the cases (10)–(15), $\gamma=0$, for (10)–(12), $\beta=0$, whereas for (13)–(15), we have $\beta=(-1)^{s+1}Dq^B$. For (16)–(18) we have $\beta=(-1)^{s+1}Dq^BF$. The formulas follow on multiplication in (20) with $x^2(x-1)(x-q)\ldots(x-q^{a_1})$ and comparing coefficients as before.

Mention should be made of another method of proof, which in the cases (1)–(5) and (7) starts with the formula

(21)

$$\sum_{k=0}^{a_1} \frac{\binom{a_1}{k}\binom{a_2}{k}\cdots\binom{a_m}{k}}{\binom{b_1}{k}\binom{b_2}{k}\cdots\binom{b_{m-1}}{k}}(-1)^k\frac{1}{k+t} = \frac{1}{t}\cdot\frac{\binom{b_1+t}{b_1}\binom{b_2+t}{b_2}\cdots\binom{b_{m-1}+t}{b_{m-1}}}{\binom{a_1+t}{a_1}\binom{a_2+t}{a_2}\cdots\binom{a_m+t}{a_m}},$$

$$s \geqq 0$$

which follows from partial fraction development of $Af(x)/(x+t)$. Formulas (1)–(5) and (7) follows on developing (21) in powers of t and $1/t$. The formula (21) and with it (1)–(5) and (7), could be generalised, using gammafunctions and residue calculus, to a summation of an infinite series, where the differences between tha a's and b's are integral numbers, but not necessarily the a's and b's themselves. For a simple example, see [1, problem 6083 p. 205].

REFERENCES

1. Amer. Math. Monthly, 83 (1976), 205.
2. H. W. Gould, *Combinatorial Identities*, Morgantown, W.Va., 1972.
3. H. W. Gould, *The q-series generalisation of a formula of Sparre Andersen*, Math. Scand. 9 (1961), 90–94.
4. Nordisk Matematisk Tidskrift, 6 (1958), 44 and 92.

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