A NOTE ON FREE DIRECT SUMMANDS

ISTVÁN BECK and PETER J. TROSBORG

1. Introduction.

In the paper [1] Bass demonstrated a useful technique for dealing with big projective modules (i.e. non-finitely generated projective modules). By the so-called “Eilenberg-swindle” he proved: Let $P$ be a big projective module which is a quotient of a big free module $F$; if $P$ has a direct summand isomorphic to $F$ then $P$ is isomorphic to $F$. With this result at hand the problem of showing that a given projective module is free naturally leads on to look for free direct summands in modules.

In the paper [2] we proved a theorem of this kind. Let $R$ be a ring with Jacobson-radical $N$, and let $F$ be a free module. If $F'$ is a submodule of $F$ such that $F = F' + NF$ then $F'$ has a direct summand isomorphic to $F$. Theorem 2 of the present paper is an improvement of this earlier result. In theorem 3 we give a necessary and sufficient condition for a module $M$ to have a direct summand isomorphic to $R^{|I|}$, where $I$ is an infinite set, and we give some applications.

In the following rings are associative with identity and modules are left unitary modules. For a free $R$-module $F$ with basis $\{e_i\}_{i \in I}$ we set the support of an element $x \in F$, $x = \sum r_i e_i$, to be

\[ \text{supp} (x) = \{i \in I \mid r_i \neq 0\}. \]

2. Free direct summands.

To simplify the proof of Theorem 2 we first prove a set-theoretic lemma.

**Lemma 1.** Let $A$ be a set and let $X$ be a subset of the cartesian product $A \times A$. There exists a subset $B$ of $A$ satisfying:

(i) $B$ has a (non-reflexive) well-ordering such that if $b_1, b_2 \in B$ and $b_1 < b_2$ then $(b_1, b_2) \in X$.

(ii) If $a \in A$ and $(b, a) \in X$ for all $b \in B$, then $a \in B$.

**Proof.** Let $A$ be non-empty and let $\Omega$ be the set of all pairs $(C, <_c)$ where $C \subseteq A$ and $<_c$ is a well-ordering of $C$ such that if $c_1, c_2 \in C$ and $c_1 <_c c_2$ then...

---

Received March 24, 1977.
$(c_1, c_2) \in X$. $\Omega$ is partially ordered by $(C, < C) \leq (D, < D)$ if $C \subseteq D$ and the restriction of $< D$ to $C$ is $< C$ and $c < D d$ whenever $c \in C$ and $d \in D \setminus C$. The set $\Omega$ is not empty and inductively ordered. It is then straightforward to verify that a maximal member $(B, < )$ of $\Omega$ satisfies conditions (i) and (ii) of Lemma 1.

For a free $R$-module $F$ with basis $\{e_i\}_{i \in I}$ let $\pi_i(x) = r_i$ $(i \in I)$ when $x = \sum r_i e_i$ is an element of $F$. These coordinate-projections are surjective $R$-homomorphisms from $F$ to $R$.

**Theorem 2.** Let $F$ be a free $R$-module with basis $\{e_i\}_{i \in I}$ where $I$ is an infinite index-set. If $\{x_i\}_{i \in I}$ is a set of elements of $F$ such that $\pi_i(x_i) = 1$ for every $i \in I$, then there exists a set $J \subseteq I$ of the same cardinality as $I$ such that $\{x_j\}_{j \in J} \cup \{e_i\}_{i \in I \setminus J}$ is a basis for $F$.

**Proof.** Let $X$ be the following subset of $I \times I$:

$$X = \{ (i, j) \in I \times I \mid j \notin \text{Supp} (x_i) \}.$$ 

According to Lemma 1 there exists a set $J \subseteq I$, well-ordered by $<$, such that

(i) If $j_1, j_2 \in J$ and $j_1 < j_2$ then $j_2 \notin \text{Supp} (x_{j_1})$.

(ii) $(\bigcup_j \text{Supp} (x_j)) \cup J = I$.

Since $i \in \text{Supp} (x_i)$ we have $\bigcup_j \text{Supp} (x_j) = I$ and as $\text{Supp} (x_j)$ is a finite set for every $j \in J$ we obtain that $\text{card} (J) = \text{card} (I)$. It remains to prove that $\{x_j\}_{j \in J} \cup \{e_i\}_{i \in I \setminus J}$ is a basis for $F$. Let $\varphi: F \to \sum_j R e_j$ be the projection, that is, $\varphi (e_j) = e_j$ for $j \in J$ and $\varphi (e_i) = 0$ for $i \in I \setminus J$. Set $y_j = \varphi (x_j)$ for $j \in J$. Due to (i) we have

$$x_j = e_j + \sum_{k \in J, k < j} r_k e_k + \sum_{i \notin J} s_i e_i$$

hence

$$y_j = e_j + \sum_{k \in J, k < j} r_k e_k .$$

Assume that

$$u_1 y_{j_1} + \ldots + u_n y_{j_n} = 0,$$

where $u_i \in R$, and $j_1 < \ldots < j_n$.

Since $e_{j_n}$ occurs only in the representation of $y_{j_n}$ and with coefficient 1 it follows that $u_n = 0$. By induction we get all the $u_i$'s to be zero. Thus: if $u_1 x_{j_1} + \ldots + u_n x_{j_n} \in \text{Ker} \varphi$ then $u_1 = \ldots = u_n = 0$, implying that $\{x_j\}_{j \in J}$ is a basis for the free module $\sum_j R x_j$, and

$$\sum_j R x_j \cap \text{Ker} \varphi = \{0\} .$$

To prove that $\sum_j R x_j + \text{Ker} \varphi = F$ it suffices to show that $\sum_j R y_j = \sum_j R e_j$. 


Obviously $\sum_j Rv_j \subseteq \sum_j Rv_j$. Suppose that $e_j \notin \sum_j Rv_j$ for some $j \in J$. Let $j_0$ be the smallest $j \in J$ with this property. But as $y_{j_0} = e_{j_0} + \sum_k r_k e_k$ where the summation is taken over values of $k \in J$ with $k < j_0$ we have a contradiction. As $\text{Ker} \varphi = \sum_{i \in J} Rv_i$ the proof is complete.

**Definition.** Let $\{ \pi_i \}_{i \in I}$ be a family of $R$-homomorphisms between $R$-modules $M$ and $N$. We shall say that the family is **locally finite** if for each $m \in M$, $\pi_i(m) = 0$ for all but a finite number of $i$'s.

**Theorem 3.** Let $M$ be an $R$-module and let $I$ be an infinite set. The following conditions are equivalent.

1° $M$ has a direct summand isomorphic to $R^{(I)}$
2° There exists a locally finite family of surjective $R$-homomorphisms $\{ \pi_i \}_{i \in I}$ from $M$ to $R$.

**Proof.** It is enough to prove 2° $\Rightarrow$ 1°. Let $\{ e_i \}_{i \in I}$ be a basis for $R^{(I)}$ and define $f: M \to R^{(I)}$ by

$$f(m) = \sum_i \pi_i(m)e_i.$$ 

For every $i \in I$ there exists an element $m_i \in M$ with $\pi_i(m_i) = 1$; letting $x_i = f(m_i)$ the family $\{ x_i \}_{i \in I}$ satisfies the condition in Theorem 2. Let $J \subseteq I$ be such that $\text{card } J = \text{card } I$ and $\{ x_j \}_{j \in J} \cup \{ e_i \}_{i \in I \setminus J}$ is a basis for $R^{(I)}$. Let $F = \sum_j Rx_j$. Then $F \subseteq f(M)$ and $F$ is a direct summand of $R^{(I)}$. Hence $F$ is also a direct summand of $f(M)$, and consequently $M$ has a surjection onto a module isomorphic to $R^{(I)}$.

Let $N$ denote the Jacobson-radical of a ring $R$ and let $F$ be a free module. If $F'$ is a submodule of $F$ such that $F = F' + NF$, then we proved in [2] that $F'$ has a surjection onto $F$. This was the keyresult in establishing that $P/NP \cong F/NF$ implies $P \cong F$ whenever $P$ is projective and $F$ is free.

**Corollary 4.** Let $F$ be a free module and $F' \subseteq F$ a submodule such that $F' + NF = F$. Then $F$ has a direct summand $F'', F'' \subseteq F'$ and $F'' \cong F$.

**Proof.** Let $\{ e_i \}_{i \in I}$ be a basis for $F$. If $I$ is finite it follows from Nakayama's lemma that $F' = F$. So assume that $I$ is infinite. Writing $e_i = y_i + z_i$, $y_i \in F'$, $z_i \in NF$, we get that $y_i$ can be written $\sum_I r_{ij} e_j$ where $r_{ii}$ is a unit in $R$. Letting $x_i = r_{ii}^{-1}y_i$ we get a family $\{ x_i \}_{i \in I}$ as in Theorem 2 and we are done.

**Definition.** Let $P$ be a projective module. A **dual basis** for $P$ is a family
$\{\pi_i, q_i\}_{i \in I}$ where $\pi_i \in \text{Hom}(P, R)$ and $q_i \in P$ such that for all $x \in P$, $x = \sum_j \pi_i(x)q_i$ ($\{\pi_i\}$ locally finite).

**Theorem 5.** Let $P$ be a big projective module of type $\alpha$ (the type is the smallest cardinal of a set of generators). The following conditions are equivalent:

1° $P$ is free
2° $P$ has a dual basis $\{\pi_i, q_i\}_{i \in I}$ where all the $\pi_i$'s are surjective.
3° $P$ has a dual basis $\{\pi_i, q_i\}_{i \in I}$ such that $\pi_i$ is surjective for all $j$ in a subset $J \subseteq I$ with card $J = \text{card } I = \alpha$.
4° There exists a locally finite family $\{\pi_i\}_{i \in I}$, where $\pi_i \in \text{Hom}(P, R)$ is surjective for all $i \in I$ and card $I = \alpha$.

**Proof.** 1° $\Rightarrow$ 2° $\Rightarrow$ 3° $\Rightarrow$ 4° are all evident.
4° $\Rightarrow$ 1°. It follows from Theorem 3 that $P$ has a surjection to $R^{(I)}$, and using the result of Bass mentioned in the introduction it follows that $P$ is free.

**Problem.** Let $P$ be finitely generated projective with dual basis $\{\pi_i, q_i\}$; $i = 1, 2, \ldots, n$ such that all the $\pi_i$'s are surjective. Is $P$ a free module? This is true if $R$ is commutative or left noetherian.

Elements of the form $x_i = e_i + \sum_{j \neq i} r_{ij}e_j$ often arise as generators of kernels in free modules. Let $Q$ be an $R$-module with the property that any finitely generated homomorphic image of $Q$ is zero. If $R$ is an integral domain with field of quotients $K$, $K \neq R$, then $K$ has this property. It is easily seen that a module $Q$ has this property if and only if $Q$ has no maximal submodules. We close this paper with an application of Theorem 2, which shows that the relations of such a module are “big”.

**Theorem 6.** Let $Q$ be an $R$-module with no maximal submodule. Let $\varphi: F \to Q$ be surjective with $F$ free. Then Ker $\varphi$ has a direct summand isomorphic to $F$.

**Proof.** Let $\{e_i\}_{i \in I}$ be a basis for $F$ and set $q_i = \varphi(e_i)$ (notice that $I$ must be infinite). Then for every $i \in I$, $q_i \in \sum_{j \neq i} Rq_j$. It follows that for every $i \in I$ there exists $x_i \in \text{Ker } \varphi$ such that

$$x_i - e_i \in \sum_{j \neq i} R e_j.$$  

Now the proof is easily completed by Theorem 2.
REFERENCES


UNIVERSITY OF BERGEN
NORWAY

AND

UNIVERSITY OF COPENHAGEN
DENMARK